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Identifiability of MIMO Linear Dynamic Systems Operating in Closed Loop*

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Necessary and sufficient identifiability conditions for multi-input multi-output linear dynamic systems operating in closed loop indicate that under certain conditions physically meaningful models for forward and reverse paths can be uniquely determined.

Key Word Index—Identification; closed loop systems; multivariable systems; stochastic systems; spectral factorization; observability.

Summary—Identification of multi-input multi-output (MIMO) linear dynamic systems is considered for the case when the measurements are obtained during closed loop operation. Both noise free and noisy feedback situations are analysed. For the case where the disturbances in the feedback path are a full rank stochastic process it is shown that, under certain mild conditions, physically meaningful models for the forward and reverse paths can be uniquely determined. For the case where the feedback path is noise free it is shown that the forward path model can be uniquely determined provided the regulator satisfies certain minimal complexity requirements.

1. INTRODUCTION

MOST commonly used identification procedures rely on the assumption, explicitly or implicitly, that the process input is independently generated[1]. However, this requirement conflicts with the frequently met practical requirement that the system cannot be operated in open loop.

The feedback may occur naturally, as is frequently the case in sociological, biological and economic modelling problems. Alternatively, the feedback may be purposefully introduced to achieve some acceptable level of process operation. For example, the output may be required to meet normal production constraints. An extreme case is when the system is open loop unstable.

The closed loop identifiability problem has been studied by a number of authors (see e.g. [2]–[9]). A comprehensive survey of recent results on this problem is contained in the paper by Gustavsson *et al.*[14]. It is known that unique identification is possible under a number of different conditions, e.g. when the feedback regulator is switched among a number of different settings[5], the disturbances in the feedback path are persistently exciting[14] or the regulator satisfies certain complexity requirements[6], [7]. The purpose of the current paper is to further study the latter two conditions especially in the multi-input multi-output case.

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Section 3 of the paper is concerned with the case where the disturbances in the feedback path are a full rank stochastic process. Other authors, e.g.[10], [11], [14] and [17] have also studied the case under different conditions. Gustavsson *et al.* in [14] show that the forward path model is strongly system identifiable under the assumption that the forward path noise model is minimum phase. Phadke and Wu[11] describe a procedure for obtaining forward and reverse path models by analysing a particular factorization of the joint input–output spectral density. Caines and Chan[10] have used a similar formulation to test for 'feedback free' processes. In both [10] and [11], the stable minimum phase spectral factor is used but no proof is given that this yields physically meaningful models for the forward and reverse paths. It is shown in Section 3 of the current paper that the forward and reverse path models will, in general, depend on the particular factorization of the joint input–output spectral density. Moreover, the stable minimum phase spectral factor can only be guaranteed to yield physically meaningful models provided the noise in the forward and reverse paths are uncorrelated and there is a delay in both the forward and reverse paths. These conditions also give further insight into the results of Vorchik[17] who has shown that truncated maximum likelihood estimates are consistent under a minimum phase assumption for the noise model.

Section 4 of the paper is concerned with the case where the feedback path is noise free. The multi-input multi-output case is considered and it is shown that the forward path model can be uniquely identified provided the maximum observability index for the forward path is less than or equal to the minimum observability index for the feedback path. This result reduces to the known results [6], [7], [14] for the single input single output case.

2. MODELS FOR FEEDBACK SYSTEMS

The class of feedback systems under consideration is depicted in Fig. 1 where

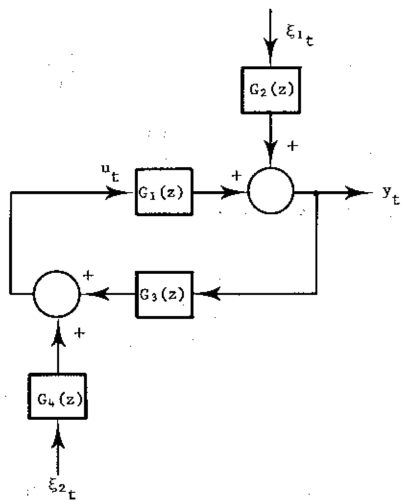

FIG. 1. A closed loop system.

$u_t \in R^m$ is the process input
$y_t \in R^r$ is the process output
$\xi_{1_t} \in R^r$ is the noise in the forward path
$\xi_{2_t} \in R^m$ is the noise in the feedback path.

$G_1(z)$, $G_2(z)$, $G_3(z)$ and $G_4(z)$ are rational transfer function matrices expressed in terms of the Z-transform variable, z. We assume that closure of the loop yields a properly defined system. This will be so if, for example,

$$\lim_{z \to \infty} G_3(z) G_1(z) = 0.$$

In our subsequent analysis we shall simplify the notation by omitting the argument z from $G(z)$ whenever no possibility of ambiguity may arise. We also use the notation $G^*(z)$ for $G^T(z^{-1})$.

We shall occasionally make a slight abuse of notation by writing equations of the form $Y(z) = G(z)U(z)$ as $y_t = G(z)u_t$. The latter equation can be thought of as a difference equation provided z^{-1} is interpreted as the unit delay operator, i.e.

$$z^{-1} y_t = y_{t-1}. \tag{2.1}$$

We assume that the white noise sequences $\{\xi_{1_t}\}$ and $\{\xi_{2_t}\}$ have a joint Gaussian distribution with zero mean and covariance given by

$$E\begin{pmatrix} \xi_{1_s} \\ \xi_{2_s} \end{pmatrix} (\xi_{1_t}^T \xi_{2_t}^T) = D\delta_{st}. \tag{2.2}$$

We assume that D is positive definite.

