INT. J. CONTROL, 1977, VOL. 26, NO. 2, 235–247

Stability of matrix polynomials†

BRIAN D. O. ANDERSON‡ and ROBERT R. BITMEAD‡

The paper considers the following question: Given a square, non-singular polynomial matrix $C(s)$, how do we check, without evaluating the determinant, whether all the zeros of det $C(s)$ are in the open left-half plane?

The approach used to answer this question is to derive from $C(s)$ a rational transfer function matrix which is lossless positive real (l.p.r.) if and only if det $C(s)$ is Hurwitz. The l.p.r. property is easily checked using the coefficients of the rational function only. The construction of the l.p.r. function requires solution of a polynomial matrix equation, and the later part of the paper discusses both existence questions and solution procedures; if no solution exists to the matrix equation then det $C(s)$ is non-Hurwitz.

The connection is also illustrated between the l.p.r. stability test and that of Shieh and Sacheti (1976). Prospects for development of the theory are discussed.

1. Introduction

In many situations of linear systems theory it is often necessary to examine the location of the zeros of the determinant of a matrix polynomial. For instance, given a rational transfer function matrix $H(s)$, it is often of interest to know whether it represents a stable system. If $H(s)$ is represented by a minimal matrix fraction description (M.F.D.) $A^{-1}(s)B(s)$ (where $A(s)$ and $B(s)$ are relatively left prime polynomial matrices with det $A(s) \not\equiv 0$), then $H(s)$ represents a stable system if and only if det $A(s)$ has all its zeros in the open left-half plane Re $[s] < 0$.

One method of localizing the roots of det $A(s)$ is to evaluate the scalar polynomial det $A(s)$ and then to apply a scalar stability test to it. However, as more sophisticated matrix methods become available, it is both reasonable and instructive to attempt to find a direct stability test which does not involve the computation of det $A(s)$. It is our intention to examine such an approach in this paper.

Consideration of the scalar polynomial stability results indicates several likely directions of approach to the matrix case. Recall that, given a scalar polynomial $p(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n$ with $a_0 > 0$, $p(s)$ has zeros only in the left-half plane:

(i) if and only if the elements constituting the first column of Routh's array are all positive (Routh 1877);

(ii) if and only if the rational function

$$w(s) = \frac{\text{even part of } p(s)}{\text{odd part of } p(s)} \quad \text{or} \quad \frac{\text{odd part of } p(s)}{\text{even part of } p(s)}$$

is a reactance function, i.e. represents the driving point impedance or admittance of an LC network (Guillemin 1957);

Received 23 February 1977.
† Work supported by the Australian Research Grants Committee.
‡ Department of Electrical Engineering, The University of Newcastle, New South Wales, 2308, Australia.

(iii) if and only if, for $w(s)$ as above (the alternative chosen which yields $w(\infty)=0$)

$$\mathop{\mathrm{I}}_{-\infty}^{\infty} jw(j\omega) = \text{degree } p(s)$$

where $\mathop{\mathrm{I}}_{b}^{a} z(s)$ is the Cauchy index of $z(s)$ over (a, b) and is defined as the number of jumps of $z(s)$ from $-\infty$ to $+\infty$ less the number from $+\infty$ to $-\infty$ as s varies from a to b – jumps at a and b neglected (Gantmacher 1959).

Recent developments in the theory of rational and polynomial matrices have produced a matrix Routh array (Shieh 1975, Shieh and Sacheti 1976) applicable to the testing of stability of a restricted class of matrices, a simple test for a rational matrix to represent the driving point impedance of an LC multiport network (R. Bitmead and B. D. O. Anderson, under review; Bitmead and Anderson 1977) and a matrix Cauchy index applicable to real symmetric or hermitian rational matrices (Bitmead and Anderson 1977). As is already known in the scalar case, the Cauchy index approach to the matrix stability question yields simply the other results (Gantmacher 1959) and we shall show that the matrix Cauchy index methods provide the link between our LC impedance matrix test for stability and the less general matrix Routh array test of Shieh and Sacheti (1976) which, in fact, represents a restricted LC test.

In this paper, we extend this viewpoint to the matrix case. More specifically, we show how the stability of a prescribed matrix polynomial can be examined by testing a rational matrix for the lossless positive real (l.p.r.) property, or equivalently another rational matrix for the lossless bounded real (l.b.r.) property (defined later). The construction of the l.p.r. or l.b.r. matrices from the prescribed matrix polynomial is not as straightforward as in the scalar case. We also connect these ideas with the Routh array of Shieh, which in fact represents a scheme for testing for a property that implies l.p.r., but is at the same time more restrictive.

