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THE MATRIX CAUCHY INDEX: PROPERTIES AND APPLICATIONS*

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Abstract. The notion of the Cauchy index of a real rational scalar function is generalized to define the Cauchy index of a real rational symmetric matrix in terms of the behavior of the matrix at real singularities of its elements. Alternative characterizations are obtained which flow from representations of the real rational matrix using a Laurent series, a matrix fraction description, and a state variable realization. These characteristics involve a Hankel and a Bezoutian matrix. A matrix Sturm theorem is obtained and its use for evaluating the index is indicated. Descriptions of certain impedance matrices arising in passive circuit theory are given using the matrix Cauchy index.

1. Introduction. The Cauchy index over (a, b) of a real rational function $z(s)=p(s)/q(s)$ where $p(\cdot), q(\cdot)$ are real polynomials is defined as the number of jumps of $z(s)$ from $-\infty$ to $+\infty$ less the number from $+\infty$ to $-\infty$ as s moves from a to b [1]. (Jumps at a and b are not counted.) The Cauchy index is useful in examining properties involving the root distribution of polynomials; for example, it can be used for determining the number of distinct real roots of a polynomial, for checking the positivity for all s of a polynomial, or for checking the number of roots of a polynomial with negative real part [1]. Such calculations are important in many problems of linear system theory.

In this paper, we define the Cauchy index over (a, b) of a symmetric real rational matrix $Z(s)$. The definition is given in terms of the behavior of the eigenvalues of $Z(s)$ as s moves from a to b. The ultimate aim of such a definition is to develop a tool useful in studying linear multivariable systems, describable by matrices of rational transfer functions, and indeed, towards the end of the paper, we indicate one or two such applications.

The bulk of the paper is however concerned with establishing the extent to which results known for the Cauchy index of a scalar transfer function can be carried over to the matrix case. The most important are those concerned with evaluation of the index of a function without determination of the points of discontinuity of the function. In fact, the index can be evaluated from the signature of a certain Hankel matrix, from the signature of a certain Bezoutian matrix, and with the aid of a Sturm sequence. Matrix generalizations of these ideas are given.

To put the paper in context we recall the following properties of the Cauchy index of a real rational scalar function $z(s)=p(s)/q(s)$, see [1]. Suppose that $p(s)$ and $q(s)$ are relatively prime polynomials with real coefficients. Without loss of generality, we may take the degree of $p(\cdot)$ to be less than that of $q(\cdot)$.

1. Suppose a partial fraction expansion of $z(s)$ is available as

$$z(s)=\sum_i\left[\frac{a_{0i}}{(s-\alpha_i)}+\frac{a_{1i}}{(s-\alpha_i)^2}+\cdots+\frac{a_{n_i i}}{(s-\alpha_i)^{n_i+1}}\right]+R(s) \tag{1.1}$$

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where α_i, a_{ji} are real and all singularities in $R(s)$ occur at strictly complex values of s. The α_i are distinct. Then

$$\text{(1.2)} \qquad I_a^b z(s) = \sum_{\substack{n_i \text{ even} \\ \alpha_i \in (a,b)}} \operatorname{sgn} a_{n_i i}.$$

2. Suppose that

$$\text{(1.3)} \qquad z(s) = \frac{p(s)}{q(s)} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \cdots$$

and consider the infinite Hankel matrix

$$\text{(1.4)} \qquad \mathcal{H} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Let $\mathcal{H}_{kk}$ denote the first k rows and columns of $\mathcal{H}$. Then if $q(\cdot)$ has degree n, one has for all $m \geqq n$

$$\text{rank } \mathcal{H}_{mm} = \text{rank } \mathcal{H}_{nn}, \qquad \text{signature } \mathcal{H}_{mm} = \text{signature } \mathcal{H}_{nn} = I_{-\infty}^{+\infty} z(s).$$

3. If we construct the Bezoutian form

$$\text{(1.5)} \qquad \frac{p(y)q(x) - q(y)p(x)}{x - y} = \sum_{i,j=1}^{n} C_{ij} x^{i-1} y^{j-1}$$

it is shown in [2] that the signature of the matrix (C_{ij}) is $I_{-\infty}^{+\infty} p(s)/q(s)$.

4. If we form a minimal realization $\{F, g, h\}$ of $z(s)$, i.e. construct an $n \times n$ matrix F and n-vectors g and h such that $z(s) = h'(sI - F)^{-1}g$ (and this is a standard construction of linear system theory; see e.g. [3]), then (see [2]) there exists a unique symmetric $n \times n$ matrix P such that $PF = F'P$, $Pg = h$, and signature $P = I_{-\infty}^{+\infty} z(s)$. For a particular choice of $\{F, g, h\}$, P is identical with the Bezoutian matrix.

5. One of the methods for calculating $I_a^b z(s)$ is based on Sturm's theorem [1], [4].

We consider a sequence of polynomials $f_1(\cdot), f_2(\cdot), \cdots, f_m(\cdot)$ satisfying the following two conditions:

(a) for any $x \in (a, b)$ at which any $f_k(x) = 0$, $f_{k-1}(x) f_{k+1}(x) < 0$;

(b) $f_m(x)$, the last member of the sequence, has no zeros in (a, b).

Such a sequence is called a *Sturm sequence in* (a, b). If all the terms are multiplied by the same polynomial, the sequence is termed a "generalized Sturm sequence".

If the two polynomials $f_1(x)$ and $f_2(x)$ are given and the degree of $f_2(x)$ is not greater than that of $f_1(x)$, then a generalized Sturm sequence may be constructed via the Euclidean algorithm, [1], [4]. One divides f_2 into f_1 obtaining remainder

$-f_3$. Then one divides f_3 into f_2, obtaining remainder $-f_4$. The procedure continues until there is no remainder. Thus, with quotient polynomials $q_i(\cdot)$, one has

$$
\begin{aligned}
f_1(s) &= q_1(s)f_2(s) - f_3(s), \\
f_2(s) &= q_2(s)f_3(s) - f_4(s), \\
&\vdots \\
f_{m-1}(s) &= q_{m-1}(s)f_m(s).
\end{aligned} \tag{1.6}
$$

THEOREM [1]. *If* $\{f_1(\cdot), \cdots, f_m(\cdot)\}$ *is a generalized Sturm sequence and* $V(x)$ *denotes the number of changes in sign of the sequence evaluated at the point* x, *then*

$$
I_a^b \frac{f_2(s)}{f_1(s)} = V(a) - V(b). \tag{1.7}
$$

This completes the resumé of those results associated with the Cauchy index of a scalar function which we seek to extend. In § 2, we define the Cauchy index of a symmetric real rational matrix and link it with a partial fraction expansion. In § 3, following some linear system theory preliminaries, the evaluation of the index by examining the signature of matrices is considered, and in § 4 the use of a Sturm sequence is described. Then, after having constructed the matrix Cauchy index, we consider some of the benefits of its application in passive circuit theory, and conclude with some suggestions of further applications.

