Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 62, No. 3, March 1, 1978

Robust Linear-Quadratic Minimization*

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Submitted by G. Leitmann

We define various ways in which linear-quadratic control problems can be robust and show that there are conditions which are simultaneously necessary and sufficient for robust problems to have a solution when such conditions do not exist for nonrobust problems.

1. INTRODUCTION

Conditions for the solvability of linear-quadratic continuous-time minimization problems have been studied in a number of papers, e.g., [1–5]. The more recent work has shown the value of characterizing problem solvability in terms of nonnegativity conditions involving certain Riemann–Stieltjes integrals. In this work, one somewhat bothersome problem remains—there are generally gaps between necessary conditions and sufficient conditions. It was the attempt to eliminate these gaps that led to the work reported in this paper.

In the broadest terms, what we find is that the gap vanishes if one is studying problems which are in some way robust, i.e., tolerant of small changes to some or all of the relevant parameters. To the extent that nonrobust problems can be regarded as qualitatively ill posed (though in strict terms, or quantitative terms, they may be well posed), we are saying that there is no gap in the case of a well-posed problem.

In Section 2, we study robustness in terms of initial time and initial state. In Section 3, we examine robustness for problems where the terminal state is nominally fixed as zero. The robustness is in terms of the value of the terminal state, the terminal time and a terminal weighting matrix for the free end-point problem.

* Work supported by the Australian Research Grants Committee.

0022-247X/78/0623–0620$02.00/0

2. Initial Condition Results

We study the system

$$\dot{x} = F(t)\,x + G(t)\,u, \qquad t_0 \leqslant t \leqslant t_f \tag{2.1}$$

with $x(t_0) = x_0$ prescribed, $x(\cdot)$ of dimension n and $u(\cdot)$ of dimension m. Associated with (2.1) is the performance index

$$V[x_0, t_0, u(\cdot)] = \int_{t_0}^{t_f} \{x'Q(t)\,x + 2u'H(t)\,x + u'R(t)\,u\}\,dt + x'(t_f)\,Sx(t_f). \tag{2.2}$$

The matrices $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, $Q(\cdot)$, and $R(\cdot)$ all have dimensions consistent with (2.1) and (2.2) and are continuous. (This has the obvious meaning; the matrices have continuous entries.) The matrix S is constant. Without loss of generality, we assume $Q(\cdot)$, $R(\cdot)$, and S are symmetric. The controls $u(\cdot)$ are assumed to be piecewise continuous on $[t_0, t_f]$. Frequently, we shall study classes of problems in which one or more of x_0, t_0, and t_f vary. In general, we shall be interested in the question of when (2.2) possesses an infimum, denoted $V^*[x_0, t_0]$, which is not $-\infty$.

For convenience, we shall assume in this section that $x(t_f)$ is free, although the results of this section carry over to the constrained endpoint problem with minor changes, and these minor changes do *not* involve the behavior of quantities in the vicinity of t_0. Since the results of this section are almost all considered with behavior in the vicinity of t_0, these minor changes are also conceptually insignificant.

We now define further notation. With $P(\cdot)$ an $n \times n$ symmetric matrix with entries of bounded variation, defined the Riemann–Stieltjes differential

$$dM(P) = \begin{bmatrix} dP + (PF + F'P + Q)\,dt & (PG + H)\,dt \\ (PG + H)'\,dt & R\,dt \end{bmatrix}. \tag{2.3}$$

Then we say that $dM(P) \geqslant 0$ on $[t_1, t_2]$ if $P(t)$ is defined on $[t_1, t_2]$ and

$$\int_{t_1}^{t_2} y'(t)\,dM(P)\,y(t) \geqslant 0 \tag{2.4}$$

for any $(n + m)$-vector function of time $y(\cdot)$, with the first n entries continuous, and the last m entries piecewise continuous. The integral is defined in the usual Riemann–Stieltjes sense. We shall say that $dM(P) \geqslant 0$ *within* any interval I, closed or open at either end, if $dM(P) \geqslant 0$ on $[t_1, t_2]$ for all $[t_1, t_2] \subset I$.