The plan of the paper is as follows. We review in § 2 the statements of the l.p.r. and l.b.r. properties for a rational transfer function matrix represented as a polynomial matrix fraction. The testing for these properties is done by simple calculations with the coefficients of the polynomials in the matrix fraction. Section 3 attempts to derive from a prescribed matrix polynomial a rational transfer function matrix which is l.p.r. if and only if the determinant of the matrix polynomial is Hurwitz. To do this, it proves necessary to introduce a dual matrix polynomial whose presence in the scalar problem is disguised since scalar polynomials are self-dual. Section 4 considers the computational aspects of the derivation of the dual polynomial and narrows down the difficulty to the solution of a certain polynomial matrix equation. However, it is then noted that other authors have investigated similar equations and their results may be modified to determine whether a solution exists and then to calculate it. We show that if no solution exists to the equation then the prescribed polynomial matrix has a non-Hurwitz determinant. In § 5 we illustrate the connection between the stability test of Shieh and Sacheti

(1976) and our l.p.r. test. We show that the tests are equivalent, except that that of Shieh and Sacheti (1976) is applicable to only a subset of those polynomial matrices dealt with by our test. We also mention prospects for further development of the theory.

2. Lossless positive real and lossless bounded real rational matrices

A real rational matrix $Z(s)$ which represents the driving point impedance (or admittance) of an LC multiport network is termed l.p.r. The properties of l.p.r. rational matrices will play a crucial role in the derivation of a direct stability test. Therefore we now summarize the matrix l.p.r. property as approached from a Cauchy index viewpoint. We note the definition and basic properties of the matrix Cauchy index as a preliminary.

Defining the Cauchy index of a rational hermitian matrix $Z(s)$ over the real interval (a, b), $\mathop{I}_{a}^{b} Z(s)$, as the number of eigenvalues of $Z(s)$ which jump from $-\infty$ to $+\infty$ less the number which jump from $+\infty$ to $-\infty$ as the independent variable s traverses the real axis from a to b (jumps at a and b are not counted) we have :

Lemma 1 (Bitmead and Anderson 1977)

Let $Z(s)$ be an hermitian matrix of rational functions of s over the complex field with $\lim_{s\to\infty} Z(s)$ finite and with a left matrix fraction description (m.f.d.) $A^{-1}(s)B(s)$. Then the matrix Δ, whose $i-j$ block entry Δ_{ij} is given by

$$\sum_i \sum_j \Delta_{ij} x^{i-1} y^{j-1} = \frac{1}{x-y}\,[A(x)B^*(y) - B(x)A^*(y)] \tag{2.1}$$

is an hermitian matrix, and

$$\mathop{I}_{-\infty}^{\infty} Z(s) = \text{signature } \Delta \tag{2.2}$$

The matrix Δ above is actually a generalized bézoutian (Anderson and Jury 1976) matrix associated with $Z(s)$. Entries of Δ are easily constructed from the coefficients of $A(\cdot)$ and $B(\cdot)$.

Another method for the evaluation of the matrix Cauchy index is via a matrix Sturm sequence. Let $Z(s)$ be a rational hermitian and now non-singular matrix with left m.f.d. $A_1^{-1}(s)A_2(s)$. A left Sturm sequence associated with $Z(s)$ is the sequence of polynomial matrices $\{A_1(s), A_2(s), \ldots, A_m(s)\}$ obtained from

$$\left.\begin{aligned} A_1(s) &= A_2(s)Q_2(s) - A_3(s) \\ A_2(s) &= A_3(s)Q_3(s) - A_4(s) \\ &\vdots \\ A_{m-1}(s) &= A_m(s)Q_m(s) \end{aligned}\right\} \tag{2.3}$$

where $A_i(s)$ is non-singular and $Q_i(s)$ is polynomial. As shown by Bitmead and Anderson (1977), it is always possible to choose the $A_i(s)$ so that $\deg\det A_{i+1}(s) < \deg\det A_i(s)$ and $A_{i-1}^{-1}(s)A_i(s)$ is hermitian if $Z(s)$ is also. We now have :

Lemma 2 (Bitmead and Anderson 1977)

Let $Z(s)$ be a rational hermitian non-singular matrix with left m.f.d. $A_1^{-1}(s)A_2(s)$. Let $\{A_1, A_2, \ldots, A_m\}$ be an associated left Sturm sequence. Then

$$\mathop{\mathrm{I}}_a^b Z(s) = \sum_{i=1}^{m-1} \Delta_a{}^b(A_i^{-1}A_{i+1}) \tag{2.4}$$

where $\Delta_a{}^b Y(s)$ for hermitian $Y(s)$ equals half the signature of $Y(b)$ less half the signature of $Y(a)$.

A minor variant on the lemma, whose details will not concern us, allows extension to singular $Z(s)$.