If prior knowledge indicates that a model structure of the form shown in Fig. 1 is appropriate, then the identifiability question can be stated as: is it possible to extract physically meaningful estimates of the transfer functions G_1 to G_4 from measurements of y_t and u_t? The purpose of the current paper is to analyze this question under a number of different conditions.

The structure depicted in Fig. 1 is not meant to preclude the possibility of considering situations in which for example, the plant noise is added at the plant input (after measurement of u_t), or indeed at some internal point of the plant. For if additive input noise is derived by passing ξ_t through a filter $\tilde{G}_2(z)$, the arrangement is equivalent (in the sense that measurements of u_t, y_t will not be able to distinguish the situation) to that shown in Fig. 1, provided that we take $G_2(z) = G_1(z)\tilde{G}_2(z)$.

3. IDENTIFICATION USING THE JOINT INPUT–OUTPUT SPECTRAL PROPERTIES

In this section we consider the case where the joint process (y_t, u_t) is a full rank stochastic process. We assume that the process (y_t, u_t) is stationary and we denote the joint spectral density by Φ_{yu}. From Fig. 1, Φ_{yu} is given by

$$\Phi_{yu}(\omega) = K(z) D K^*(z), z = e^{j\omega} \tag{3.1}$$

where D is defined in (2.2) and where $K(z)$ is given by

$$K = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right]$$
$$= \left[\begin{array}{c|c} (I - G_1 G_3)^{-1} G_2 & (I - G_1 G_3)^{-1} G_1 G_4 \\ \hline (I - G_3 G_1)^{-1} G_3 G_2 & (I - G_3 G_1)^{-1} G_4 \end{array}\right]. \tag{3.2}$$

(The inverses of $I - G_1 G_3$ and $I - G_3 G_1$ exist when a system is properly defined after loop closure.)

We now have the following result regarding the relationship between K and G_1 to G_4.

Lemma 3.1. There is a one to one correspondence between K, as defined in (3.2), and the quadruple (G_1, G_2, G_3, G_4). Furthermore, the values of G_1 to G_4 are uniquely expressible in terms of K as follows

$$G_1 = K_{12} K_{22}^{-1} \tag{3.3}$$

$$G_2 = K_{11} - K_{12} K_{22}^{-1} K_{21} \tag{3.4}$$

$$G_3 = K_{21} K_{11}^{-1} \tag{3.5}$$

$$G_4 = K_{22} - K_{21} K_{11}^{-1} K_{12} \tag{3.6}$$

Proof. We first note that the assumption that Φ_{yu} has full normal rank implies that K_{11} and K_{22} are nonsingular. For if K_{11} is singular, (3.2) shows that G_2 is singular and thus $\left[\begin{smallmatrix} K_{11} \\ K_{21} \end{smallmatrix}\right]$ fails to have full normal rank.. Thus K is singular and so Φ_{yu} is singular. Likewise Φ_{yu} is singular if K_{22} is singular. In either case we have a contradiction. Therefore the quantities on the right hand side of (3.3) to (3.6) are well defined. Then it is obvious that K is uniquely determined by G_1 to G_4. To prove the converse, we assume that there exists another set (H_1, H_2, H_3, H_4) giving rise to K. Then from (3.2) we have:

$$K_{11}=(I-G_1G_3)^{-1}G_2=(I-H_1H_3)^{-1}H_2 \quad (3.7)$$

$$K_{12}=(I-G_1G_3)^{-1}G_1G_4=(I-H_1H_3)^{-1}H_1H_4 \quad (3.8)$$

$$K_{21}=(I-G_3G_1)^{-1}G_3G_2=(I-H_3H_1)^{-1}H_3H_2 \quad (3.9)$$

$$K_{22}=(I-G_3G_1)^{-1}G_4=(I-H_3H_1)^{-1}H_4 \quad (3.10)$$

Now using the fact that $(I-AB)^{-1}A=A(I-BA)^{-1}$ for rectangular matrices A and B, it follows that equations (3.7) to (3.10) have a unique solution

$$G_1=H_1=K_{12}K_{22}^{-1} \quad (3.11)$$

$$G_2=H_2=K_{11}-K_{12}K_{22}^{-1}K_{21} \quad (3.12)$$

$$G_3=H_3=K_{21}K_{11}^{-1} \quad (3.13)$$

$$G_4=H_4=K_{22}-K_{21}K_{11}^{-1}K_{12} \quad (3.14)$$

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Lemma 3.1 indicates that it is possible to uniquely recover G_1 to G_4 from K. However, there is a fundamental and unavoidable nonuniqueness associated with the spectral factorization given in equation 3.1. Thus for a given Φ_{yu} there will be many transfer function matrices G_1 to G_4 corresponding to the different spectral factorizations of Φ_{yu}. Of course, unique values for G_1 to G_4 can be determined by using a particular factorization of Φ_{yu}. For example, the stable minimum phase spectral factor is used in [10], [11]. However, it is not obvious that this will necessarily lead to physically meaningful values for G_1 to G_4. The conditions under which the stable minimum phase spectral factor leads to physically meaningful values for G_1 to G_4 are studied in the following theorem.