Several points should be noted. First, $dM(P) \geqslant 0$ on $[t_1, t_2]$ and $dM(P) \geqslant 0$ on $[t_2, t_3]$imply $dM(P) \geqslant 0$ on $[t_1, t_3]$. Second, $dM(P) \geqslant 0$ within I if and only if $\int_{t_1}^{t_2} dM(P) = M(t_2) - M(t_1)$ is nonnegative for all $[t_1, t_2] \subset I$. Third, if $\dot{P}$ exists in $[t_1, t_2]$, then $dM(P) \geqslant 0$ on $[t_1, t_2]$ if and only if

$$\begin{bmatrix} \dot{P} + PF + F'P + Q & PG + H \\ (PG + H)' & R \end{bmatrix} \geqslant 0. \tag{2.5}$$

Our starting point will be the following result, an amalgam of results in [4, 5].

PROPOSITION 2.1. *Suppose that $V[0, t_0, u(\cdot)] \geqslant 0$ for all $u(\cdot)$ and that*

$$\int_{t_0}^{t_1} \Phi(t_1, \tau)\, G(\tau)\, G'(\tau)\, \Phi'(t_0, \tau)\, d\tau > 0 \qquad \forall t_1 \in (t_0, t_f] \tag{2.6}$$

(so that all states are reachable from $x(t_0) = 0$ at any time $t_1 > t_0$). Then,

$$\infty > V^*[x_1, t_1] > -\infty \qquad \forall x(t_1) = x_1, \quad \forall t_1 \in (t_0, t_f]$$

and there exists a symmetric $P(\cdot)$ of bounded variation defined on $(t_0, t_f]$ such that $P(t_f) \leqslant S$ and $dM(P) \geqslant 0$ within $(t_0, t_f]$. Conversely, if there exists a symmetric $P(\cdot)$ of bounded variation defined on $[t_0, t_f]$ with $P(t_f) \leqslant S$ and with $dM(P) \geqslant 0$ within $[t_0, t_f]$, then $V[0, t_0, u(\cdot)] \geqslant 0$ for all $u(\cdot)$.

Observe that the two parts of Proposition 2.1 are not true converses. In the first part but not the second, reachability is required; in the first part, $dM(P) \geqslant 0$ within $(t_0, t_f]$ while in the second, $dM(P) \geqslant 0$ within $[t_0, t_f]$.

Note that the condition $V[0, t_0, u(\cdot)] \geqslant 0$ for all $u(\cdot)$ is equivalent to $V^*[0, t_0] = 0$. (If $V[0, t_0, u(\cdot)] < 0$ for some particular $u(\cdot)$, scaling that $u(\cdot)$ scales V, and so V can be made arbitrarily negative, i.e., $V^*[0, t_0] = -\infty$. Hence $V^*[0, t_0] > -\infty$ implies $V[0, t_0, u(\cdot)] \geqslant 0$ for all $u(\cdot)$. Then, $V[0, t_0, u(\cdot)] \geqslant 0$ for all $u(\cdot)$ and the observation $V[0, t_0, u(t) \equiv 0] = 0$ shows that $V^*[0, t_0] = 0$.) Instead then of seeking conditions for $V^*[0, t_0] = 0$, we suggest a change of viewpoint, based on seeking conditions for a more robust property. Thus we seek conditions not merely for $V^*[0, t_0]$ to be zero, but conditions for $V^*[0, t']$ to be zero for t' in a neighborhood of t_0 (robustness as far as initial time is concerned); also, we seek conditions for $V^*[x_0, t_0]$ to be finite for x_0 in a neighborhood of 0. (Because of the linear-quadratic nature of the problem, this means that $V^*[x_0, t_0]$ is finite for all x_0.)

As will be seen, the gap between necessary conditions and sufficient conditions in Proposition 2.1 vanishes with this change of viewpoint.

We first study problems with arbitrary x_0.