Having briefly described the matrix Cauchy index we turn now to its application with l.p.r. matrices. A *real* rational matrix $Z(s)$ is termed l.p.r. if and only if $Z(s)+Z'(-s)=0$ and all poles of entries of $Z(s)$ are either pure imaginary or at infinity, are simple, and have the associated residue matrix non-negative definite hermitian (Anderson and Vongpanitlerd 1973, Newcomb 1966). Or, alternatively (Anderson and Vongpanitlerd 1973, Newcomb 1966),

$$Z(s)+Z'(-s)=0 \tag{2.5}$$

and

$$Z(s)+Z'(s^*) \geqslant 0 \quad \text{in} \quad \mathrm{Re}\,[s]>0 \tag{2.6}$$

In terms of the Cauchy index we have the following theorem and important corollary.

Theorem 1 (Bitmead and Anderson 1977)

The real rational matrix $Z(s)$ is l.p.r. if and only if $Z(s)+Z'(-s)=0$, entries of $Z(s)$ may have a pole at ∞ and that pole is simple with non-negative definite symmetric residue matrix Z_∞, and

$$\mathop{\mathrm{I}}_{-\infty}^{\infty} W(\omega) = \delta[Z(s)] - \mathrm{rank}\, Z_\infty \tag{2.7}$$

where $W(\omega)=jZ(j\omega)$ is an hermitian rational matrix and $\delta[\ ]$ is the McMillan degree (Anderson and Vongpanitlerd 1973, Newcomb 1966).

This theorem has the following computationally important corollary when the result of Lemma 1 is applied to Theorem 1. (A minor result of (Bitmead and Anderson (to be published) and Anderson and Jury (1976) is used to eliminate the condition that $Z(s)$ be finite at ∞.)

Corollary 1 (Bitmead and Anderson, to be published)

The $r \times r$ real rational matrix $Z(s)$, with left matrix fraction decomposition $A^{-1}(s)B(s)$ is l.p.r. if and only if $Z(s)+Z'(-s)=0$ and, with Δ defined as the matrix whose $i-j$ block element is Δ_{ij}, where

$$\sum_i \sum_j \Delta_{ij} x^{i-1} y^{j-1} = \frac{1}{x-y}\,[A(x)B'(-y)+B(x)A'(-y)] \tag{2.8}$$

and $\Sigma = \mathrm{diag}\,[I_r, -I_r, I_r, -I_r, \ldots]$, $\Sigma\Delta$ is non-negative definite symmetric.

This last corollary presents a simple way of testing the l.p.r. property given a left m.f.d., since the entries of Δ can be found easily from the coefficients of the polynomials of the m.f.d. As a left m.f.d. arises naturally in the following work, this test is crucial.

Allied to l.p.r. rational matrices are lossless bounded real (l.b.r.) rational matrices. A real rational matrix $S(s)$ is termed l.b.r. if (Anderson and Vongpanitlerd 1973, Newcomb 1966):

$$S(s) \text{ has no poles in } \operatorname{Re}[s] \geqslant 0 \tag{2.9}$$

and

$$S(s)S'(-s)=I \tag{2.10}$$

It is shown in (Newcomb 1966) that the real rational matrix

$$S(s)=[I+Z(s)]^{-1}[I-Z(s)]$$

is l.b.r. if and only if the real rational matrix $Z(s)$ is l.p.r. With this in mind and using some of the elementary properties of the generalized bézoutian matrix (Bitmead and Anderson, to be published; Anderson and Jury 1976), we may construct the l.b.r. analogue of Corollary 1:

Corollary 2

The real rational matrix $S(s)$ with left m.f.d. $C^{-1}(s)D(s)$ is lossless bounded real if and only if $S(s)S'(-s)=I$ and, with Δ defined as the matrix whose $i-j$ block element is Δ_{ij}, where

$$\sum_i \sum_j \Delta_{ij} x^{i-1} y^{j-1} = \frac{1}{x-y}[C(x)C'(-y)-D(x)D'(-y)] \tag{2.11}$$

and $\Sigma=\operatorname{diag}[I_r, -I_r, I_r, -I_r, \ldots]$, $\Sigma\Delta$ is non-negative definite symmetric.

In the light of the definition of the l.b.r. property, and noting that, given a m.f.d. of a rational function $Z(s)=A^{-1}(s)B(s)$ where A and B are relatively left prime, the poles of $Z(s)$ occur at zeros of $\det A(s)$, we have the following theorem:

Theorem 2

If the real rational matrix $Z(s)=A^{-1}(s)B(s)$, where $A(s)$ and $B(s)$ are relatively left prime polynomial matrices, if l.p.r. then the polynomial matrix $C(s)=A(s)+B(s)$ is Hurwitz, i.e. all zeros of $\det C(s)$ are in the open left half plane.