Theorem 3.1. Consider the process depicted in Fig. 1 under the following assumptions

(a) the closed-loop system is asymptotically stable
(b) G_1 and G_3 are strictly proper, while G_2 and G_4 are proper, i.e.

$$G_1(\infty)=0 \qquad G_3(\infty)=0 \quad (3.15)$$

$$\|G_2(\infty)\|<\infty \qquad \|G_4(\infty)\|<\infty \quad (3.16)$$

Let $\bar{K}(z)$ be specified by the requirements that $\bar{K}(z)$ be stable, $\bar{K}(z)$ be nonsingular for $|z|>1$, $\bar{K}(\infty)=I$ and $\Phi_{yu}(z)=\bar{K}(z)\bar{D}\bar{K}^{\mathrm{T}}(z^{-1})$ for some $\bar{D}>0$. Thus $\bar{K}(z)$ is the unique minimum phase spectral factor of $\Phi_{yu}(z)$, as shown in the Appendix. Define $\bar{G}_i$ by (3.3) through (3.6) with K_{ij} replaced by $\bar{K}_{ij}$. Let

$$\bar{D}=\begin{bmatrix}\bar{D}_{11} & \bar{D}_{12}\\ \bar{D}_{21} & \bar{D}_{22}\end{bmatrix} \quad (3.17)$$

The open loop transfer functions are recoverable if

$$\left.\begin{array}{ll} G_1=\bar{G}_1 & G_3=\bar{G}_3\\ G_2=\bar{G}_2V & G_4=\bar{G}_4W \end{array}\right\} \quad (3.18)$$

where $V(z)$ and $W(z)$ are rational matrices with

$$V(z)D_1V^*(z)=\bar{D}_1 \qquad W(z)D_2W^*(z)=\bar{D}_2 \quad (3.19)$$

(i) A sufficient condition for recoverability of G_1 to G_4 is that D be block diagonal, $D_{12}=D_{21}^{\mathrm{T}}=0$.

(ii) A necessary and sufficient condition for recoverability of G_1 to G_4 is that the transformation from K to $\bar{K}$ be block diagonal.

Proof. (i) Let $A^{-1}(z)[B(z)\,\vdots\,C(z)]$ be any left prime polynomial matrix fraction decomposition of $[G_1(z)\,\vdots\,G_2(z)]$, see [12], and let $E^{-1}(z)[F(z)\,\vdots\,L(z)]$ be a left prime decomposition of $[G_3(z)\,\vdots\,G_4(z)]$. Then

$$K(z)=\begin{bmatrix}A & -B\\ -F & E\end{bmatrix}^{-1}\begin{bmatrix}C & 0\\ 0 & L\end{bmatrix} \quad (3.20)$$

and the closed loop is stable if and only if

$$\det\begin{bmatrix}A(z) & -B(z)\\ -F(z) & E(z)\end{bmatrix}\neq 0 \qquad \text{for } |z|\geq 1 \quad (3.21)$$

Let $V_A(z)$, $V_E(z)$ be polynomial matrices corresponding to $A(z)$, $E(z)$ as per Lemma 4 of the appendix. Then

$$K(z)=\begin{bmatrix}V_AA & -V_AB\\ -V_EF & V_EE\end{bmatrix}^{-1}\begin{bmatrix}V_AC & 0\\ 0 & V_EL\end{bmatrix} \quad (3.22)$$

with

$$\det\begin{bmatrix}V_AA & -V_AB\\ -V_EF & V_EE\end{bmatrix} = \det V_A \det V_E \det\begin{bmatrix}A & -B\\ -F & E\end{bmatrix} \neq 0 \text{ for } |z|\geq 1 \quad (3.23)$$

by (3.21) and Lemma 4.

Because

$$G_2(\infty)=\lim_{z\to\infty}(V_AA)^{-1}(V_AC),$$

Lemma 4 implies that V_AC has degree $\leqq n_A$. Similarly, V_EL has degree $\leqq n_E$.

Now apply Lemma 2 of the Appendix to

$$\Psi_A(z)=V_A(z)C(z)D_1C^{\mathrm{T}}(z^{-1})V_A^{\mathrm{T}}(z^{-1})$$

and to

$$\Psi_E(z)=V_E(z)L(z)D_2L^{\mathrm{T}}(z^{-1})V_E^{\mathrm{T}}(z^{-1}).$$

Then there exists a polynomial matrix $\bar{C}(z)$ degree n_A and $\bar{D}_1>0$ such that

$$\Psi_A(z)=V_A(z)C(z)D_1C^{\mathrm{T}}(z^{-1})V_A^{\mathrm{T}}(z^{-1})$$
$$=\bar{C}(z)\bar{D}_1\bar{C}^{\mathrm{T}}(z^{-1}) \quad (3.24)$$

$$\det\bar{C}(z)\neq 0 \quad \text{for} \quad |z|>1 \quad (3.25)$$

$$\lim_{z\to\infty} z^{-n_A}\bar{C}(z)=I \quad (3.26)$$

Similarly there exists a polynomial matrix $\bar{L}$ of degree n_E and $\bar{D}_2>0$ such that

$$\Psi_E(z)=V_E(z)L(z)D_2L^{\mathrm{T}}(z^{-1})V_E^{\mathrm{T}}(z^{-1})$$
$$=\bar{L}(z)\bar{D}_2\bar{L}^{\mathrm{T}}(z^{-1}) \quad (3.27)$$

$$\det\bar{L}(z)\neq 0 \quad \text{for} \quad |z|>1 \quad (3.28)$$

$$\lim_{z\to\infty} z^{-n_E}\bar{L}(z)=I \quad (3.29)$$

Define

$$\hat{K}(z)=\begin{bmatrix} V_AA & -V_AB \\ -V_EF & V_EE\end{bmatrix}^{-1}\begin{bmatrix}\bar{C} & 0\\ 0 & \bar{L}\end{bmatrix} \quad (3.30)$$

We shall show that $\hat{K}(z)$ is the unique minimum phase spectral factor of $\Phi_{yu}(z)$ with $\hat{K}(\infty)=I$, i.e., $\bar{K}=\hat{K}$ with $\bar{D}_{11}=\bar{D}_1$, $\bar{D}_{12}=\bar{D}_{21}^{\mathrm{T}}=0$, $\bar{D}_{22}=\bar{D}_2$. Equations (3.30), (3.22), (3.24), (3.27) show that

$$\Phi_{yu}(z)=K(z)DK^*(z)=\bar{K}(z)\bar{D}\bar{K}^*(z)$$

with D and $\bar{D}$ block diagonal.