2. The matrix Cauchy index and partial fraction decomposition.

DEFINITION. The *Cauchy index* of a real rational symmetric matrix $Z(s)$ over the real interval (a, b), denoted $I_a^b Z(s)$, is the number of eigenvalues of $Z(s)$ which jump from $-\infty$ to $+\infty$ minus the number which jump from $+\infty$ to $-\infty$ as the independent variable s traverses the real axis from a to b. Jumps at a and b are not counted.

This definition is obviously consistent with that for scalar functions. Further, diagonal matrices have an index equal to the sum of the indices of the diagonal elements.

Our aim in this section is to indicate how the index might be determined if a partial fraction decomposition of $Z(s)$ is available. Thus we suppose that

$$
Z(s) = \sum_i \frac{A_{0i}}{s - \alpha_i} + \frac{A_{1i}}{(s - \alpha_i)^2} + \cdots + \frac{A_{n_i i}}{(s - \alpha_i)^{n_i + 1}} + R(s) \tag{2.1}
$$

where the α_i are real distinct scalars, the A_{ji} are real symmetric matrices, and any singularities in $R(s)$ occur at strictly complex values of s or at $s = \infty$.

It is immediately evident that if $\alpha_i \in (a, b)$, there is a contribution (maybe zero) to $I_a^b Z(s)$ from the pole at $s = \alpha_i$; in fact, since any infinite eigenvalues at $s = \alpha_i$ must stem solely from those summands in $Z(s)$ with a pole at α_i, we must have

$$\int_a^b Z(s) = \sum_{\alpha_i \in (a,b)} \int_a^b \left[\frac{A_{0i}}{s-\alpha_i} + \frac{A_{1i}}{(s-\alpha_i)^2} + \cdots + \frac{A_{n_i i}}{(s-\alpha_i)^{n_i+1}} \right]. \tag{2.2}$$

Without loss of generality, it is clear that we can focus on the problem of computing the Cauchy index of

$$V(s) = \frac{A_0}{s} + \frac{A_1}{s^2} + \cdots + \frac{A_{n-1}}{s^n} \tag{2.3}$$

provided $0 \in (a, b)$. This we now do.

If A_{n-1} has full rank then the last term of $V(s)$ dominates the behavior near to the origin of all eigenvalues of $V(s)$. In this case the index of $V(s)$ is simply equal to the signature of A_{n-1}, $\sigma(A_{n-1})$, if n is odd, or zero if n is even. However, if A_{n-1} has less than full rank, some of the eigenvalues of $V(s)$ near $s=0$ will be determined by some or all of the A_{n-i-1} in addition to A_{n-1}.

In the remainder of this section, we discuss the calculations required to evaluate $\int_a^b V(s)$ in case A_{n-1} is singular. We shall use the fact that for any polynomial matrix $U(s)$ nonsingular at all singularities of $V(s)$

$$\int_a^b V(s) = \int_a^b U(s)V(s)U'(s). \tag{2.4}$$

This follows because nonsingular congruency transformations preserve signature, rank and symmetry.

With A_{n-1} singular, choose a constant nonsingular T such that

$$TA_{n-1}T' = \begin{bmatrix} A_{n-1}^{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad |A_{n-1}^{11}| \neq 0 \tag{2.5}$$

and replace $V(s)$ by $TV(s)T'$ for which

$$A_{n-1} = \begin{bmatrix} A_{n-1}^{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_i = \begin{bmatrix} A_i^{11} & A_i^{12} \\ A_i^{12'} & A_i^{22} \end{bmatrix}, \qquad i = 0, 1, 2, \cdots, n-2. \tag{2.6}$$

Let us force the A_i^{12} to be zero via a congruency transformation. Define matrices $C_1, C_2, \cdots, C_{n-1}$ by

$$\begin{aligned} A_{n-1}^{11}C_1 &= A_{n-2}^{12}, \\ A_{n-1}^{11}C_2 + A_{n-2}^{11}C_1 &= A_{n-3}^{12}, \\ &\vdots \\ A_{n-1}^{11}C_{n-1} + A_{n-2}^{11}C_{n-2} + \cdots + A_1^{11}C_1 &= A_0^{12} \end{aligned} \tag{2.7}$$

or equivalently, with $C(s) = C_1 s + C_2 s^2 + \cdots + C_{n-1}s^{n-1}$,

$$\begin{aligned} \left[\frac{A_{n-1}^{11}}{s^n} + \frac{A_{n-2}^{11}}{s^{n-1}} + \cdots + \frac{A_0^{11}}{s} \right] C(s) &= \frac{A_{n-2}^{12}}{s^{n-1}} + \frac{A_{n-3}^{12}}{s^{n-2}} + \cdots + \frac{A_0^{12}}{s} \\ &\quad + \text{matrix polynomial in } s. \end{aligned} \tag{2.8}$$

It follows from (2.8) that

$$\begin{bmatrix} I & 0 \\ -C'(s) & I \end{bmatrix} V(s) \begin{bmatrix} I & -C(s) \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{n-1}^{11} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{s^n} + \sum_{i=1}^{n-1} \begin{bmatrix} B_{i-1}^{11} & 0 \\ 0 & B_{i-1}^{22} \end{bmatrix} \frac{1}{s^i}$$

$$+ \text{polynomial in } s.$$

This equation shows that under a nonsingular congruency transformation $V(s)$ is equivalent to the direct sum of two matrices

$$V_1(s) = \frac{A_{n-1}^{11}}{s^n} + \sum_{i=1}^{n-1} \frac{B_{i-1}^{11}}{s^i}$$

and

$$V_2(s) = \sum_{i=1}^{n-1} \frac{B_{i-1}^{22}}{s^i}.$$

Evidently,

$$\mathrm{I}_a^b\, V(s) = \mathrm{I}_a^b\, V_1(s) + \mathrm{I}_a^b\, V_2(s).$$

In view of the nonsingularity of A_{n-1}^{11}, $\mathrm{I}_a^b\, V_1(s)$ is immediately obtainable. The matrix $V_2(s)$ is then treated in the same way as $V(s)$ above.

