Stabilization of Certain Two-Dimensional Recursive Digital Filters

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Abstract—A possible extension of a well-known stabilization technique for one-dimensional recursive digital filters to the two-dimensional case was proposed by Shanks via a conjecture, stating that the planar least squares inverse of a two-dimensional filter polynomial is minimum phase and hence stable. In the present work, the conjecture has been verified first for a class of polynomials which are linear in one variable and quadratic in the other and then extended to a class of polynomials of higher degrees in the same variables. Though the conjecture is known to be false, in general, some conditions under which the conjecture is valid are explored.

I. Introduction

DIGITAL signal processing is concerned with the representation of signals by sequences of numbers or symbols and the processing of these sequences. The purpose of such processing may be to estimate characteristic parameters of a signal or to transform a signal into a form which is in some sense more desirable. The classical numerical analysis formulae, such as those designed for interpolation, integration, and differentiation are certainly digital signal processing algorithms. On the other hand, the availability of high-speed digital computers has fostered the development of increasingly complex and sophisticated signal processing algorithms, and recent advances in integrated circuit technology promise economical implementations of very complex digital signal processing systems. Real time implementations of very complex digital signal processing algorithms are now prevalent in such diverse areas as biomedical engineering, acoustics, sonar, radar, seismology, speech communication, data communication, nuclear science, image processing, and many others [1]. Signal processing problems are, of course, not confined to one-dimensional signals. Many picture processing applications require the use of two-dimensional signal processing techniques. This is the case with X-rays enhancement, the enhancement and analysis of aerial photographs for detection of forest fires or crop damage, the analysis of satellite weather photos, and the enhancement of the television transmissions from lunar and deep space probes. Processing of seismic records, gravity, and magnetic data also utilizes two-dimensional signal processing techniques [2].

Using two-dimensional fast Fourier transforms, conventional (nonrecursive) two-dimensional filtering can be implemented. A different method, which has often proved to be more efficient than the previous one, is recursive filtering. In this technique, the filter algorithm uses previously computed output values as well as the input values. The basic problems in two-dimensional recursive filtering are those of stability and synthesis.

Since a portion of the output values are used by the recursive algorithm, it is possible for the output values to become arbitrarily large independent of size of the input values. A filter of this type is said to be unstable, a condition which is undesirable. Shanks *et al.* [3] proposed an extension of a well known stabilization technique for one-dimensional recursive digital filters [4] to the two-dimensional case. They conjectured that the planar least square inverse (PLSI) of any arbitrary recursive two-dimensional filter is minimum phase and hence stable. Making use of the above conjecture, they suggested a stabilization technique, which involves taking a double PLSI (DPLSI) of the original (unstable) filter polynomial. They gave many examples from which, they concluded that the DPLSI of an unstable filter polynomial gives its minimum phase version with an amplitude spectrum roughly equal to that of the original filter. By examples they also verified that the larger the dimension of the intermediate PLSI filter, the better is the resemblance of the amplitude spectra. But they could not prove the conjecture.

Jury and Anderson [5] verified the conjecture for a special low-order polynomial, which is linear in the two variables. The key to their verification of the conjecture lies in utilizing the centrosymmetric properties of the particular Toeplitz matrix, which arises in the equations of the approximate inverse.

Meanwhile, Kamp and Genin came up with a counterexample for the above conjecture, by showing that a polynomial of third degree in the two variables allows an inverse polynomial of lower degree (linear) in the two variables, which violates the stability conditions [6]. They also made use of properties of orthogonal polynomials of two variables to disprove the conjecture in general [7]. But they did not investigate the conditions under which the conjecture is valid, which is crucial for the design of stable two-dimensional recursive digital filters and for the stabilization of the unstable ones.

In this present work, the conjecture has been first proved to be true for a class of polynomials, which are linear in one variable and quadratic in the other (Section II) and then extended to a class of polynomials of higher degrees in the same variables (Section III). The conditions under which the conjecture is valid for the special class of polynomials are investigated. Finally the difficulties involved in the proof of the conjecture for the general case and the suggestions for further research in this area are mentioned (Section IV).