Equations (3.23) and (3.25) and (3.28) together with this definition of $\hat{K}(z)$ show that $\hat{K}(z)$ is analytic in $|z|\geqq 1$ and nonsingular in $|z|>1$.

Equation (3.30) also yields

$$\lim_{z\to\infty}\hat{K}(z)=\lim_{z\to\infty}\begin{bmatrix} I & -A^{-1}B\\ -E^{-1}F & I\end{bmatrix}^{-1}$$
$$\lim_{z\to\infty}\begin{bmatrix}(V_AA)^{-1}\bar{C} & 0\\ 0 & (V_EE)^{-1}\bar{L}\end{bmatrix}$$

By (3.15), the first matrix is I, while the definition of V_A, V_E (see Lemma 3) and (3.26) and (3.29) ensure the second matrix is I. So $\hat{K}(\infty)=I$. Thus $\hat{K}=\bar{K}$.

Define

$$V(z)=\bar{C}^{-1}(z)V_A(z)C(z)$$
$$W(z)=\bar{L}^{-1}(z)V_L(z)L(z) \quad (3.31)$$

Then (3.24) and (3.27) yield (3.19). Also, because

$$\bar{K}(z)\begin{bmatrix}V(z) & 0\\ 0 & W(z)\end{bmatrix}=K(z) \quad (3.32)$$

we have, as required,

$$G_1=K_{12}K_{22}^{-1}=\bar{K}_{12}\bar{K}_{22}^{-1}=\bar{G}_1$$

with $G_2=\bar{G}_2V$, $G_3=\bar{G}_3$, $G_4=\bar{G}_4W$ following similarly.

(ii) This is immediate since (3.18) implies and is implied by (3.32).

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The above theorem shows that, provided the noise sequences $\{\xi_{it}\}$, $i=1,2$, in the forward and reverse paths are uncorrelated, then the correct values of G_1 and G_3 can be obtained by applying equations (3.11) and (3.13) to the stable minimum phase spectral factor of Φ_{yu}. Furthermore, the values of $\bar{G}_2$ and $\bar{G}_4$ obtained from $\bar{K}$ will differ from the true values by, at most, right multiplication by a scaled para-unitary matrix. This latter ambiguity is of a fundamental nature and occurs even in the open loop case. This ambiguity does not influence the input–output characteristics of either the forward or reverse path and therefore is of no practical importance.

In many practical situations, it is reasonable to assume that the disturbance in the forward and reverse paths are uncorrelated. Furthermore, D block diagonal implies that the likelihood function splits into the product of two terms, one for the forward path and one for the feedback path. Thus provided G_1, G_2 have no parameters in common with G_3, G_4 (this will invariably be the case in practice) maximum likelihood type procedures can be applied to the data (u_t, y_t) as if it were open loop data. Alternatively, standard procedures exist for estimating $\bar{K}(z)$ for the joint process (y_t, u_t) and then equations (3.11) and (3.12) can be used to determine G_1 and G_2. The principle of invariance [16] for maximum likelihood estimators ensures that these two approaches yield identical estimates.

It should be noted that Theorem 3.1 does not require $\det(A)\neq 0$, $\det(E)\neq 0$ in $|z|\geqq 1$. Thus there is no restriction against open loop unstable systems.

There is no difficulty identifying open loop unstable systems provided the closed loop system is stable.

If it is known that G_2 and G_4 are minimum phase, i.e. are stable and nonsingular in $|z|>1$, and if G_1 and G_3 are stable, then the use of the spectral factor $\bar{K}$ will yield $\bar{G}_2$ and $\bar{G}_4$ identical with G_2, G_4 except possibly for differing poles at $z=0$ and a scaling constant. For $\bar{G}_2=(V_A A)^{-1}\bar{C}$ will be stable and is nonsingular in $|z|>1$, and from this fact, the claim follows.

For the case where the noise sequence $\{\xi_{it}\}$, $i=1,2$ in the forward and feedback paths are correlated, then the models obtained for G_1 to G_4 from (3.11) to (3.14) will depend in general upon the particular factorization of Φ_{yu} used. Among all possible stable spectral factors of Φ_{yu} only one will yield appropriate values for G_1 to G_4. In practice there is no way of knowing which factor should be used and use of the wrong factor will lead to incorrect conclusions regarding the forward and feedback paths, except in the non-generic cases covered by part (ii) of Theorem 3.1.

We remark that many recent papers on closed loop identification have assumed that there is a delay in only one of G_1 or G_3 with $D=I$, whereas we have assumed that both G_1 and G_3 have at least one delay. However, if the delay is absent from either G_1 or G_3, then this is equivalent to having correlated noise in the forward and feedback paths even though $D=I$. Thus the unique determination of G_1 and G_3 from $\Phi(z)$ will again be impossible, and we illustrate this below in Example 3.2. Of course the assumption of a delay in G_1 and G_3 is often physically reasonable since it is never possible to have instantaneous transmission of signals over a finite distance; nevertheless, a number of important practical situations have $G_3(\infty)\neq 0$.