PROPOSITION 2.2. *$V^*[x_0, t_0] > -\infty$ for all x_0 if and only if there exists on $[t_0, t_f]$ a symmetric $P(\cdot)$ of bounded variation such that $P(t_f) \leqslant S$ and $dM(P) \geqslant 0$ on $[t_0, t_f]$.*

Proof. Suppose $P(\cdot)$ exists satisfying the conditions listed. As noted in [4, 5], it follows from an easy calculation that

$$V[x_0, t_0, u(\cdot)]$$
$$= \int_{t_0}^{t_f} [x' \ u']\, dM(P) \begin{bmatrix} x \\ u \end{bmatrix} + x'(t_f)\,[S - P(t_f)]\, x(t_f) + x'(t_0)\, P(t_0)\, x(t_0).$$

Thus $x'(t_0)\, P(t_0)\, x(t_0)$ lower bounds $V[x_0, t_0, u(\cdot)]$, and existence of $V^*[x_0, t_0] > -\infty$ is immediate.

For the converse, the arguments of [4, Proposition 3, 4, and Theorem 2] or of [5, Section 3] can be carried through, provided that $V^*[x_0, t_0]$ is quadratic in x_0, to conclude the existence of $P(\cdot)$ with the requisite properties. That $V^*[x_0, t_0]$ has the quadratic property is proved in the Appendix.

Remarks. (1) There exist problems without the robustness property, i.e., $V^*[0, t_0] = 0$ but $V^*[x_0, t_0] = -\infty$ for some x_0, or equivalently there exists no $P(\cdot)$ of bounded variation such that $P(t_f) \leqslant S$ and $dM(P) \geqslant 0$ on $[t_0, t_f]$: see [4] for an example.

(2) Either by using the proposition or simple first principles arguments, one can see that if $V^*[x_0, t_0] > -\infty$ for all x_0, then $V^*[x(t), t] > -\infty$ for all $x(t)$, $t \in [t_0, t_f]$.

It is somewhat harder to obtain robustness with respect to initial time.

PROPOSITION 2.3. *Suppose there exist $t_{-1} < t_0$ and definitions on $[t_{-1}, t_0)$ of $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, $Q(\cdot)$, and $R(\cdot)$ such that these quantities are continuous on $[t_{-1}, t_f]$, such that*

$$\int_{t_{-1}}^{t_1} \Phi(t_1, \tau)\, G(\tau)\, G'(\tau)\, \Phi'(t_1, \tau)\, d\tau > 0 \qquad \forall t_1 \in (t_{-1}, t_f] \tag{2.7}$$

and such that $V[0, t_{-1}, u(\cdot)] \geqslant 0$ for all $u(\cdot)$. Then there exists on $[t_0, t_f]$ a symmetric $P(\cdot)$ of bounded variation such that $P(t_f) \leqslant S$ and $dM(P) \geqslant 0$ on $[t_0, t_f]$.

Conversely, if such a $P(\cdot)$ exists, then there exist extensions of the domain of definition of $F(\cdot)$, etc., as specified, and such that not only is $V[0, t_{-1}, u(\cdot)] \geqslant 0$ for all $u(\cdot)$, but $V^*[x_{-1}, t_{-1}]$ exists for all x_{-1}.

Proof. The first part of the proposition is immediate from the first part of Proposition 2.1. We proceed to the second part. Suppose for the moment, that $\dot{P}$ exists in a neighborhood of t_0. The result will be first proved for this case, and then the restriction will be removed. Take $t_{-1} < t_0$ and otherwise arbitrary, and take $F(\cdot)$, $G(\cdot)$ on $[t_{-1}, \frac{1}{2}(t_{-1} + t_0)]$ to be any constant, completely controllable pair; define $F(\cdot)$, $G(\cdot)$ on $(\frac{1}{2}(t_{-1} + t_0), t_0)$ so as to ensure smooth joins, and continuity on $[t_{-1}, t_f]$.