II. Verification of Shanks' Conjecture for a Class of Low-Degree Polynomials

In this section, the outline of the proof of Shanks' conjecture for a class of low-degree polynomials, which are linear in one variable and quadratic in the other, is presented. The key

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to the verification of the conjecture lies in utilizing the properties of the particular Toeplitz matrix which arises in the equations of the approximate inverse. The main results are presented in this section and for the related algebraic manipulations the readers are referred to [8].

$$\begin{bmatrix} (\Gamma_{00}+\Gamma_{21}) & (\Gamma_{01}+\Gamma_{20}) & (\Gamma_{10}+\Gamma_{11}) & 0 & 0 & 0 \\ (\Gamma_{01}+\Gamma_{20}) & (\Gamma_{00}+\Gamma'_{20}) & (\Gamma_{10}+\Gamma'_{10}) & 0 & 0 & 0 \\ (\Gamma_{10}+\Gamma_{11}) & (\Gamma_{10}+\Gamma'_{10}) & (\Gamma_{00}+\Gamma_{01}) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\Gamma_{00}-\Gamma_{21}) & (\Gamma_{01}-\Gamma_{20}) & (\Gamma_{10}-\Gamma_{11}) \\ 0 & 0 & 0 & (\Gamma_{01}-\Gamma_{20}) & (\Gamma_{00}-\Gamma'_{20}) & (\Gamma'_{10}-\Gamma_{10}) \\ 0 & 0 & 0 & (\Gamma_{10}-\Gamma_{11}) & (\Gamma'_{10}-\Gamma_{10}) & (\Gamma_{00}-\Gamma_{01}) \end{bmatrix} \begin{bmatrix} b_{00}+b_{21} \\ b_{01}+b_{20} \\ b_{10}+b_{11} \\ b_{21}+b_{00} \\ b_{20}-b_{01} \\ b_{11}-b_{10} \end{bmatrix} = \begin{bmatrix} a_{00} \\ 0 \\ 0 \\ -a_{00} \\ 0 \\ 0 \end{bmatrix}. \quad (2.4)$$

Suppose that there is a prescribed polynomial

$$A(z_1, z_2) = a_{00} + a_{01}z_2 + a_{10}z_1 + a_{11}z_1z_2 + a_{20}z_1^2 + a_{21}z_1^2z_2 \quad (2.1)$$

where not all a_{ij} are zero. The stability of an appropriate inverse (i.e., of the same order as the original filter polynomial) of the form

$$B(z_1, z_2) = b_{00} + b_{01}z_2 + b_{10}z_1 + b_{11}z_1z_2 + b_{20}z_1^2 + b_{21}z_1^2z_2 \quad (2.2)$$

will be established.

Using two-dimensional convolution

$$A(z_1, z_2)\,B(z_1, z_2) = C(z_1, z_2) = \sum_{i=0}^{4}\sum_{j=0}^{2} c_{ij} z_1^i z_2^j$$

where the c_{ij}'s are readily obtainable in terms of the a_{ij}'s and b_{ij}'s [8, Appendix I].

The planar least square inverse is obtained after forming

$$Q = (1 - c_{00})^2 + \sum_{\substack{i=0\\ i+j>0}}^{4}\sum_{j=0}^{2} c_{ij}^2 = (1 - a_{00}b_{00})^2 + \sum_{\substack{i=0\\ i+j>0}}^{4}\sum_{j=0}^{2} c_{ij}^2.$$

The equations that result from minimization of Q with respect to b_{ij}, [8, Appendix II] can be expressed in matrix form

$$\begin{bmatrix} \Gamma_{00} & \Gamma_{01} & \Gamma_{10} & \Gamma_{11} & \Gamma_{20} & \Gamma_{21} \\ \Gamma_{01} & \Gamma_{00} & \Gamma'_{10} & \Gamma_{10} & \Gamma'_{20} & \Gamma_{21} \\ \Gamma_{10} & \Gamma'_{10} & \Gamma_{00} & \Gamma_{01} & \Gamma_{10} & \Gamma_{11} \\ \Gamma_{11} & \Gamma_{10} & \Gamma_{01} & \Gamma_{00} & \Gamma'_{10} & \Gamma_{10} \\ \Gamma_{20} & \Gamma'_{20} & \Gamma_{10} & \Gamma'_{10} & \Gamma_{00} & \Gamma_{01} \\ \Gamma_{21} & \Gamma_{20} & \Gamma_{11} & \Gamma_{10} & \Gamma_{01} & \Gamma_{00} \end{bmatrix} \begin{bmatrix} b_{00} \\ b_{01} \\ b_{10} \\ b_{11} \\ b_{20} \\ b_{21} \end{bmatrix} = \begin{bmatrix} a_{00} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.3)$$

where

$$\Gamma_{00} = \sum_{i=0}^{2}\sum_{j=0}^{1} a_{ij}^2; \quad \Gamma_{01} = a_{00}a_{01} + a_{10}a_{11} + a_{20}a_{21}$$

$$\Gamma_{10} = a_{00}a_{10} + a_{10}a_{20} + a_{01}a_{11} + a_{11}a_{21};$$

$$\Gamma_{11} = a_{00}a_{11} + a_{10}a_{21}$$

$$\Gamma_{11} = a_{00}a_{11} + a_{10}a_{21}; \quad \Gamma_{20} = a_{00}a_{20} + a_{01}a_{21}$$

$$\Gamma_{21} = a_{00}a_{21}; \quad \Gamma'_{10} = a_{01}a_{10} + a_{11}a_{20}; \quad \Gamma'_{20} = a_{01}a_{20}$$

or more compactly as

$$\Gamma b = a.$$

Here it is observed that if $a_{20} = a_{21} = 0$, then Γ_{20} and Γ_{21} will also become zero, in which case the above matrix equation reduces to the one presented in [5].