Example 3.1

Consider a single-input single-output feedback system of the form illustrated in Fig. 1 with the following values for the transfer functions G_1 to G_4 and for the covariance D

$$G_1(z)=\frac{1}{z-0.5} \tag{3.33}$$

$$G_2(z)=\frac{z-2}{z-0.5} \tag{3.34}$$

$$G_3(z)=\frac{-0.5}{z} \tag{3.35}$$

$$G_4(z)=1 \tag{3.36}$$

$$D=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix} \tag{3.37}$$

The corresponding value of K is given by

$$K(z)=\frac{1}{z^2-0.5z+0.5}\left[\begin{array}{c|c} z^2-2z & z\\ \hline -0.5z+1 & z^2-0.5z\end{array}\right] \tag{3.38}$$

It can be seen that $K(z)$ given above is a stable, but non-minimum phase, spectral factor of Φ_{yu}. The minimum phase spectral factor of Φ_{yu} turns out to be

$$\bar{K}(z)=\frac{1}{z^2-0.5z+0.5}\left[\begin{array}{c|c} z^2-0.5z & 0.25z\\ \hline -0.5z+0.25 & z^2-0.5z+0.375\end{array}\right] \tag{3.39}$$

with $\Phi_{yu}=\bar{K}(z)\bar{D}\bar{K}^*(z)$ and

$$\bar{D}=\begin{bmatrix}2.5 & 1\\ 1 & 2\end{bmatrix} \tag{3.40}$$

The unique values of $\bar{G}_1$ to $\bar{G}_4$ corresponding to $\bar{K}$ are

$$\bar{G}_1=\frac{0.25z}{z^2-0.5z+0.375} \tag{3.41}$$

$$\bar{G}_2=\frac{z^2-0.5z}{z^2-0.5z+0.375} \tag{3.42}$$

$$\bar{G}_3=\frac{-0.5}{z} \tag{3.43}$$

$$\bar{G}_4=1 \tag{3.44}$$

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Comparing equations (3.41) to (3.44) with equations (3.34) to (3.37) shows that use of the minimum phase spectral factor when D is not block diagonal can give incorrect estimates for G_1 to G_4. Of course, the model defined by $\bar{G}_1$, $\bar{G}_2$, $\bar{G}_3$, $\bar{G}_4$ and $\bar{D}$ is in the equivalence class of systems having the structure depicted in Fig. 1 and giving rise to the same value of Φ_{yu}. However the model would lead to incorrect conclusions regarding the forward and reverse path transfer functions. This could be important in practice. For example if one were interested in predicting the effect of changes in the feedback law, then it is important to have the correct forward path model.

Example 3.2

Suppose one takes $D=I$ and

$$G_1(z)=\frac{1}{z-0.5} \qquad G_2(z)=\frac{z-2}{z-0.5}$$
$$G_3(z)=\frac{z^2-z+1}{z^2-z} \qquad G_4(z)=\frac{z-2}{z-1} \tag{3.45}$$

Then one obtains the same spectrum as in Example 3.1. Then use of a minimum phase spectral factor $\bar{K}$

leads to different $\bar{G}_1, \bar{G}_3$

$$\bar{G}_1 = \frac{0.25z}{z^2 - 0.5z + 0.375} \qquad \bar{G}_3 = \frac{z^2 - 1.75z + 1}{2.5z^2 - z} \tag{3.46}$$

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The above example illustrates the need for the assumption that there is a delay in both the forward and reverse paths even when D is block diagonal.

4. CONDITIONS ON REGULATOR COMPLEXITY FOR IDENTIFIABILITY—NOISE-FREE CASE

In this section, we assume that $G_4 \equiv 0$ and G_3 is fully known. (This is in contrast to Section 3 where G_4 was assumed nonsingular and G_3 was unknown.) As before, we are interested in estimating G_1 and G_2 from measurements on (u_t, y_t).

The result in the previous section does not require knowledge of the structure of G_1 or G_2. Here, however, we shall need to assume certain structural information regarding the pair $[G_1 \vdots G_2]$. In particular, we shall assume knowledge of the observability index of any minimal state-space model for $[G_1 \vdots G_2]$. It is well known [1], [2] that for a simple feedback transfer function G_3, ambiguity may arise in identifying parameters in G_1 and G_2. It has been shown for the single input single output case [6], [7] that identifiability of G_1 and G_2 can be achieved by use of a sufficiently complex feedback transfer function G_3. We extend this result to the multi-input, multi-output case in the remaining part of this section.

Without loss of generality, we assume that G_1 and G_2 are represented by a left matrix fraction decomposition (MFD) [12] of the form

$$[G_1 \vdots G_2] = A^{-1}[B \vdots C] \tag{4.1}$$

where A, B, C are polynomial matrices in z with A row proper [12], A and $[B \vdots C]$ relatively left prime and

$$\lim_{z\to\infty} A^{-1}B = 0, \quad \lim_{z\to\infty} A^{-1}C = I.$$

We further assume that G_3 is represented by a right MFD of the form

$$G_3 = FE^{-1} \tag{4.2}$$

where E is column proper and with the column degrees of F less than corresponding column degrees of E[12]. [The results to follow can be adjusted to cover the case of $0 \neq G_3(\infty) < \infty$.] Lemma 4 of the Appendix applied to G_3^{T} shows that, without loss of generality, E can be expressed in the form $E_0 z^{\ell} + \ldots, E_{\ell}$ with $E_0 = I$.

We assume that $(AE - BF)$ and C are relatively left prime (i.e. we assume that the feedback does not introduce additional pole zero cancellations): We also assume that the closed loop system is stable i.e. $\det(AE - BF) \neq 0$ in $|z| \geq 1$.

For given G_3 there exists some flexibility in the choice of E and F to satisfy (4.2). However, this flexibility does not influence the conclusions that we shall reach.