Proof

If $Z(s)=A^{-1}(s)B(s)$ is l.p.r. then

$$\begin{aligned} S(s) &= [I+Z(s)]^{-1}[I-Z(s)] \\ &= [A(s)+B(s)]^{-1}[A(s)-B(s)] \end{aligned}$$

is l.b.r. Since $\{A(s), B(s)\}$ relatively left prime implies the same of the pair $\{A+B, A-B\}$, and $S(s)$ has no poles in $\operatorname{Re}[s] \geqslant 0$, $A(s)+B(s)$ is Hurwitz.

From a heuristic viewpoint the result is also evident, since we know that an $r \times r$ l.p.r. $Z(s)$ represents the impedance of a r-port network of lossless elements and should r unit shunt impedances be connected across the network ports the resultant network is no longer lossless but is stable (see Fig. 1). The resultant rational impedance matrix $W(s) = (I+Z)^{-1}Z = [A(s)+B(s)]^{-1}B(s)$ has no poles in Re $[s] \geq 0$. Since $\{A, B\}$ relatively left prime implies $\{A+B, B\}$ relatively left prime, the Hurwitz nature of $A(s)+B(s)$ is clear.

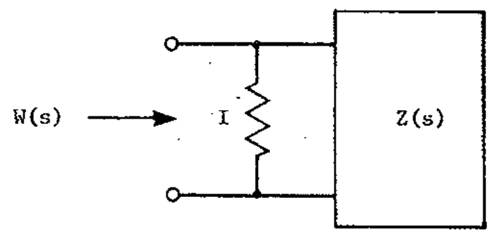

Figure 1. Loading of lossless network to produce stable network.

3. The stability problem—the dual polynomial matrix

One method of attempting to exploit the result of Theorem 2 in developing a matrix stability test is to ask : Given the polynomial matrix $C(s)$, how do we decide whether there exists a decomposition $C(s) = A(s) + B(s)$, with $Z(s) = A^{-1}(s)B(s)$ l.p.r. if $C(s)$ is Hurwitz ? Recall that the scalar answer to this question is that the polynomial $c(s)$ is Hurwitz if and only if the rational function $w(s)$ given by the even part of c divided by the odd part (or perhaps vice versa) is scalar l.p.r.—the l.p.r. test corresponding to that of Corollary 2 is then just the Hermite stability test (Jury 1974). Unfortunately, *the same decomposition is in general unsatisfactory for matrix polynomials.* Such a decomposition almost never permits $Z(s) + Z'(-s) = 0$, so there is no chance of obtaining $Z(s)$ l.p.r. One situation where $Z(s) + Z'(-s) = 0$ occurs is when $A^{-1}(s)B(s)$ is symmetric with A odd and B even, which always holds in the scalar case, but obviously will not hold in general.

With the direction of development of the l.p.r. property unclear, we approach the stability question on a different tack. We first introduce the notion of a dual polynomial matrix $D(s)$ to our given polynomial matrix $C(s)$.

Definition

The dual polynomial matrix to a given non-singular polynomial matrix $C(s)$ is that matrix $D(s)$ such that

$$C(s)C'(-s) = D'(-s)D(s) \tag{3.1}$$

and

$$\det C(s) = \det D(s) \tag{3.2}$$

Observe that all scalar polynomials are necessarily self-dual. Similar dual polynomials have appeared in Levinson filtering (Kailath 1974)—in the scalar case these polynomials reduce to the Szego polynomials where the duality is simply characterized.

We shall leave until later the discussion of the existence and computation of the dual polynomial matrix as its calculation may become quite involved; in this section we shall stress its significance to the stability problem.

Theorem 3

Given a polynomial matrix $C(s)$ with dual polynomial matrix $D(s)$, i.e. such that (3.1) and (3.2) hold, then $\det C(s)$ is Hurwitz if and only if $S(s) = C^{-1}(s)D'(-s)$ represents a minimal left m.f.d. of a lossless bounded real rational matrix.

Proof

Suppose $\det C(s)$ is Hurwitz; then clearly $S(s)$ has all poles in $\mathrm{Re}\,[s] < 0$ and

$$S(s)S'(-s) = C^{-1}D'(-s)D(s)C'^{-1}(-s) = I \tag{3.3}$$

Hence $S(s)$ is l.b.r. Furthermore, since $\det C(s)$ is Hurwitz and $\det C = \det D$, $\det C$ and $\det D'(-s) = \det C'(-s)$ have no common factors—one having zeros in $\mathrm{Re}\,[s] < 0$, the other in $\mathrm{Re}\,[s] \geqslant 0$—and $\{C(s), D'(-s)\}$ form a relatively left prime pair.