Let $\dot{P}(t) = \dot{P}(t_0)$ on $[t_{-1}, t_0)$; this ensures that $P(t)$ is continuous on $[t_{-1}, t_0]$. Choose $Q(\cdot)$ on $[t_{-1}, t_0)$ such that $\dot{P} + PF + F'P + Q$ is constant on $[t_{-1}, t_0]$; this ensures that $Q(\cdot)$ is continuous on $[t_{-1}, t_f]$. Choose $H(\cdot)$ on $[t_{-1}, t_0)$ so that $PG + H$ is constant on $[t_{-1}, t_0]$; again, $H(\cdot)$ is continuous on $[t_{-1}, t_f]$. Finally, choose $R(\cdot)$ on $[t_{-1}, t_0)$ to be constant and equal to $R(t_0)$, ensuring thereby continuity on $[t_{-1}, t_f]$.

These choices quarantee that

$$\dot{M} = \begin{bmatrix} \dot{P} + PF + F'P + Q & PG + H \\ (PG + H)' & R \end{bmatrix}$$

is constant on $[t_{-1}, t_0]$. The fact that $dM(P) \geqslant 0$ on $[t_0, t_1]$ for all $t_1 \in [t_0, t_f]$ and that $\dot{P}$ exists in a neighborhood of t_0 ensures that $\dot{M}(t_0) \geqslant 0$. Consequently $\dot{M} \geqslant 0$ on $[t_{-1}, t_0]$. Then $dM(P) \geqslant 0$ on $[t_{-1}, t_0]$ and $[t_0, t_1]$, and therefore on $[t_{-1}, t_f]$. By Proposition 2.2, $V^*[x_{-1}, t_{-1}] > -\infty$ for all x_{-1}, while (2.7) holds because of the choice of $F(\cdot)$, $G(\cdot)$ on $[t_{-1}, \frac{1}{2}(t_{-1} + t_0)]$.

Now suppose that $\dot{P}$ does not exist in a neighborhood of t_0. Consider the following equation for $t \leqslant t_0$.

$$\dot{P} + PF + F'P + Q - (PG + H)[R_0 + (t_0 - t)^{1/2} I]^{-1} (PG + H)' = 0, \tag{2.8}$$

where $R_0 = R(t_0)$, and $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, and $Q(\cdot)$ are arbitrary continuous extensions of these quantities into $t < t_0$, chosen to ensure continuity at t_0. Equation (2.8) is initialized by the known quantity $\Pi_0 = P(t_0)$.

In case R_0 is nonsingular, $P(t)$ is guaranteed to exist in some interval $[t_{-2}, t_0]$, with $\dot{P}$ guaranteed to exist in $[t_{-2}, t_0]$. However, in case R_0 is singular, $\dot{P}$ is unbounded as $t \uparrow t_0$, and so an existence question arises, which we now resolve. Set $\tau = (t_0 - t)^{1/2}$. Then

$$\frac{dP}{dt} = \frac{dP}{d\tau}\frac{d\tau}{dt} = -\frac{1}{2(t_0 - t)^{1/2}}\frac{dP}{d\tau}$$

and with τ the new independent variable, (2.8) becomes

$$-\frac{dP}{d\tau} + 2\tau[PF + F'P + Q] - (PG + H)\, 2\tau(R_0 + \tau I)^{-1} (PG + H)' = 0. \tag{2.9}$$

This equation is defined in the interval $\tau \geqslant 0$; strictly, we should have used different symbols for $F(\cdot)$, etc., to reflect their change of independent variable. The equation has $P(\tau)|_{\tau=0} = \Pi_0$. Now $\tau(R_0 + \tau I)^{-1}$ is obviously continuous for $\tau > 0$, and it is not hard to check that it is continuous at $\tau = 0$. Therefore, $P(\tau)$ exists in some interval $[0, \tau_2]$ with $dP(\tau)/d\tau$ continuous there. It follows that (2.8) has a solution in some interval $[t_{-2}, t_0]$ with $dP(t)/dt$ existing on $[t_{-2}, t_0)$ and, in fact, in a neighborhood of t_{-2}.

Now (2.8) implies that

$$\begin{bmatrix} \dot{P} + PF + F'P + Q & PG + H \\ (PG + H)' & R \end{bmatrix} \geqslant 0$$

on $[t_{-2}, t_0)$, where $R = R_0 + (t_0 - t)^{1/2} I$ and is nonsingular. Since $\lim_{t \uparrow t_0} P(t) = \Pi_0 = P(t_0)$, it is clear that $dM(P) \geqslant 0$ within $[t_{-2}, t_f]$. Now we can use the first part of the proof to further extend on $[t_{-1}, t_{-2}]$, since $\dot{P}(t)$ exists in a neighborhood of t_{-2}. In this way, the controllability assumption is fulfilled, and the proposition is proved.