We also note that there exists a polynomial $U(s)$, nonsingular at $s = 0$, such that

$$U(s)\left[\frac{A_{n-1}^{11}}{s^n} + \sum_{i=1}^{n-1} \frac{B_{i-1}^{11}}{s^i}\right] U'(s) = \frac{A_{n-1}^{11}}{s^n} + \text{polynomial terms}$$

(this is easily checked). This means that the original $V(s)$ of (2.3) is congruent to a matrix of the form $\oplus_j D_j s^{-j} + R(s)$ where the D_j are symmetric nonsingular matrices or zero, and $R(s)$ is a polynomial matrix. (Here $\oplus$ denotes a direct sum.) The congruency transformation is of course nonsingular at $s = 0$. In summary we have:

THEOREM 2.1. *Let $V(s)$ be as in* (2.3). *Then there exists a polynomial matrix $U(s)$, nonsingular at $s = 0$, such that $U(s)V(s)U'(s) = \oplus_j D_j s^{-j} + R(s)$ where the D_j are symmetric nonsingular matrices or zero, $R(s)$ is a polynomial matrix, and, if $0 \in (a, b)$,*

$$\mathrm{I}_a^b\, V(s) = \sum_{j \text{ odd}} \sigma(D_j).$$

3. Hankel and related matrix descriptions. Before presenting a characterization of the Cauchy index using a Hankel matrix, we review a number of results concerning rational matrices, many of which have been highlighted in recent work on linear system theory. For references, see [3], [5], [6].

Let $W(s)$ be a $q \times r$ real rational matrix. The proper part of $W(s)$ (call it $\hat{W}(s)$) is that real rational matrix such that $\lim_{s\to\infty} \hat{W}(s) < \infty$ and $W(s) - \hat{W}(s) = W_{-k}s^{k-1} + \cdots + W_{-2}s$ for some $W_{-2}, W_{-3}, \cdots$ and the strictly proper part of $W(s)$ is $\hat{W}(s) - \hat{W}(\infty)$. If $A(s)$ and $B(s)$ are real polynomial matrices such that

$W = A^{-1}B$, A and B define a left matrix fraction decomposition of $W(s)$. In case A, B are relatively left prime, i.e. their greatest common left divisor is a unimodular matrix (a polynomial matrix of constant determinant), the decomposition is termed minimal, and the degree of $\det A(s)$ is termed the *(McMillan) degree of* $\hat{W}(s)$, written $\delta[\hat{W}]$. This quantity is of course the same for all relatively prime left matrix fraction decompositions. Various tests for relative primeness exist; in particular, relative primeness holds if and only if $[A(s) \quad B(s)]$ has full rank for all s. The same statements hold for right matrix fraction decompositions DC^{-1} and mixed matrix fraction decompositions $W = DA^{-1}B$, mutatis mutandis.

The quantity $\delta[\hat{W}(s)]$ is obtainable in several other ways. If for constant F, G, H, J one has $\hat{W}(s) = J + H'(sI - F)^{-1}G$ with F of least possible dimension, then this dimension is $\delta[\hat{W}(s)]$ and $\operatorname{rank}[G \quad FG \quad F^2G \quad \cdots] = \operatorname{rank}[H \quad F'H \quad (F')^2H \quad \cdots] = \delta[\hat{W}(s)]$. Again if

$$\hat{W}(s) = W_{-1} + \frac{W_0}{s} + \frac{W_1}{s^2} + \cdots$$

define the Hankel matrix associated with W (or $\hat{W}$) as

$$\mathcal{H} = \begin{bmatrix} W_0 & W_1 & W_2 & \cdots \\ W_1 & W_2 & W_3 & \cdots \\ W_2 & W_3 & W_4 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then $\delta[\hat{W}] = \operatorname{rank} \mathcal{H}$. If $\mathcal{H}_\rho$ denotes the matrix formed from the first ρ block rows and columns of $\mathcal{H}$, then $\operatorname{rank} \mathcal{H}_\rho = \operatorname{rank} \mathcal{H}_{\rho+k} = \operatorname{rank} \mathcal{H}$ for all k with $\rho =$ degree of least common denominator of entries of $\hat{W}(s)$.

Finally, we recall that if $W(s) = W_1(s) + W_2(s)$ and the entries of W_1 and W_2 have no poles in common, then $\delta[\hat{W}] = \delta[\hat{W}_1] + \delta[\hat{W}_2]$.

With these preliminaries in hand, we now consider symmetric matrices $Z(s)$ and the associated Hankel matrix, which will also be symmetric. The main result is given in Theorem 3.1 below, which is proved with several lemmas.

LEMMA 3.1. *Let $Z(s)$ be a $q \times q$ symmetric real rational matrix and let $U(s) = U_0 + U_{1s} + \cdots + U_k s^k$ be a $q \times q$ polynomial matrix, nonsingular at all poles of elements of $Z(s)$. Let*

$$\mathcal{U} = \begin{bmatrix} U_0 & U_1 & \cdots & U_k & O & \cdots \\ 0 & U_0 & \cdots & U_{k-1} & U_k & \cdots \\ 0 & 0 & \cdots & U_{k-2} & U_{k-1} & \cdots \\ \vdots & \vdots & & & & \vdots \end{bmatrix}$$

and let $\mathcal{H}$ be the Hankel matrix associated with $Z(s)$. Then the Hankel matrix associated with $U(s)Z(s)U'(s)$ is $\mathcal{U}\mathcal{H}\mathcal{U}'$, and it has the same rank and signature as $\mathcal{H}$.

Proof. The first claim follows by direct calculation of the coefficient of s^{-i} in $U(s)Z(s)U'(s)$.

For the second claim, let $Z(s) = A^{-1}(s)B(s)$ with degree $[\det A] = \delta[\hat{Z}]$. Then $[A(s) \quad B(s)]$ has full rank [5]. Since $U(s)$ is nonsingular whenever $A(s)$ is singular, singularities of $A(s)$ corresponding to poles of elements of $Z(s)$, $[A(s) \quad B(s)U'(s)]$ has full rank. Therefore $A^{-1}(s)B(s)U'(s)$ is a minimal matrix fraction decomposition of $Z(s)U'(s)$, and $\delta[\hat{Z}] = \delta[ZU']$. Repeating the argument gives $\delta[\hat{Z}] = \delta[UZU']$. Hence the Hankel matrices associated with $Z(s)$ and $U(s)Z(s)U'(s)$ have the same ranks.

To show that they have the same signature, we consider truncated matrices. Let $\mathcal{U}_{\rho\rho+k}$ denote the first ρ block rows and $\rho+k$ block columns of $\mathcal{U}$. Then $(\mathcal{U}\mathcal{H}\mathcal{U}')_\rho = \mathcal{U}_{\rho\rho+k}\mathcal{H}_{\rho+k}\mathcal{U}'_{\rho\rho+k}$, and so for suitably large ρ,

$$\operatorname{rank} \mathcal{H}_{\rho+k} = \operatorname{rank} \mathcal{H} = \operatorname{rank} \mathcal{U}\mathcal{H}\mathcal{U}' = \operatorname{rank} (\mathcal{U}\mathcal{H}\mathcal{U}')_\rho = \operatorname{rank} \mathcal{U}_{\rho\rho+k}\mathcal{H}_{\rho+k}\mathcal{U}'_{\rho\rho+k}.$$

Therefore $\mathcal{H}_{\rho+k}$ and $\mathcal{U}_{\rho\rho+k}\mathcal{H}_{\rho+k}\mathcal{U}'_{\rho\rho+k}$ have the same signature. Also $(\mathcal{H}_{\rho+k}) = \sigma(\mathcal{H})$ because $\mathcal{H}_{\rho+k}$ is a principal submatrix of $\mathcal{H}$ with the same rank. Similarly $\sigma(\mathcal{U}_{\rho\rho+k}\mathcal{H}_{\rho+k}\mathcal{U}'_{\rho\rho+k}) = \sigma(\mathcal{U}\mathcal{H}\mathcal{U}')$. Therefore $\sigma(\mathcal{H}) = \sigma(\mathcal{U}\mathcal{H}\mathcal{U}')$.