Suppose $S(s) = C^{-1}(s)D(s)$ is a minimal left m.f.d. of an l.b.r. rational matrix; then all the poles of $S(s)$ occur at the zeros of $\det C(s)$ and consequently the l.b.r. property ensures $\det C$ is Hurwitz.

We note that, since the lossless bounded real property is easily checked by straightforward investigations of a finite dimensional real matrix formed from the coefficients of C and D as per Corollary 2, and the relative primeness also checked from the same matrix (Anderson and Jury 1976), *once the dual polynomial matrix $D(s)$ is obtained, the stability test for the polynomial matrix $C(s)$ is straightforward.*

We now turn to the significant problem of determining the dual polynomial.

4. Computational aspects of the dual polynomial

In this section we approach the problems of existence and computation of the dual polynomial $D(s)$ to a given non-singular polynomial matrix $C(s)$. We shall show that, although the l.b.r. test involves simple calculations on a real matrix, and the construction of the dual polynomial requires only rational calculations, the derivation of the dual is not necessarily straightforward and may even prove more difficult than the calculation of $\det C(s)$. However, the solution of certain equations arising in the derivation appears to be a matter of current interest and, should simple methods be developed for the solution of these equations, the l.b.r. approach to matrix stability may prove easier in general.

Suppose we have the polynomial matrix $X(s)$, which represents the solution of the polynomial matrix equation

$$I = X_* C + C_* X \quad \text{where } B_* \text{ denotes } B'(-s) \tag{4.1}$$

(The existence and construction of solution of this equation are discussed below.) Now (4.1) is equivalent to

$$C_*^{-1}\,C^{-1} = C_*^{-1}\,X_* + XC^{-1} \tag{4.2}$$

Now let us write a minimal left m.f.d. $F^{-1}G$ of the rational function XC^{-1}. That is, we find relatively left prime polynomial matrices F, G such that $XC^{-1}=F^{-1}G$. This is a standard construction of linear systems theory and may be performed with operations upon a matrix of real constants formed from the coefficients of X and C (Wolovich 1974). Notice, that, since (4.1) implies that C and X are relatively right prime, $XC^{-1}=F^{-1}G$ with F, G relatively left prime implies that without loss of generality $\det C=\det F$. In other words

$$C_*^{-1}C^{-1}=G_*F_*^{-1}+F^{-1}G \tag{4.3}$$

and $T=FC_*^{-1}C^{-1}F_*$ is a unimodular polynomial matrix such that $T=T_*$ with $T(s)$ non-negative definite hermitian on the $\mathrm{Re}\,[s]=0$ axis.

Under these conditions on T, there exists (Yakubovich 1970, Youla 1961) a factorization $T=UU_*$ with $U(s)$ a unimodular polynomial matrix. Writing $V=U^{-1}$, V is also polynomial and unimodular and we have

$$T=FC_*^{-1}C^{-1}F_*=V^{-1}V_*^{-1} \tag{4.4}$$

or

$$CC_*=(VF)_*(VF) \tag{4.5}$$

with

$$\det C=\det VF. \tag{4.6}$$

Therefore we have constructed the dual polynomial matrix $D(s)(=V(s)F(s))$ to the given polynomial matrix $C(s)$.

There are three separate steps in the above procedure : solving (4.1), finding the left m.f.d. $F^{-1}G$ given the right m.f.d. XC^{-1}, and factoring the unimodular $T=T_*$ as UU_*. We comment on these steps in reverse order. As noted above, the existence of U given T with the properties stated is not in dispute ; Davis (1963) illustrates a simple, algorithmic method for the factorization of the unimodular polynomial matrix T which generates the real coefficient matrices of U in succession via operations on real matrices. A similar method may also be used to general $V=U^{-1}$ from U.

Passing from the right m.f.d. to the left m.f.d. is a standard problem (Wolovich 1974).

As for (4.1), results are available in the literature for the case when C is strictly regular, i.e. the coefficient of the highest-order term in $C(s)$ is non-singular. In Barnett (1971) and Gohberg and Lerer (1976), one finds that (4.1) has a solution X, unique if X is required to have degree less than that of C, if and only if $|C(s)|$ and $|C_*(s)|$ are relatively prime. (Of course, if $|C(s)|$ is Hurwitz, the relative primeness holds.) Equation (4.1) may be rearranged as an equation $Ax=b$, with A a known matrix, b a known vector, and x a vector to be found, whose entries constitute a rearrangement of the entries of X. The matrix A is a generalized Sylvester matrix. Because of its shift invariant structure, the evaluation of x as $A^{-1}b$ can proceed by much faster algorithms than normal.