Remarks. (1) This proposition introduces an *extendability* criterion. The first use of the extendability idea of which we are aware is in [1], where nonsingular problems only were discussed.

(2) In case $R(t)$ is nonsingular throughout $[t_0, t_f]$, the above proposition is much easier to prove, for $V^*[x_1, t_1] = x_1' P(t_1) x_1$ with $P(t_1)$ the solution of a Riccati equation, $t_1 \in [t_0, t_f]$. Then $\dot{P}(t_1)$ automatically exists for all $t_1 \in [t_0, t_f)$.

(3) The result contained in the preceding proposition might lead one to make the following conjecture, which we can readily show is false: Suppose that $V^*[x_0, t_0] > -\infty$ for all x_0, and that $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, $Q(\cdot)$, $R(\cdot)$ are defined on $[t_{-1}, t_0]$ such that these quantities are continuous on $[t_{-1}, t_f]$; then there exists $t_{-2} \in [t_{-1}, t_0)$ such that $V^*[x_{-2}, t_{-2}] > -\infty$. (Effectively, it is being claimed that the set of t_0 for which $V^*[x_0, t_0] > -\infty$ is open.) By way of counterexample, consider $\dot{x} = u$, $V[x(\tau), \tau, u(\cdot)] = \int_\tau^{t_f} [xu + \rho(t) u^2] \, dt$, where $\rho(t) = 0$, $t \in [t_0, t_f]$, and $\rho(t) < 0$ for $t \in [t_{-1}, t_0)$, with $\rho(\cdot)$ continuous on $[t_{-1}, t_f]$. Certainly then, it is impossible that $V^*[x_{-2}, t_{-2}] > -\infty$ for $t_2 \in [t_{-1}, t_0)$, since ρ is negative on $[t_2, t_0)$. However, $V[x(t_0), t_0, u(\cdot)] = \frac{1}{2}x^2(t_f) - \frac{1}{2}x^2(t_0)$ for all $u(\cdot)$, so that $V^*[x_0, t_0] = -\frac{1}{2}x_0^2$. Under the restriction $R(t) > 0$ on $[t_0, t_f]$, the above conjecture is however true, for $V^*[x(\tau), \tau]$ is defined via the solution to a Riccati equation which, if it exists at t_0, exists in a neighborhood around t_0, *including points to the left of* t_0.

The main result of this section is that it is possible to tie together the Riemann–Stieltjes condition with the V^* existence and V nonnegativity coupled with extendability. We collect preceding results:

PROPOSITION 2.4. *With notation as previously, the following conditions are equivalent.*

(a) $V^*[x_0, t_0]$ *is finite for all* x_0.

(b) $V^*[x(t), t]$ *is finite for all* $x(t)$ *and for all* $t \in [t_0, t_f]$.

(c) $V[0, t_{-1}, u(\cdot)] \geqslant 0$ *for some* $t_{-1} < t$, *and one has controllability on* $[t_{-1}, t]$ *for all* $t \in (t_{-1}, t_f]$.

(d) *There exists on* $[t_0, t_f]$ *a symmetric* $P(\cdot)$ *of bounded variation with* $P(t_f) \leqslant S$ *and* $dM(P) \geqslant 0$ *within* $[t_0, t_f]$.

We emphasize the fact that the conditions involving the Riemann–Stieltjes integral are simultaneously necessary and sufficient.

3. End Point Constraints

In this section, we consider problems associated with the constraint $x(t_f) = 0$. We conclude equivalence of a Riemann–Stieltjes integral inequality condition, and robustness with respect to each of terminal state, terminal weighting time, and terminal weighting matrix in a free end-point problem with penalty function.