The spectral density for the process (y_t) is given by:

$$\Phi_y = TD_1 T^* \tag{4.3}$$

where

$$T = E(AE - BF)^{-1}C \tag{4.4}$$

Given Φ_y we can readily compute Φ_ω with $\omega_t = z^{\ell}E^{-1}y_t$. This gives

$$\Phi_\omega = SD_{11}S^* \tag{4.5}$$

with

$$S = z^{\ell}E^{-1}T = z^{\ell}(AE - BF)^{-1}C \tag{4.6}$$

Using similar arguments to those used in the proof of Theorem (3.1) we can assume, without loss of generality, that $\det C \neq 0$ for $|z| > 1$ and hence that S given in (4.6) is stable and minimum phase with $\lim_{z\to\infty} S = I$ and is uniquely determined by Φ_ω. (see Lemma 3 of the Appendix)

We now represent S by a left irredicible MFD of the form

$$S = Q^{-1}R \tag{4.7}$$

Since $(AE - BF)$ and C have been assumed to be relatively left prime, then we know[12] that

$$[Q \vdots R] = U[AE - BF \vdots C] \tag{4.8}$$

for some unimodular matrix U.

From equation (4.8) we have

$$[Q \vdots R] = [A'E - B'F \vdots C'] \tag{4.9}$$

where $A' = UA$, $B' = UB$, $C' = UC$.

We now investigate the conditions under which equation (4.9) can be uniquely solved for A', B' and C', knowing Q, R, E and F. We remark that it is immaterial whether we obtain estimates for A', B', C' or A, B, C since $(A')^{-1}B' = (A)^{-1}B = G_1$ and $(A')^{-1}C' = (A)^{-1}C = G_2$.

It is clear from (4.9) that $C' = R$. Also expressing $A'(z) = A'_0 z^k + A'_1 z^{k-1} \ldots + A'_k$, $B'(z) = B'_1 z^{k-1} + \ldots + B'_k$, $E(z) = E_0 z^{\ell} + E_1 z^{\ell-1} + \ldots + E_{\ell}$ and $F(z) = F_1 z^{\ell-1} + \ldots F_{\ell}$, we have from (4.9) that $A'E - B'F = Q$ and hence equating coefficients we have

$$[A'_0 \vdots A'_1 \vdots \ldots A'_k \vdots B'_1 \vdots \ldots B'_k]
\left[\begin{array}{cc|cccc}
E_0 & E_1 & \cdots & E_{\ell} & & \\
0 & E_0 & E_1 & \cdots & E_{\ell} & \\
\hline
0 & 0 & E_0 & \ddots & E_{\ell} & \\
 & & & \ddots & E_0 & E_{\ell} \\
\hline
0 & 0 & F_1 & \cdots & F_{\ell} & \\
0 & 0 & & \ddots & F_1 \cdots & F_{\ell}
\end{array}\right]
= [Q_0 \quad Q_1 \ \ldots \ Q_{k+\ell}] \tag{4.10}$$

or

$$\theta N = \theta\left[\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline 0 & S^k(E,F) \end{array}\right] = \beta \qquad (4.11)$$

where

$$\theta = [A_0' \vdots A_1' \vdots \ldots A_k' \vdots B_1' \vdots \ldots B_k'] \qquad (4.12)$$

$$\beta = [Q_0 \vdots Q_1 \ \ldots \ Q_{k+\ell}] \qquad (4.13)$$

$$M = \begin{bmatrix} E_0 E_1 \ \ldots & E_\ell 0 \ \ldots \ 0 \\ 0 \quad E_0 E_1 \ \ldots & E_\ell 0 \ \ldots \ 0 \end{bmatrix} \qquad (4.14)$$

$$S^k(E,F) = \begin{bmatrix} E_0 E_1 \ \ldots \ E_\ell & \\ \ddots & \\ E_0 E_1 \ \ldots \ E_\ell & \\ F_1 \ \ldots \ F_\ell & \\ \ddots & \\ F_1 \ \ldots \ F_\ell & \end{bmatrix} \begin{matrix} (k-1) \text{ block rows} \\ \\ (k) \text{ block rows} \end{matrix} \qquad (4.15)$$

The matrix $S^k(E,F)$ is known as the Generalized Sylvester matrix[18] for the matrix polynomial pair (E,F). The following lemma gives elementary properties of the matrix N.

Lemma 4.1. Properties of the matrix N are

(i) Rank $S^k(E,F)$ is independent of the particular realization used in equation (4.2.)

(ii) Rank $[S^k(E,F)] \leqq (k-1)r + p$ (4.16)

with equality iff $k \geqq \nu_{\max}(G_3)$
where
r is dimension of E (number of system outputs)
p is McMillan degree of $G_3 = FE^{-1}$
$\nu_{\max}(G_3)$ is the maximum observability index for any irreducible representation for G_3.

(iii) Rank $[S^k(E,F)] = (k-1)r + km$ (4.17)

provided $k \leqq \nu_{\min}(G_3)$
where
$\nu_{\min}(G_3)$ is the minimum observability index for any irreducible representation for G_3.

(iv)

$$\text{Rank } N = \text{Rank}\left[\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline 0 & S^k(E,F) \end{array}\right]$$

$$= \text{Rank}\,[S^k(E,F)] + 2r \qquad (4.18)$$

Proof

(i), (ii) see [18], [12]

(iii) Let $\Gamma^{-1}\Omega$ be a row proper left MFD corresponding to FE^{-1}. Since $\Gamma^{-1}\Omega = FE^{-1}$, we have

$$\Omega E - \Gamma F = 0 \qquad (4.19)$$

or

$$[\Omega \vdots -\Gamma]\begin{bmatrix} E \\ F \end{bmatrix} = 0 \qquad (4.20)$$