The above lemma allows us to evaluate the Hankel matrix of a particular $Z(s)$.

LEMMA 3.2. *Let*

$$Z(s) = \frac{Z_0}{s} + \frac{Z_1}{s^2} + \cdots + \frac{Z_{n-1}}{s^n}$$

be a $q \times q$ real symmetric matrix with $\mathcal{H}$ the associated Hankel matrix.

Then if $0 \in (a, b)$,

$$I_a^b Z(s) = \sigma(\mathcal{H}).$$

Proof. Let $U(s)$ be a polynomial matrix, existing by Theorem 2.1, nonsingular at $s = 0$ and such that $U(s)Z(s)U'(s) = \oplus_j D_j s^{-j} + R(s)$ where R is a polynomial matrix and the D_j are nonsingular or zero. As we know,

$$I_a^b Z(s) = \sum_{j \text{ odd}} \sigma(D_j).$$

Also, $\sigma(\mathcal{H}) = \sigma(\mathcal{U}\mathcal{H}\mathcal{U}')$. Now the Hankel matrix $\mathcal{U}\mathcal{H}\mathcal{U}'$ is the direct sum of Hankel matrices

$$\mathcal{H}^j = \begin{bmatrix} 0 & 0 & \cdots & 0 & D_j & 0 & \cdots \\ 0 & 0 & \cdots & D_j & 0 & 0 & \cdots \\ \vdots & \vdots & & & & & \\ 0 & D_j & & & & & \\ D_j & 0 & & & & & \\ 0 & 0 & & & & & \\ \vdots & \vdots & & & & & \end{bmatrix},$$

where D_j in the first row is in the jth block column. Evidently, $\sigma(\mathcal{U}\mathcal{H}\mathcal{U}') = \sum_j \sigma(\mathcal{H}^j) = \sum_j \sigma(\mathcal{H}_j^j)$. Now $\mathcal{H}_j^j$ is the Kronecker product $J_j \otimes D_j$ where

$$J_j = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and is $j \times j$. By [7, Thm. 4.3.6], $\sigma(\mathcal{H}_j^j) = \sigma(J_j)\sigma(D_j)$ and $\sigma(J_j)$ is easily seen to be $+1$ if j is odd, zero if j is even. Therefore $\sigma(\mathcal{U}\mathcal{H}\mathcal{U}') = \sum_{j \text{ odd}} \sigma(D_j)$, and the result follows.

It is immediate from the above proof that if $-\alpha \in (a, b)$, the symmetric real rational matrix

$$Z(s) = \frac{Z_0}{s+\alpha} + \frac{Z}{(s+\alpha)^2} + \cdots + \frac{Z_{n-1}}{(s+\alpha)^n} \tag{3.1}$$

has

$$\underset{a}{\overset{b}{I}}\, Z(s) = \sigma(\hat{\mathcal{H}}_n)$$

where

$$\hat{\mathcal{H}}_n = \begin{bmatrix} Z_o & Z_1 & \cdots & Z_{n-2} & Z_{n-1} \\ Z_1 & Z_2 & \cdots & Z_{n-1} & 0 \\ \vdots & & & & \vdots \\ Z_{n-1} & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

The next lemma relates the signature of this matrix to that of the usual Hankel matrix of $Z(s)$.

LEMMA 3.3. *Let $Z(s)$ be as in (3.1) and let $Z(s) = \sum_{i=0}^{\infty} A_i/s^{i+1}$. Then $\hat{\mathcal{H}}_n$ and*

$$\mathcal{H}_n = \begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1} \\ \vdots & & & \vdots \\ A_{n-1} & A_n & \cdots & A_{2n-2} \end{bmatrix}$$

are related by a nonsingular congruency transformation, and have the same rank and signature as $\mathcal{H}$, the Hankel matrix associated with $Z(s)$.

Proof. Let $\{F, G, H\}$ be a triple of matrices such that $H'(sI-F)^{-1}G = Z(s)$. It follows that $H'F^iG = A_i$ and, from (3.1), that

$$H'(sI - \overline{F + \alpha I})^{-1}G = \frac{Z_0}{s} + \frac{Z_1}{s^2} + \cdots + \frac{Z_{n-1}}{s^n}$$

so that $H'(F+\alpha I)^i G = Z_i$, with $H'(F+\alpha I)^{n+r}G = 0$ for all $r \geqq 0$.

Define the $q\times q$ matrix T_{ij} to be zero for $j>i$ and $\alpha^{i-j}\binom{i-1}{i-j}I$ for $j \leqq i$. Observe then that

$$\sum_{k=1}^{\infty} T_{ik}A_{k+l-2} = \sum_{k=1}^{i} \alpha^{i-k}\binom{i-1}{i-k}H'F^{k+l-2}G = H'F^{l-1}(F+\alpha I)^{i-1}G.$$

Similarly,

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} T_{ik}A_{k+l-2}T_{jl} = H'(F+\alpha I)^{i+j-2}G = Z_{i+j-2}.$$

Let $\mathcal{T}$ be the lower triangular matrix with block $i-j$ entry T_{ij}:

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 & \cdots \\ \alpha I & I & 0 & \cdots \\ \alpha^2 I & 2\alpha I & I & \cdots \\ \alpha^3 I & 3\alpha^2 I & 3\alpha I & \cdots \end{bmatrix}.$$

Then we have just shown that $\hat{\mathcal{H}} = \mathcal{T}\mathcal{H}\mathcal{T}'$, whence $\hat{\mathcal{H}}_n = \mathcal{T}_{nn}\mathcal{H}_n\mathcal{T}'_{nn}$. The lemma is then immediate.

COROLLARY 3.1. *With $Z(s)$ as in* (3.1), *$\mathcal{H}$ the associated Hankel matrix and $-\alpha \in (a, b)$,*

$$I_a^b\, Z(s) = \sigma(\mathcal{H}).$$

Up until now we have been dealing exclusively with rational symmetric matrices with single real poles. We next show that complex conjugate poles do not contribute to the signature of the Hankel matrix, before going on to discuss matrices with many poles.