The authors have checked that this existence result carries over to the case when C is not strictly regular, see the Appendix ; X is uniquely specified if one requires XC^{-1} to be strictly proper. Again a linear equation $Ax=b$ will yield the entries of X, and again A will have some shift structure which will speed up the evaluation of x. An alternative possible computational scheme is also described in the Appendix.

If (4.1) is not solvable, $|C(s)|$ and $|C_*(s)|$ must have a non-trivial greatest common divisor, and thus $|C(s)|$ cannot be Hurwitz.

As an example, consider the linear case with non-singular leading element. Let $C(s)=sI-A$ be the polynomial matrix in question. We seek a dual polynomial of the form $D(s)=sI-B$ so that

$$CC_*=D_*D$$
$$\det C=\det D$$

Since the solution $X(s)$ of (4.1) must have XC^{-1} strictly proper, we see that $X(s)$ must be a constant matrix X. Then (4.1) yields simply $I=-XA-A'X$, a standard Lyapunov equation. The right m.f.d. $X(sI-A)^{-1}$ has a left m.f.d. $(sI-B)^{-1}Y$ if and only if $sX-BX=sY-YA$ or $B=XAX^{-1}$. This defines the dual polynomial.

Checking stability with the aid of $C(s)$ and $D(s)$, if $D(s)$ has to be computed, may be pointless in view of the fact that X has to be found, with X positive definite if and only if $C(s)$ is stable. Nevertheless, it is instructive to see what happens.

Since $(sI-B)$ is the dual of $(sI-A)$, by Theorem 3 we only need to check the relative primeness of these two matrices and whether the rational matrix $S(s)=(sI-A)^{-1}(-sI-B')$ is l.b.r. Both these aims may be realized using the bézoutian matrix Δ, which has rank equal to the McMillan degree of $S(s)$, which, in turn equals deg det $(sI-A)$ if and only if the two polynomial matrices are relatively left prime. The bézoutian is given by

$$\sum_i \sum_j \Delta_{ij} x^{i-1} y^{j-1} = \frac{1}{x-y}\left[(xI-A)(-yI-A')-(-xI-B')(yI-B)\right]$$
$$=-(A'+B)$$

Thus, $\Sigma\Delta=-(A'+B)=-A'-XAX^{-1}$ must have full rank and be non-negative definite symmetric. From the calculations above we see that $\Sigma\Delta=X^{-1}$ and the result for stability then becomes identical to that of Lyapunov's theorem, i.e. the polynomial matrix $(sI-A)$ has a Hurwitz determinant if and only if there exists a positive definite symmetric matrix X satisfying $XA+A'X=-I$.

We note now that, although this particular example does not involve rational operations with polynomials, we do in fact have to solve a Lyapunov equation.

We would have hoped to be able to present a simple derivation of a stability criterion for the matrix polynomial $s^2I+A_1s+A_2$, at least with $A_1=A_1'$. (Stability follows if $A_1>0$, $A_2+A_2'>0$.) The fact that a derivation is not *simply* obtainable using the dual polynomial concept suggests that our formulation is not yet optimum.

5. Complements, remarks and conclusions

We first show that the existing sufficient stability criterion of Shieh and Sacheti (1976), which is valid for a restricted class of polynomial matrices, is actually an l.p.r. test utilizing a matrix Routh array, which is the equivalent to a Cauchy index calculation using a Sturm sequence.

In Shieh and Sacheti (1976) it is stated that, given a polynomial matrix $B(s)=Is^n+B_ns^{n-1}+\ldots+B_1$, we form a matrix Routh array, C_{ij} according to the rules :

$$C_{1,j}=B_{n+3-2j}, \quad j=1,2,3,\ldots,l \qquad \text{where } l=\begin{cases}\dfrac{n}{2}+1 & (n \text{ even})\\[2ex] \dfrac{n+1}{2} & (n \text{ odd})\end{cases}$$

$$C_{2,j}=B_{n+2-2j}, \quad j=1,2,3,\ldots,l$$

$$C_{11}=I$$

$$C_{i,j}=C_{i-2,j+1}-H_{i-2}C_{i-1,j+1}, \quad j=1,2,\ldots, \quad i=3,4,\ldots$$

$$H_i=C_{i,1}(C_{i+1,1})^{-1}, \quad i=1,2,\ldots,n$$

$$\det(C_{i+1,1})\neq 0$$

Then a sufficient condition for stability of $\det B(s)$ is that all the 'matrix quotients' H_i be real symmetric positive definite matrices.