Thus the columns of $[{}^{\Omega^T}_{-\Gamma^T}]$ are in the null space of $[E^T \vdots F^T]$. Since E is $r \times r$ and nonsingular, the range space of $[{}^E_F]$ has dimension r which must also be the dimension of the orthogonal complement of the null space of $[E^T \vdots F^T]$. Now the domain of $[E^T \vdots F^T]$ has dimension $(m+r)$, and thus the dimension of the null space of $[E^T \vdots F^T]$ must have dimension $(m+r) - r = m$. Moreover, Γ is $(m \times m)$ and nonsingular and thus the columns of $[{}^{\Omega^T}_{-\Gamma^T}]$ span the null space of $[E^T \vdots F^T]$. Since $\Gamma^{-1}\Omega = FE^{-1} = G_3$ and the smallest column degree in $[{}^{\Omega^T}_{-\Gamma^T}]$ = the smallest row degree in $[\Omega - \Gamma]$ = the least observability index for $G_3 = \nu_1$, it is now clear that two arbitrary non-zero row polynomials α and β ($\alpha: 1 \times r, \beta: 1 \times m$)

$$\alpha(z) = \alpha_0 z^k + \alpha_1 z^{k-1} + \ldots \alpha_k \qquad (4.21)$$

$$\beta(z) = \beta_1 z^{k-1} + \beta_2 z^{k-2} + \ldots \beta_k \qquad (4.22)$$

with $k < \nu_1$ satisfy

$$[\beta(z) \vdots -\alpha(z)]\begin{bmatrix} E(z) \\ F(z) \end{bmatrix} \neq 0 \quad \text{for} \quad \begin{cases} \alpha(z) \not\equiv 0 \\ \beta(z) \not\equiv 0 \end{cases} \qquad (4.23)$$

Hence

$$\phi^T S^{k+1}(E,F) \neq 0 \quad \text{for} \quad y \not\equiv 0 \qquad (4.24)$$

with

$$\phi^T = [\beta_1, \beta_2, \ldots, \beta_k, \ \alpha_0, \alpha_1,,, \alpha_k] \qquad (4.25)$$

Thus $S^{k+1}(E,F)$ has full row rank for $k < \nu_1$ or

$$\text{Rank } S^k(E,F) = r(k-1) + mk \quad \text{for} \quad k \leqq \nu_1 \qquad (4.26)$$

(iv) The proof of this result relies upon the structure of the matrix N when the column degrees of F are less than the corresponding column degrees of E. Since E is column proper, then the scalar matrix Γ_c, with elements in $\mathscr{R}$ consisting of the coefficients of the highest degree of z in each column of $E(z)$ will be nonsingular. It therefore follows that, if block rows of the form $[E_0 \ldots E_\ell 0 \ldots 0]$ are added to the top of $S^k(E,F)$, then the rank increases by r since the columns of Γ_c appear above zero columns in $S^k(E,F)$. The same argument applies when $[E_0 \ldots E_\ell 0 \ldots 0]$ is added to the top of

$$\left[\begin{array}{c|c} 0 & E_0 \ldots E_\ell 0 \ldots 0 \\ 0 & S^k(E,F) \end{array}\right].$$

Thus

$$\text{Rank}\, N = \text{Rank}\left[\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline 0 & S^k(E,F) \end{array}\right]$$

$$= \text{Rank}\,[S^k(E,F)] + 2r \qquad (4.27)$$

∇∇∇

We can now use the above lemma to establish necessary and sufficient conditions for identifiability of A', B'.

Theorem 4.1. Let

$\nu_{max}(G)$ = maximum observability index of any irreducible representation for G [12].

$\nu_{min}(G)$ = minimum observability index of any irreducible representation for G [12].

Then

(i) A necessary condition for identifiability is

$$\nu_{max}(G_1) \leqq \nu_{max}(G_3)$$

(ii) A sufficient condition for identifiability is

$$\nu_{max}(G_1) \leqq \nu_{min}(G_3)$$

Proof

(i) If $\nu_{max}(G_1) > \nu_{max}(G_3)$

then k, the highest power of z in A, B where $A^{-1}B = G_1$, is greater than $\nu_{max}(G_3)$.

Hence from Lemma (4.1) part (ii)

$$\text{Rank}\ [S^k(E,F)] = (k-1)r + p \qquad (4.28)$$

But,

$$p = \sum_{i=1}^{m} \nu_i(G_3) \leqq m\nu_{max}(G_3) < mk \qquad (4.29)$$

Hence

$$\text{Rank}\ [S^k(E,F)] < (k-1)r + mk \qquad (4.30)$$

Using part (iv) of Lemma 4.1 we have

$$\text{Rank}\left[\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline 0 & S^k(E,F) \end{array}\right] < (k+1)r + mk \quad (4.31)$$

However

$$\left[\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline 0 & S^k(E,F) \end{array}\right]$$

has $(k+1)r + mk$ rows, and it therefore follows from (4.31) and (4.10) that $A'(z), B'(z)$ cannot be uniquely determined.

(ii) If $\nu_{max}(G_1) \leqq \nu_{min}(G_3)$, then $k \leqq \nu_{min}(G_3)$ and hence from Lemma (4.1) parts (iii) and (iv)

$$\text{Rank}\left[\begin{array}{c|c} \multicolumn{2}{c}{M} \\ \hline 0 & S^k(E,F) \end{array}\right] = (k+1)r + km \quad (4.32)$$

Hence equation (4.10) can be solved uniquely for A' and B'.

∇∇∇

The above result can be extended to the case where $0 \neq G_3(\infty) < \infty$. The essential modifications are to write

$$F(z) = F_0 z^{\ell} + F_1 z_{\ell-1} + \ldots F_{\ell}$$

and to replace $(k-1)$ by k on the right hand side of (4.16) and (4.17) and to replace $2r$ by r on the right hand side of (4.18).

Our proof of the above theorem has been indirect in so far that we have argued via a closed loop model from which the open loop model is subsequently determined. However, provided the feedback regulator is sufficiently complex, as measured by the observability indices, then the data (u_t, y_t) can be analysed using maximum likelihood or prediction error methods as if it were open loop to yield a model for the forward path, cf. similar remarks in Section 3.