LEMMA 3.4. *Let*

$$Z(s) = \frac{Z_0}{s+\alpha+j\beta} + \frac{Z_1}{(s+\alpha+j\beta)^2} + \cdots + \frac{Z_{n-1}}{(s+\alpha+j\beta)^n} \tag{3.2}$$
$$+ \frac{Z_0^*}{s+\alpha-j\beta} + \frac{Z_1^*}{(s+\alpha-j\beta)^2} + \cdots + \frac{Z_{n-1}^*}{(s+\alpha-j\beta)^n}$$

be a real symmetric matrix with associated Hankel matrix $\mathcal{H}$. Here α, β are real scalars, $\beta \neq 0$, and the Z_i are complex symmetric matrices. Then $\sigma(\mathcal{H}) = 0$.

Proof. Let $Z_i(s)$ denote the complex symmetric matrix obtained as the sum of the first n terms on the right side of (3.2). Let $\mathcal{H}^1$ denote the associated complex symmetric Hankel matrix. As we know, rank $\mathcal{H}_n^1 =$ rank $\mathcal{H}^1$; further, there exists [4, p. 340] a nonsingular complex P and integer t such that

$$\mathcal{H}_n^1 = P\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}P' = \sum_{i=1}^{t} p_i' p_i'$$

where p_i denotes the ith column of P. Therefore, $\mathcal{H}_n^{1*}=\sum_{i=1}^t p_i^* p_i^{*\prime}$ and

$$\mathcal{H}_n=\sum_{i=1}^t (p_i p_i' + p_i^* p_i^{*\prime}).$$

Because $Z_1(s)$ and $Z_1^*(s)$ have no poles in common, rank $(H_n)=\delta[Z(s)]=\delta[Z_1(s)]+\delta[Z_1^*(s)]=2$ rank $\mathcal{H}_n^1=2t$. Therefore $p_1,\cdots,p_t,p_1^*,\cdots,p_t^*$ are all linearly independent. Let $p_i=a_i+jb_i$ for real a_i, b_i. Then

$$\mathcal{H}_n=\sum_{i=1}^t (a_i a_i' - b_i b_i')$$

and since rank $\mathcal{H}_n=2t$, it follows that $\sigma(\mathcal{H}_n)=0$. Then $\sigma(\mathcal{H})=0$.

COROLLARY 3.2. *With $Z(s)$ as in Lemma 3.4, with $\mathcal{H}$ the associated Hankel matrix and with (a,b) arbitrary,*

$$\mathop{\mathrm{I}}_a^b Z(s)=\sigma(\mathcal{H})=0.$$

We are now in a position to state and prove the main result of the paper. In this result we use Lemma 3.2 and Corollaries 3.1 and 3.2 and generalize our symmetric real rational matrix to have no restrictions on pole position or pole multiplicity.

THEOREM 3.1. *Let $Z(s)$ be a $q\times q$ real rational symmetric matrix, and let $\mathcal{H}$ be the associated Hankel matrix. Then*

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} Z(s)=\sigma(\mathcal{H}).$$

Proof. Let $Z(s)$ be written as

$$Z(s)=Z_0(s)+Z_1(s)+\cdots+Z_p(s), \tag{3.3}$$

in which $Z_0(s)$ is polynomial, $Z_i(s)$ for $i\geqq 1$ is finite at infinity and has a least common denominator of the form $(s+\alpha)^n$ or $(s^2+2\alpha s+\alpha^2+\beta^2)^n$, $\beta\neq 0$; moreover, poles of entries of $Z_i(s)$ and $Z_j(s)$ are distinct. Now each $Z_i(s)$ is obviously symmetric, and evidently,

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} Z(s)=\sum_{i=1}^p \mathop{\mathrm{I}}_{-\infty}^{+\infty} Z_i(s).$$

Moreover, by Lemma 3.2, and Corollaries 3.1 and 3.2, for each i we have

$$\mathop{\mathrm{I}}_{-\infty}^{+\infty} Z_i(s)=\sigma(\mathcal{H}^i)$$

where $\mathcal{H}^i$ is the Hankel matrix associated with $Z_i(s)$.

From the Hankel matrix definition and (3.3), it follows that $\mathcal{H}=\sum_{i=1}^p \mathcal{H}^i$. Because poles of entries of $Z_i(s)$ and $Z_j(s)$ are disjoint for $i\neq j$, rank $\mathcal{H}=\delta[\sum_{i=1}^p Z_i(s)]=\sum_{i=1}^p \delta[Z_i(s)]=\sum_{i=1}^p$ rank $\mathcal{H}^i$. As noted in the Appendix, if the rank of the sum of two symmetric matrices is the sum of their ranks, then the signature is the sum of the signatures and accordingly,

$$\sigma(\mathcal{H})=\sum_{i=1}^{p}\sigma(\mathcal{H}^i)=\mathop{I}_{-\infty}^{+\infty} Z(s).$$

In the remainder of this section, we indicate in Corollaries 3.3 and 3.4 two alternative characterizations of the Cauchy index which flow from this theorem. The advantage of these corollaries is that they apply specifically to evaluation of the Cauchy index given a state space realization or matrix fraction description of the rational function.

COROLLARY 3.3. *Let $Z(s)$ be a $q\times q$ real rational symmetric matrix with proper part $\hat{Z}(s)$. Let $\hat{Z}(s)=J+H'(sI-F)^{-1}G$ with F of dimension $\delta[\hat{Z}(s)]$. Then there exists a unique symmetric P such that*

$$PF=F'P,\qquad PG=H \tag{3.4}$$

and

$$\sigma(P)=\mathop{I}_{-\infty}^{+\infty} Z(s).$$

Proof. From (3.4) it follows that

$$\begin{bmatrix} G' \\ G'F' \\ \vdots \\ G'F'^{n-1}\end{bmatrix} P[G\quad FG\quad\cdots\quad F^{n-1}G]=\begin{bmatrix} A_0 & A_1 & \cdots & A_{n-1}\\ \vdots & \vdots & & \vdots \\ A_{n-1} & A_n & \cdots & A_{2n-2}\end{bmatrix}=\mathcal{H}_n$$

where $A_i=H'F^iG$. Taking $n=\delta[\hat{Z}(s)]$ ensures that $\sigma(\mathcal{H}_n)=\sigma(\mathcal{H})$ and $[G\quad FG\quad\cdots\quad F^{n-1}G]$ has full rank. The result is then immediate.

Strictly, the above proof does not require minimality of the dimension of F, but simply the complete reachability [3] of the pair $[F, G]$.

The next corollary involves the generalized Bezoutian matrix associated with a matrix fraction decomposition of $Z(s)$ [8]. Suppose that left and right matrix fraction decompositions of an arbitrary $q\times r$ real rational $W(s)$ are available as $W=A^{-1}B=DC^{-1}$. Define matrices Δ_{ij} by

$$\sum_i\sum_j \Delta_{ij}x^{i-1}y^{j-1}=\frac{1}{x-y}[A(x)D(y)-B(x)C(y)]$$

and define the matrix Δ, the generalized Bezoutian matrix, as the matrix with block i-j entry Δ_{ij}.