Notice that this result is restricted in two ways ; first, if $B(s)$ has a singular leading coefficient or indeed has any of the C_{i1} singular then the Routh array breaks down, and, second, since the first column of the matrix Routh algorithm above represents the terms of the continued fraction expansion of the rational function formed by $E(s)O^{-1}(s)$, where $E(s)$ is the even part of B and $O(s)$ is the odd part, and these terms are required symmetric, the test is also restricted to those $B(s)$ such that $E(s)O^{-1}(s)$ is symmetric. This test is thus restricted to a fairly narrow class of matrix polynomials.

The test can be seen in a different light. Suppose that the polynomial matrix $B(s)$ does obey the requirements, then $Z(s)=E(s)O^{-1}(s)$ satisfies $Z(s)+Z'(-s)=0$ and is expressible as $Z(s)=H_1s+[H_2s+[H_3s+\ldots]^{-1}]^{-1}$ in continued fraction form, with H_i positive definite symmetric. That this rational function is realizable as the driving point impedance of an LC network, and is hence l.p.r., is at once evident when a Cauer synthesis (Anderson and Vongpanitlerd 1973) is attempted (see Fig. 2).

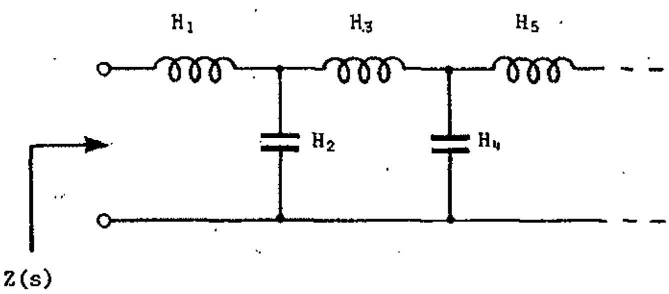

Figure 2. Multiport Cauer synthesis.

Consequently, we have shown that the stability test of Shieh and Sacheti (1976) represents an l.p.r. test on the restricted class of polynomial matrices $B(s)$ such that $Z(s)=EO^{-1}=-Z'(-s)=-Z(-s)$. That the continued fraction result above represents an l.p.r. test may also be established by noticing that $\{H_i s\}$ is the sequence of quotient polynomials, $Q_i(s)$, from a Sturmian division of polynomial matrices E and O. Lemma 2 and Theorem 1 then combine to complete the proof. We use the fact that the McMillan degree of EO^{-1} equals the sum of the ranks of the H_i (this is easily checked).

Several other points should be made. First, we might ask whether a Liénard–Chipart (Gantmacher 1959) type of simplification of the stability test might evolve. Looked at using the Hermite matrix in the scalar case, it transpires that this simplification depends on approximately half the entries of the Hermite matrix being zero. Since the Hermite matrix is nothing but a special Bézout matrix, one would imagine that for a simplification to exist in the matrix polynomial stability problem, the Bézout matrix would have to have many zero entries. This is not generally the case. Thus a Liénard–Chipart simplification does seem unlikely.

Generalizations of scalar polynomial results which might have more chance of being achieved could include tests for the stability of complex matrix polynomials, and tests for the zeros of the determinant of a matrix polynomial to all be negative real.

The most pressing problem, however, would seem to be one of evolving rapid means for performing the computations outlined in § 4. And, should these be developed, as appears likely using a generalized Sylvester matrix having a high degree of structure, then our method of l.p.r. testing should provide not only a method for testing stability of the polynomial matrices using only the real matrices of the coefficients but also a fair degree of computational flexibility which should allow the use of well-conditioned standard linear algebra packages.

Appendix

The equation $AX+YB=I$

We shall establish the following result.

Theorem

Let $A(s)$ and $B(s)$ be prescribed $n\times n$ polynomial matrices with $|A(s)|$ and $|B(s)|$ coprime. Then there exist polynomial matrices $X(s)$, $Y(s)$ satisfying

$$AX+YB=I \tag{1}$$

and if X, Y are constrained to be such that XB^{-1} and $A^{-1}Y$ are strictly proper, then X, Y are unique.

We shall present two quite distinct proofs, both of which contain a constructive procedure.

First proof

By right multiplication by a unimodular matrix U, we can obtain AU as row proper. Likewise, for a unimodular V, we can obtain VB column proper. Since $AX+YB=(AU)(U^{-1}X)+(YV^{-1})(VB)$, there is no loss of generality in simply assuming A and B are row and column proper. Now let

$$A^{-1}(s)=H_1'(sI-F_1)^{-1}G_1, \quad B^{-1}(s)=H_2'(sI-F_2)^{-1}G_2$$

define state variable realizations of A^{-1}, B^{-1} such that any strictly proper transfer function matrices $A^{-1}(s)L(s)$ and $M(s)B^{-1}(s)$ have the form $H_1'(sI-F_1)^{-1}M$ and $N'(sI-F_2)^{-1}G_2$, respectively, and conversely. Such realizations can always be found (Wolovich 1974). Let P solve