For the single input single output case, we have $\nu_{max}(G_3) = \nu_{min}(G_3)$ and thus the conditions given in Theorem 4.1 reduce to the known [6], [7], [14] necessary and sufficient conditions for identifiability in the single input single output case.

An important distinction between Theorem 3.1 and Theorem 4.1 is that Theorem 3.1 does not depend upon knowledge of the system order whereas Theorem 4.1 applies only in the case where there is a definite upper bound on the complexity of the forward path model.

5. CONCLUSION

This paper has discussed conditions under which a multiple-input multiple-output linear dynamic system can be identified from closed loop measurements. The results are believed to be of considerable practical importance since many processes are either intrinsically closed loop or have undesirable operating characteristics in open loop. The conditions established in this paper can be applied either to test identifiability for some given system or as an aid in the design of a closed loop identification experiment in which the feedback may be chosen by the experimenter.

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APPENDIX

MATRIX SPECTRAL FACTORIZATION

In this appendix, we recall several facts concerning matrix spectral factorization in the z-plane. The results for the s-plane are well documented[15], but a little harder to find for the z-plane. We start with the following result, quoted by Popov[19]:

Lemma 1. (Popov) Let $\Psi(z)=\Psi^{\mathrm{T}}(z^{-1})$ be real rational, such that $z^m\Psi(z)$ is polynomial for some m, nonsingular almost everywhere and nonnegative for $|z|=1$. Then there exists a real polynomial $D_0(z)$ with $\det D_0(z)\neq 0$ for $|z|\leq 1$ such that

$$\Psi(z)=D_0^{\mathrm{T}}(z^{-1})D_0(z).$$

∇∇∇

This allows us to prove the following result, used in the proof of Theorem 3.1.

Lemma 2. Let $\Psi(z)$ be as above. Let $n\geq m$ but otherwise be arbitrary. Then there exists a polynomial $M(z)$ of degree n and $\bar{D}=\bar{D}^{\mathrm{T}}>0$ such that

$$\Psi(z)=M(z)\bar{D}M^{\mathrm{T}}(z^{-1})$$

$$\det M(z)\neq 0 \quad \text{for} \quad |z|>1$$

$$\lim_{z\to\infty} z^{-n}M(z)=I$$

Proof. Check that $M(z)=z^nD_0^{\mathrm{T}}(z^{-1})[D_0^{\mathrm{T}}(0)]^{-1}$ and $\bar{D}=D_0^{\mathrm{T}}(0)D_0(0)$ work.

∇∇∇

We also have the following better known result.

Lemma 3. Let $\Phi(z)=\Phi^{\mathrm{T}}(z^{-1})$ be real rational, nonsingular almost everywhere and bounded and nonnegative for $|z|=1$. Then there exists a real rational $\bar{K}(z)$, analytic in $|z|\geq 1$, nonsingular in $|z|>1$ and with $\bar{K}(\infty)=I$ and a positive definite $\bar{D}$ such that

$$\Phi(z)=\bar{K}(z)\bar{D}\bar{K}^{\mathrm{T}}(z^{-1})$$

Furthermore, $\bar{K}$ and $\bar{D}$ are unique.

Proof. Let $\psi(z)=\Pi(z-z_i)$ where z_i is a pole of $\Phi(z)$ with $|z_i|<1$. Then set $\Psi(z)=\psi(z)\psi(z^{-1})\Phi(z)$. Let $M(z)$ and $\bar{D}$ be as in Lemma 2 for some arbitrary n. Then take

$$\bar{K}(z)=\psi^{-1}(z)M(z)z^{\deg\psi-n}$$

Suppose that $\bar{K}(z)\bar{D}\bar{K}^{\mathrm{T}}(z^{-1})=\hat{K}(z)\hat{D}\hat{K}^{\mathrm{T}}(z^{-1})$ with $\hat{K}$ and $\hat{D}$ satisfying the same properties as $\bar{K}$ and $\bar{D}$. Then $S=\hat{K}^{-1}\bar{K}$ is analytic and nonsingular in $|z|>1$. Also, we have $S=\hat{D}[S^{\mathrm{T}}(z^{-1})]^{-1}\bar{D}^{-1}$, showing that S is analytic and nonsingular in $|z|<1$ and that, since $\bar{D}^{-1}=S^{\mathrm{T}}(z^{-1})\hat{D}^{-1}S(z)$ on $|z|=1$, $S(z)$ is bounded on $|z|=1$ and is thus analytic there. Hence S is analytic everywhere, with $S(\infty)=I$ by the conditions on $\bar{K}$ and $\hat{K}$. By Liouville's theorem, $S(z)=I$.

∇∇∇

For the proof of Theorem 3.1, we shall also require the following lemma concerning polynomial matrices.

Lemma 4. Let $A(z)$ be a square matrix polynomial. Then there exists a polynomial $V_A(z)$ and an integer n_A such that

$$V_A(z)A(z)=z^{n_A}I+\text{lower order terms}$$

$$\det V_A(z)\neq 0 \quad \text{for} \quad |z|\geq 1.$$

Proof. There exists[12] $V_1(z)$ with $V_1(z)$ unimodular, i.e. of constant determinant, polynomial, and such that $V_1(z)A(z)$ is row proper, i.e. there exists a set of indices n_i such that

$$\lim_{z\to\infty} \operatorname{diag}[z^{-n_1}, z^{-n_2}\ldots]\, V_1(z)A(z)=\Gamma$$

is finite and nonsingular. Set $V_2(z)=\operatorname{diag}[z^{n-n_1}\, z^{n-n_2}\ldots]$ where $n=\max n_i$ and take $V_A(z)=\Gamma^{-1}V_2(z)V_1(z)$ and $n_A=n$ to obtain the lemma.