In case now $Z(s)$ is a $q\times q$ symmetric matrix with left matrix fraction decomposition $A^{-1}(s)B(s)$, it is natural to take as an associated right-matrix fraction decomposition $B'(s)A'^{-1}(s)$. Then

$$\sum_i\sum_j \Delta_{ij}x^{i-1}y^{j-1}=\frac{1}{x-y}[A(x)B'(y)-B(x)A'(y)] \tag{3.5}$$

and it is evident that $\Delta_{ij}=\Delta'_{ji}$, so that $\Delta=\Delta'$.

COROLLARY 3.4. *Let $Z(s)$ be a $q \times q$* proper *real rational symmetric matrix, with left matrix fraction decomposition $A^{-1}(s)B(s)$. Define Δ_{ij} by (3.5) as the i-j block entry of Δ. Then*

$$I_{-\infty}^{+\infty} Z(s) = \sigma(\Delta).$$

Before proving the corollary, we remark that, as shown in [8], simple formulas exist for obtaining Δ; also, $\delta[\hat{Z}(s)] = \operatorname{rank} \Delta$.

Proof. Suppose that $A(s) = A_0 s^n + A_1 s^{n-1} + \cdots + A_n$. Then, as shown in [8] specializing to the case of $C(s) = A'(s)$,

$$\Delta = \begin{bmatrix} A_{n-1} & A_{n-2} & \cdots & A_1 & A_0 \\ A_{n-2} & A_{n-3} & \cdots & A_0 & 0 \\ \vdots & & & & \vdots \\ A_1 & A_0 & \cdots & 0 & 0 \\ A_0 & 0 & \cdots & 0 & 0 \end{bmatrix} \mathcal{H}_n \begin{bmatrix} A'_{n-1} & A'_{n-2} & \cdots & A'_1 & A'_0 \\ A'_{n-1} & A'_{n-3} & \cdots & A'_0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ A'_1 & A'_0 & \cdots & 0 & 0 \\ A'_0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Because $\operatorname{rank} \Delta = \delta[Z(s)] = \operatorname{rank} \mathcal{H}_n$, this congruency relationship implies that $\sigma(\Delta) = \sigma(\mathcal{H}_n) = I_{-\infty}^{+\infty} Z(s)$.

Using a result linking the generalized Bezoutian matrix to the generalized Sylvester matrix [8]—a matrix depending just on either a left or a right matrix fraction description—it is possible to formulate a Cauchy index result in terms of the inners [14] of the latter matrix for the case where the denominator polynomial matrix has the unit matrix as the coefficient of the highest degree term. This result is neither fully stated nor proved here as it appears to be a little restricted.

Theorem 3.1, Corollary 3.3 and Corollary 3.4 are appropriate for calculating $I_{-\infty}^{+\infty} Z(s)$ in case a description of $Z(s)$ is given in terms of Markov parameters, state variable equations, or a matrix fraction decomposition. In case entries of $Z(s)$ are given as rational functions, it is not clear what method is the best, and if the denominators of the entries of $Z(s)$ are easily factored, the ideas of § 2 may be the most effective. Note incidentally that the signature of a symmetric matrix can be evaluated using Jacobi's theorem [1], provided it is regularly arranged [7] first.

In the next section, results are given which will allow evaluation of the index over an arbitrary interval.

4. Matrix Sturm theorem. Here we are concerned with developing the matrix analogue to Sturm's theorem for scalar polynomials, as is briefly outlined in § 1 of this paper. We treat first the definition and existence of Sturmian sequences of polynomial matrices. We then show a connection between these sequences and the index of a rational symmetric matrix over an arbitrary real interval.

The following theorem is proved in the Appendix and is based on a result of [7].

THEOREM 4.1. *If $A(s)$ and $B(s)$ are $q \times q$ polynomial matrices in s and if $B(s)$ is nonsingular, then there exist polynomial matrices $Q(s)$ and $C(s)$ such that*

$$A(s) = B(s)Q(s) + C(s)$$

and either $C(s)=0$ *or else* $C(s)$ *is nonsingular with* degree [det $C(s)$] < degree [det $B(s)$]. *Moreover, if* $B^{-1}(s)A(s)$ *is symmetric, one can ensure that* $Q(s)$ *and* $B^{-1}(s)C(s)$ *are symmetric.*

Because either $C(s)=0$ or $C(s)$ is nonsingular, the formal development of a Euclidean algorithm is possible.

By analogy with the scalar case, we have the following.

DEFINITION. Let $Z(s)$ be a real rational nonsingular matrix with left matrix fraction decomposition $A_1^{-1}(s)A_2(s)$. A *left Sturm sequence associated with* $Z(s)$ is the sequence of real polynomial matrices $\{A_1(s), A_2(s), \cdots, A_m(s)\}$ obtained from

$$\begin{aligned} A_1(s) &= A_2(s)Q_2(s) - A_3(s), \\ A_2(s) &= A_3(s)Q_3(s) - A_4(s), \\ &\vdots \\ A_{m-1}(s) &= A_m(s)Q_m(s). \end{aligned} \tag{4.1}$$

Notice that existence (but not uniqueness) of the sequence is guaranteed by Theorem 4.1. Also, a typical equation of (4.1) can be written as

$$A_{k-1}^{-1}(s)A_{k-2}(s) = Q_{k-1}(s) - A_{k-1}^{-1}(s)A_k(s) \tag{4.2}$$

and using Theorem 4.1, we see that if $Z(s)$ is symmetric, each term in (4.2) is symmetric. As shown in the Appendix, $Q_{k-1}(s)$ is polynomial and $A_{k-1}^{-1}(s)A_k(s)$ is the sum of a strictly proper part and a symmetric singular constant term. An immediate consequence is that for arbitrary (a, b)

$$\mathop{I}_a^b A_{k-1}^{-1}(s)A_{k-2}(s) = -\mathop{I}_a^b A_{k-1}^{-1}(s)A_k(s). \tag{4.3}$$

In order to use (4.3) recursively to obtain $I_a^b Z(s) = I_a^b A_1^{-1}(s)A_2(s)$, for nonsingular $Z(s)$ we need to relate $I_a^b A_{k-1}^{-1}(s)A_k(s)$ to $I_a^b A_k^{-1}(s)A_{k-1}(s)$. This we now do.

Now $I_a^b Z(s)$ is defined as half the sum of signature changes of $Z(s)$ due to eigenvalues passing through ∞ at real poles between a and b. For any signature change of $Z(s)$, at all, one of the eigenvalues of $Z(s)$ must change sign. Since $Z(s)$ is a rational matrix, all eigenvalues vary continuously with s, except at a finite number of poles. Hence a change of signature of $Z(s)$ may only occur when one of the eigenvalues of $Z(s)$ passes through zero or through ∞. Thus, writing $Z(s)$ as $A^{-1}(s)B(s)$, the signature may only change when either $\det A(s)=0$ or $\det B(s)=0$ or both.