$$PF_2-F_1P=-G_1H_2'$$

Because $|A(s)|$ and $|B(s)|$ are coprime, F_1 and F_2 have no eigenvalues in common and thus P exists; it follows that

$$\begin{aligned} A^{-1}(s)B^{-1}(s) &= H_1'(sI-F_1)^{-1}G_1H_2'(sI-F_2)^{-1}G_2 \\ &= H_1'(sI-F_1)^{-1}[P(sI-F_2)-(sI-F_1)P](sI-F_2)^{-1}G_2 \\ &= H_1'(sI-F_1)^{-1}PG_2-H_1'P(sI-F_2)^{-1}G_2 \\ &= A^{-1}(s)X(s)+Y(s)B^{-1}(s) \end{aligned}$$

for some X, Y. Equation (1) is immediate.

Second proof

Without loss of generality, we can assume that A has upper triangular Hermite form (Gantmacher 1959, Wolovich 1974) and B has lower triangular Hermite form. For if this is not the case for A say, we may find a unimodular U such that AU has this form, and then replace A by AU and X by $U^{-1}X$.

The (n, n) term of (1) then becomes $a_{nn}x_{nn}+b_{nn}y_{nn}=1$ and with a_{nn}, b_{nn} relatively prime—as they must be with $|A|$ and $|B|$ coprime—x_{nn} and y_{nn} can be obtained. The $(n-1, n)$ term then gives

$$a_{n-1,n-1}x_{n-1,n}+y_{n-1,n}b_{nn}=-a_{n-1,n}x_{nn}$$

and relative primeness of $a_{n-1,n-1}$ and b_{nn} allows computation of $x_{n-1,n}$ and $y_{n-1,n}$. One proceeds looking successively at terms $(n-2, n)$, $(n-3, n)$ ... identifying thereby the entries of the last column of X and Y. Then one examines the terms in positions $(n, n-1)$, $(n-1, n-1)$, $(n-2, n-1)$, ... and so on. In this way, all entries of X and Y are obtained. The polynomial equations encountered at each stage are all solvable precisely because each diagonal entry of A is coprime with each diagonal entry of B.

It is possible at each stage when x_{ij} and y_{ij} are determined to take these quantities to have least degree. This will ensure the proper character of $A^{-1}Y$ and XB^{-1}.

To recover the result required for § 4, take $A=B_*=C_*$. Thus $C_*X+YC=I$. Immediately $C_*Y_*+X_*C=I$, and uniqueness gives $Y=X_*$.

References

ANDERSON, B. D. O., and JURY, E. I., 1976, *I.E.E.E. Trans. autom. Control*, **21**, 551.

ANDERSON, B. D. O., and VONGPANITLERD, S., 1973, *Network Analysis and Synthesis—A Modern Systems Theory Approach* (New Jersey: Prentice-Hall).

BARNETT, S., 1971, *Matrices in Control Theory* (London: Van Nostrand Reinhold).

BITMEAD, R., and ANDERSON, B. D. O., 1977, *SIAM J. appl. Math.* (in the press).

DAVIS, M. C., 1963, *I.E.E.E. Trans. autom. Control*, **8**, 296.

GANTMACHER, F. R., 1959, *The Theory of Matrices* (New York: Chelsea).

GOHBERG, I. C., and LERER, L. E., 1976, *Bull. Am. math. Soc.*, **82**, 565.

GUILLEMIN, E. A., 1957, *Synthesis of Passive Networks* (New York: Wiley).

JURY, E. I., 1974, *Inners and Stability of Dynamic Systems* (New York: John Wiley).

KAILATH, T., 1974, *I.E.E.E. Trans. Inf. Theory*, **20**, 146.

NEWCOMB, R. W., 1966, *Linear Multiport Synthesis* (New York: McGraw-Hill).

ROUTH, E. J., 1877, *A Treatise on the Stability of a Given State of Motion* (*Adams Prize Essay*) (London: Macmillan). Reprinted 1975, *Stability of Motion*, edited by A. T. Fuller (London: Taylor & Francis Ltd.).

SHIEH, L. S., 1975, *Int. J. Control*, **22**, 861.

SHIEH, L. S., and SACHETI, S., 1976, *Proc. 10th Asilomar Conf. on Circuits and Systems* (Hollywood, California: Western Periodicals Co.).

WOLOVICH, W. A., 1974, *Linear Multivariable Systems* (New York: Springer-Verlag).

YAKUBOVICH, V. A., 1970, *Dokl. Akad. Nauk.*, **11**, 1261.

YOULA, D. C., 1961, *I.R.E. Trans. Inf. Theory*, **7**, 172.