Throughout the section, we adopt the controllability assumption

$$\int_t^{t_f} \Phi(t, \tau)\, G(\tau)\, G'(\tau)\, \Phi'(t, \tau)\, d\tau > 0 \qquad \forall t \in [t_0, t_f) \tag{3.1}$$

guaranteeing that the constraint $x(t_f) = 0$ can be met starting from arbitrary $x(t)$, $t \in [t_0, t_f)$. We adopt the notation

$$V[x(t), t, u(\cdot); x(t_f) = x_f] \qquad \text{and} \qquad V^*[x(t), t; x(t_f) = x_f]$$

to denote the value of the performance index $V[x(t), t, u(\cdot)]$ and the infimum of such values, subject to the constraint $x(t_f) = x_f$.

One form of the result available in the literature is as follows [4, 5].

Proposition 3.1. *Assume that $V^*[x(t), t; x(t_f) = 0]$ exists for all $x(t)$ and $t \in [t_0, t_f)$. Then there exists on $[t_0, t_f)$ a symmetric $P(t)$ of bounded variation such that $dM(P) \geqslant 0$ within $[t_0, t_f)$. Conversely, if there exists on $[t_0, t_f]$ a symmetric $P(t)$ of bounded variation such that $dM(P) \geqslant 0$ within $[t_0, t_f]$ then $V^*[x(t), t; x(t_f) = 0]$ exists for all $x(t)$ and $t \in [t_0, t_f)$.*

Observe the difference between the necessity and sufficiency condition. To remove this condition, a robust problem statement is adopted. The following proposition summarizes the results for robust problems.

Proposition 3.2. *The following conditions are equivalent.*

(a) (Terminal state robustness.) *$V^*[x(t), t; x(t_f) = x_f]$ exists for all $x(t)$, $t \in [t_0, t_f)$ and all x_f.*

(b) (Terminal time robustness.) *There exist $t_f' > t_f$ and definitions of $F(\cdot)$, $G(\cdot)$, etc., on $[t_f, t_f']$ such that these matrices are continuous on $[t_0, t_f']$, such that the controllability assumption (3.1) holds with t_f replaced by t_f' and such that $V^*[x(t), t; x(t_f') = 0]$ exists for all $x(t)$, $t \in [t_0, t_f]$. In fact, $V^*[x(t), t; x(t_f') = x_f]$ exists for all $x(t)$, $t \in [t_0, t_f')$, and all x_f.*

(c) (Terminal weighting matrix robustness.) *For S sufficiently large in* (2.2), $V^*[x(t), t]$ *exists for all* $x(t)$, $t \in [t_0, t_f]$, *with* $x(t_f)$ *free.*

(d) (Riemann–Stieltjes Integral Inequality Condition.) *There exists on* $[t_0, t_f]$ *a symmetric* $P(\cdot)$ *of bounded variation such that* $dM(P) \geqslant 0$ *within* $[t_0, t_f]$.

Proof. The equivalence of (a), (b), and (d) follows by the same type of arguments as used in the last section. That (c) implies (d) is immediate from Proposition 2.4. To show that (d) implies (c), choose $S \geqslant P(t_f)$ with $P(\cdot)$ given as in (d). Then Proposition 2.4 again applies to yield (c).

Note that there are problems lacking the type of robustness considered here. For example, with $\dot{x} = (t-1)\,u$, $V[x_0, 0, u(\cdot)] = \int_0^1 xu\,dt$, one can show that $V[x_0, 0, u(\cdot); x(1) = 0] \geqslant 0$ for all $u(\cdot)$ and x_0, so that $V^*[x_0, 0; x(1) = 0]$ exists for all x_0. One can also show that the only solution of $dM(P) \geqslant 0$ within $[0, 1)$ is $P(t) = \frac{1}{2}(1-t)^{-1}$, and because of the behavior of $P(t)$ as $t \to 1$, it is impossible to secure $dM(P) \geqslant 0$ within $[t_0, t_f]$.

4. Conclusions

The main thrust of paper has been to show there exist conditions involving the nonnegativity of certain Riemann–Stieltjes integrals which are both necessary and sufficient for certain linear-quadratic optimization problems to have a solution. These problems are not identical, though they are closely related, to those conventionally examined in the literature; rather, they have an inherent quality of robustness, which makes them qualitatively well posed.