Half the change in signature of $Z(s)$ over (a, b) due to sign change at eigenvalue discontinuities is $I_a^b Z(s)$; it is clear that half the change in signature of $Z(s)$ over (a, b) due to sign changes at eigenvalue points of continuity (where eigenvalues become instantaneously zero) is $I_a^b Z^{-1}(s)$. Hence, half the total change of signature of $Z(s)$ over (a, b) is

$$\tfrac{1}{2}[\sigma(Z(b)) - \sigma(Z(a))] = \mathop{I}_a^b Z(s) + \mathop{I}_a^b Z^{-1}(s).$$

[In case there is an eigenvalue change at a or b, we use $Z(b-\varepsilon)$ or $Z(a+\varepsilon)$ for arbitrarily small nonzero ε]. Denote the left side by $\Delta_a^b Z(s)$. We have

$$I_a^b Z(s) = \Delta_a^b Z(s) = I_a^b Z^{-1}(s). \tag{4.4}$$

(A scalar version of this formula is known; see [1].)

Now tie together (4.3) and (4.4). We have

$$I_a^b Z(s) = I_a^b A_1^{-1}(s)A_2(s) = \Delta_a^b Z(s) - I_a^b A_2^{-1}(s)A_1(s) = \Delta_a^b Z(s) + I_a^b A_2^{-1}(s)A_3(s).$$

Repeating the argument over the entire Sturm sequence yields

$$I_a^b Z(s) = \Delta_a^b(A_1^{-1}A_2) + \Delta_a^b(A_2^{-1}A_3) + \cdots + \Delta_a^b(A_{m-1}^{-1}A_m).$$

In summary, we have

THEOREM 4.2. *Let $Z(s)$ be a real rational nonsingular matrix with left matrix decomposition $A_1^{-1}(s)A_2(s)$.* Let $\{A_1(s), A_2(s), \cdots, A_m(s)\}$ *be an associated left Sturm sequence. Then*

$$I_a^b Z(s) = \sum_{i=1}^{m-1} \Delta_a^b(A_i^{-1}A_{i+1})$$

where $\Delta_a^b Y(s)$ for symmetric $Y(s) = \frac{1}{2}[\sigma(Y(b)) - \sigma(Y(a))]$.

The theorem is easily extended to singular $Z(s)$: use the fact that for constant symmetric D, $I_a^b [Z(s)+D] = I_a^b Z(s)$ and $Z(s)+D$ is nonsingular for almost all such D.

5. Application to circuit theory. In this section, we characterize properties of several classes of linear, lumped, passive circuits in terms of the Cauchy index of the impedance matrix of the circuit. In particular, we look at circuits comprising resistors, capacitors and transformers (RC networks), resistors, inductors and transformers (RL networks) and inductors, capacitors, transformers and gyrators (lossless networks). All network elements are assumed passive. Consideration of the latter class of networks requires us to introduce the notion of a matrix Cauchy index for Hermitian rational matrices over the complex field. This extension to Hermitian matrices is straightforward because Hermitian matrices have real eigenvalues and well-defined signature. Moreover, it is easily shown that, if the original rational matrix is Hermitian, then so is the corresponding Hankel matrix and the rational matrices arising from the matrix Sturm theorem (see Appendix).

LEMMA 5.1. *Let $V(s)$ be a proper real rational symmetric matrix. Then*

$$I_a^b V(s) = \delta[V(s)] \tag{5.1}$$

if and only if all poles of elements of $V(s)$ lie in (a, b), and at any such pole $s=\alpha$ say, $\lim_{s\to\alpha}(s-\alpha)V(s)$ exists and is nonnegative definite symmetric. Otherwise

$$I_a^b V(s) < \delta[V(s)]. \tag{5.2}$$

Proof. Letting $V(s)=V_1(s)+V_2(s)$ where $V_1(s)$ has poles only in (a,b) while $V_2(s)$ has no poles in (a,b) and letting $\mathcal{H}_1$ and $\mathcal{H}_2$ be the associated Hankel matrices we have:

$$\mathop{I}_a^b V(s)=\mathop{I}_a^b V_1(s)=\mathop{I}_{-\infty}^{\infty} V_1(s)=\sigma(\mathcal{H}_1)$$

$$\leqq \operatorname{rank}\mathcal{H}_1=\delta(V_1(s))\leqq\delta(V(s))=\delta(V_1(s))+\delta(V_2(s)).$$

Combining this with (5.1), we see that $\delta[V_2(s)]=0$, i.e. $V_2(s)$ is a constant matrix. Then $I_{-\infty}^{+\infty} V(s)=\delta[V(s)]=n$, say. Let $\{F,G,H,J\}$ be a quadruple with $V(s)=J+H'(sI-F)^{-1}G$ and F of dimension $\delta[V(s)]$. By Corollary 3.1 and (5.1) with (a,b) replaced by $(-\infty,\infty)$, there exists a positive definite P such that $PF=F'P$, $PG=H$. Then $\{F_1=P^{1/2}FP^{-1/2},\ G_1=P^{1/2}G,\ H_1=P^{-1/2}H,\ J\}$ defines another state-variable realization for which $F_1=F_1'$, $G_1=H_1$. Let U be an orthogonal matrix such that $U'F_1U=A$, a diagonal matrix $\{\alpha_1,\cdots,\alpha_n\}$. Set $F_2=A$, $G_2=U'G_1$, $H_2=U'H_1=G_2$. Then

$$V(s)=J+\sum_i\frac{1}{s-\alpha_i}g_ig_i'.$$

Here, g_i is the ith column of G_2 and the α_i are not necessarily distinct. The conclusions of the lemma are immediate.

Conversely, assume that all poles of $V(s)$ are in (a,b) and the residue condition holds. Thus

$$V(s)=J+\sum_i\frac{1}{s-\alpha_i}V_i$$

where $V_i=V_i'\geqq 0$. It is easily checked that $I_a^b V(s)=\delta[V(s)]=\sum_i \operatorname{rank} V_i$. Finally, (5.2) follows from the fact that for any $V(s)$, as noted above, $I_a^b V(s)\leqq\delta[V(s)]$.

With this lemma, we now have the following characterization of the driving point impedance of a q-port passive network comprising resistors, capacitors and transformers.

THEOREM 5.1. *The real rational symmetric $q\times q$ matrix $Z(s)$ is realizable as the impedance of a q-port* RC *network, possibly containing ideal transformers, if and only if*

$$\mathop{I}_{-\infty}^{\varepsilon} Z(s)=\delta[Z(s)]\quad \textit{for all } \varepsilon>0.$$

Proof. A standard condition [9] for the realizability is that $Z(\infty)$ be finite, that all poles of entries of $Z(s)$ be simple and confined to $(-\infty,0]$, and that the associated residue matrix be nonnegative definite at every pole. The lemma then yields the result. In a similar manner, we have

THEOREM 5.2. *The real rational symmetric $q\times q$ matrix $Z(s)$ is realizable as the impedance of a q-port* RL *network, possibly containing ideal transformers, if and only if*

(a) *any pole of entries of $Z(s)$ at $s=\infty$ is of order at most one, and the associated residue matrix Z_∞ is nonnegative definite symmetric.*

(b) $I_{-\infty}^{\varepsilon} Z(s)=-\delta[Z(s)]+\operatorname{rank} Z_\infty$ *for all $\varepsilon>0$.*

Proof. Let $\hat{Z}(s)=Z(s)-sZ_\infty$. Then $\delta[\hat{Z}(s)]=\delta[Z]-\text{rank } Z_\infty$ by a property of degree, and so (b) is equivalent to demanding that all poles of entries of $Z(s)$ other than $s=\infty$ be in $(-\infty, 0]$ and have nonpositive definite residue matrices. This reformulation of (b) together with (a) are necessary and sufficient for the RL realizability property; see [9].

We turn now to impedance matrices of lossless networks. A real rational $q\times q$ matrix $Z(s)$ is termed *lossless positive real* and is realizable as the impedance of a q-port network comprising inductors, transformers and capacitors if and only if [10], [11] $Z(s)+Z'(-s)=0$ and all poles of entries of $Z(s)$ are either pure imaginary or at infinity, are simple, and have the associated residue matrix as nonnegative definite Hermitian (symmetric in the case of a pole at infinity or $s=0$). We have the following alternative characterization, effectively in terms of the Cauchy index of $Z(s)$ along the imaginary, rather than real, axis:

THEOREM 5.3. *The real rational $q\times q$ matrix $Z(s)$ is lossless positive real if and only if $Z(s)+Z'(-s)=0$, entries of $Z(s)$ may have a pole at ∞ and that pole is simple with nonnegative definite symmetric residue matrix Z_∞, and*

$$\text{(5.3)} \qquad I_{-\infty}^{+\infty} W(\omega)=\delta[Z(s)]-\text{rank } Z_\infty$$

where $W(\omega)=jZ(j\omega)$ is a Hermitian rational matrix.

The proof is similar to that of Theorem 5.2, again making use of Lemma 5.1. The signatures of Hankel and Bezoutian matrices associated with $W(\omega)$ also will yield $I_{-\infty}^{+\infty} W(\omega)$, and the two matrices are easily related to Hankel and Bezoutian matrices associated with $Z(s)$. Since these matrices have certain properties if and only if $Z(s)$ is lossless positive real (see [12]) one can in another way obtain (5.3).

6. Conclusion. In this paper we have generalized the notion of the Cauchy index of a scalar rational function to include rational real symmetric and Hermitian matrices. We have presented various equivalent formulations for its determination. These different presentations have basically been oriented to characterizations of the Cauchy index using various descriptions of rational matrices such as: the generalized Bezoutian matrix associated with a matrix fraction description, the P-matrix of Corollary 3.3 associated with a state-space description, the Hankel matrix and shifted Hankel matrices associated with partial fraction and unfactored descriptions, etc. The major result of the earlier sections is that the Cauchy index over $(-\infty, \infty)$ of any rational Hermitian matrix is given by the signature of the associated Hankel matrix.

Furthermore, we have developed a well-defined Euclidean algorithm amongst polynomial matrices and used this to define a Sturm sequence of polynomial matrices without reference to root locations. A quasi-Sturm theorem was constructed for the evaluation of the Cauchy index over an arbitrary interval in terms of the signature changes in a sequence of rational matrices derived from the Sturm sequence of polynomial matrices.

The whole aim of this paper has been to consider the Cauchy index and to attempt to extract from the scalar theory those features which are specializations of the broader multivariable theory. With this in mind we have examined several realizability constraints from circuit synthesis. The wealth of application of the

Cauchy index to the study of stability of scalar linear systems [1] leads the authors to suspect that the matrix Cauchy index will also have wide application in multivariable linear system stability problems.

Stability problems for discrete-time control systems have been handled by use of a Cauchy index evaluated round the unit circle [13], and we would anticipate that matrix versions of these ideas should be available.

Appendix. In this appendix, we prove a simple result on the signature of the sum of two symmetric matrices, and we review and extend a result given in [7] used in constructing the matrix Sturm sequence.

THEOREM A1. *Let A, B be $n \times n$ symmetric matrices and $C = A + B$. If* rank C = rank A + rank B, *then* $\sigma(C) = \sigma(A) + \sigma(B)$.

Proof. Let rank $A = n_1$, rank $B = n_2$. We can write $A = \sum_{i=1}^{n_1} \lambda_i t_i t_i'$, $B = \sum_{j=1}^{n_2} \mu_j s_j s_j'$ where $\{t_i\}$ and $\{s_j\}$ are orthornormal sets and $\{\lambda_i\}$, $\{\mu_j\}$ are the nonzero eigenvalues of A and B. Then $C = \sum_{i=1}^{n_1} \lambda_i t_i t_i' + \sum_{j=1}^{n_2} \mu_j s_j s_j'$ and the set $\{t_1, \cdots, t_{n_1}, s_1, \cdots, s_{n_2}\}$ is linearly independent since rank $C = n_1 + n_2$. Thus C is congruent to $\operatorname{diag}\{\lambda_1, \cdots, \lambda_{n_1}, \mu_1, \cdots, \mu_{n_2}, 0, \cdots, 0\}$ and so $\sigma(C) = \sigma(A) + \sigma(B)$.

THEOREM A2. *Let $A(s)$, $B(s)$ be $q \times q$ polynomial matrices with $B(s)$ of nonzero determinant. Then there exist polynomial matrices $Q(s)$ and $C(s)$ such that*

$$A(s) = B(s)Q(s) + C(s)$$

and either $C(s) = 0$ or $C(s)$ is nonsingular with degree[det $C(s)$] < degree[det $B(s)$]. *Moreover, if $B^{-1}A$ is symmetric, one can ensure that Q and $B^{-1}C$ are symmetric.*

Proof. Write $B^{-1}A = P(s) + R(s)$ where $P(s)$ is polynomial, including a constant term, and $R(s)$ is strictly proper, i.e. $R(s) \to 0$ as $s \to \infty$. If $R(s) \equiv 0$, take $P(s) = Q(s)$. If $\det R(s) \not\equiv 0$, take $P(s) = Q(s)$ and $C(s) = B(s)R(s)$. In case $R(s) \not\equiv 0$ but $\det R(s) \equiv 0$, there clearly exists symmetric constant E such that $\det E = 0$, $\det[R(s) + E] \not\equiv 0$. [Almost all symmetric E of rank $q - 1$ will work.] In particular if $R(s)$ is symmetric E may be simply chosen as diagonal. Then set $Q(s) = P(s) - E$ and $C(s) = B(s)[R(s) + E]$. The remaining requirements of the theorem are easily checked.

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