One set of results relates to robustness around the initial time or state, and a second set to robustness around a final time or constrained state. In the latter context, robust problems are those for which penalty function ideas are applicable.

On the grounds then of mathematical tidiness and the rational appeal of qualitatively well-posed problems, we suggest a change of viewpoint as to which linear-quadratic minimization problems should be thought of as standard.

Appendix

Let us drop the variable t_0. By elementary algebra, [6] shows that $V^*(x)$: $R^n \to R$ is quadratic if and only if, for all $\xi_1, \xi_2 \in R^n$ and $\lambda \in R$,

$$V^*(\lambda\xi_1) = \lambda^2 V^*(\xi_1), \tag{A1}$$

$$V^*(\xi_1 + \xi_2) + V^*(\xi_1 - \xi_2) = 2[V^*(\xi_1) + V^*(\xi_2)], \tag{A2}$$

$$V^*(\xi_1 + \lambda\xi_2) - V^*(\xi_1 - \lambda\xi_2) = \lambda[V^*(\xi_1 + \xi_2) - V^*(\xi_1 - \xi_2)]. \tag{A3}$$

The first two equalities are proved in [7] (see also [4]) and flow from the quadratic nature of $V[x(t_0), t_0, u(\cdot)]$ and the assumed existence of $V^*(x)$ for all x. We turn to (A3). For $\lambda = \pm 1$, (A3) is immediate. So assume $|\lambda| \neq 1$. Let $u_1(\cdot)$ and $u_2(\cdot)$ be arbitrary controls. It is straightforward to check that the linear-quadratic nature of the problem yields

$$\begin{aligned} V(\xi_1 + \lambda\xi_2, u_1 + \lambda u_2) &- V(\xi_1 - \lambda\xi_2, u_1 - \lambda u_2) \\ &= \lambda V(\xi_1 + \xi_2, u_1 + u_2) - \lambda V(\xi_1 - \xi_2, u_1 - u_2). \end{aligned} \tag{A4}$$

Assume for the moment that $\lambda \geqslant 0$. Let $\epsilon > 0$ be arbitrary. Let $v(\cdot)$, $w(\cdot)$ be controls such that $V(\xi_1 + \lambda\xi_2, v) \leqslant V^*(\xi_1 + \lambda\xi_2) + \epsilon$, $V(\xi_1 - \xi_2, w) \leqslant V^*(\xi_1 - \xi_2) + \epsilon$ and define u_1, u_2 by $u_1 + \lambda u_2 = v$, $u_1 - u_2 = w$. Since $\lambda \neq -1$, u_1 and u_2 are well defined. Then (A4) yields

$$\begin{aligned} \lambda V(\xi_1 + \xi_2, u_1 + u_2) &+ V(\xi_1 - \lambda\xi_2, u_1 - \lambda u_2) \\ &\leqslant V^*(\xi_1 + \lambda\xi_2) + \lambda V^*(\xi_1 - \xi_2) + (1 + \lambda)\epsilon \end{aligned}$$

so that, since ϵ is arbitrary,

$$\lambda V^*(\xi_1 + \xi_2) + V^*(\xi_1 - \lambda\xi_2) \leqslant V^*(\xi_1 + \lambda\xi_2) + \lambda V^*(\xi_1 - \xi_2).$$

A similar argument provides the reverse inequality, so that (A3) is verified for $\lambda \geqslant 0$. (One must use $\lambda \neq 1$ in this argument.) Suppose now $\lambda \leqslant 0$. Set $\mu = -\lambda$. Then

$$\begin{aligned} V^*(\xi_1 + \lambda\xi_2) - V^*(\xi_1 - \lambda\xi_2) &= -[V^*(\xi_1 + \mu\xi_2) - V^*(\xi_1 - \mu\xi_2)] \\ &= -\mu[V^*(\xi_1 + \xi_2) - V^*(\xi_1 - \xi_2)] \\ &= \lambda[V^*(\xi_1 + \xi_2) - V^*(\xi_1 - \xi_2)]. \end{aligned}$$

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Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.