The matrix Γ is a block Toeplitz and centrosymmetric matrix. Utilizing centrosymmetry, the above equation yields (2.4).

Denoting the top left 3×3 submatrix by A, and the lower right 3×3 submatrix by C, it can be shown that

$$b_{00} = \frac{1}{2}a_{00}\left[\frac{A_{11}}{\Delta_1} + \frac{C_{11}}{\Delta_2}\right] \quad b_{01} = \frac{1}{2}a_{00}\left[\frac{A_{12}}{\Delta_1} + \frac{C_{12}}{\Delta_2}\right]$$

$$b_{10} = \frac{1}{2}a_{00}\left[\frac{A_{13}}{\Delta_1} + \frac{C_{13}}{\Delta_2}\right] \quad b_{11} = \frac{1}{2}a_{00}\left[\frac{A_{13}}{\Delta_1} + \frac{C_{13}}{\Delta_2}\right]$$

$$b_{20} = \frac{1}{2}a_{00}\left[\frac{A_{12}}{\Delta_1} + \frac{C_{12}}{\Delta_2}\right] \quad b_{21} = \frac{1}{2}a_{00}\left[\frac{A_{11}}{\Delta_1} - \frac{C_{11}}{\Delta_2}\right]$$

where $\Delta_1 = \det A$, $\Delta_2 = \det C$, and A_{11}, A_{12}, A_{13}, and C_{11}, C_{12}, C_{13} are the corresponding cofactors [8, Appendix III].

It can be shown that [8, Appendix IV] $A_{11} > 0\, |A_{11}| > |A_{12}| + |A_{13}|$ and $|C_{11}| > 0; |C_{11}| > |C_{12}| + |C_{13}|$ and also that

$$\Delta_1 > 0; \quad \Delta_2 > 0 \quad (2.6)$$

for a special case[1] when $a_{10} = a_{11} = 0$. In this particular case it has been incidentally observed that the corresponding coefficients in the least square inverse polynomial b_{10} and b_{11} also become zero. This property is generally true as will be proven in the next part.

Now the problem left is to show that the least square inverse of the given polynomial is stable.

Writing (2.2) once again

$$B(z_1, z_2) = b_{00} + b_{01}z_2 + b_{10}z_1 + b_{11}z_1z_2 + b_{20}z_1^2 + b_{21}z_1^2z_2.$$

For stability it is necessary and sufficient to show that i) and ii) below hold [9].

i) $B(z_1, 0) \neq 0$ for $|z_1| \leq 1$.

Let $B_1(z_1, 0) = z_1^2 B(z_1^{-1}, 0)$. For zeros of $B_1(z_1, 0)$ to lie inside unit circle it is necessary and sufficient to show that a), b), and c) are satisfied [9].

a) $B_1(1, 0) = b_{00} + b_{10} + b_{20} > 0$.

Using (2.5)

$$B_1(1, 0) = \frac{1}{2}a_{00}\, \frac{A_{11} + A_{12} + A_{13}}{\Delta_1} + \frac{C_{11} - C_{12} + C_{13}}{\Delta_2}. \quad (2.7)$$

[1] For this special case the proofs in [5] can be readily applied. This will be indicated in the next part.

Using (2.6), it is trivial to show that

$$B_1(1,0) > 0. \tag{2.8}$$

b) To show: $B_1(-1,0) = b_{00} - b_{10} + b_{20} > 0$.

Using (2.5)

$$B_1(-1,0) = \frac{1}{2} a_{00} \left[\frac{A_{11} + A_{12} - A_{13}}{\Delta_1} + \frac{C_{11} - C_{12} - C_{13}}{\Delta_2} \right]. \tag{2.9}$$

From inequalities (2.6) it is again trivial to show that $B(-1, 0) > 0$

c) $b_{00} > b_{20}$.

From (2.5) and (2.6) it is again obvious

$$b_{20} = \frac{1}{2} a_{00} \left[\frac{A_{12}}{\Delta_1} - \frac{C_{12}}{\Delta_2} \right] < \frac{1}{2} a_{00} \left[\frac{A_{11}}{\Delta_1} + \frac{C_{11}}{\Delta_2} \right] = b_{00}.$$

Hence

$$b_{00} > b_{20}.$$

ii) $B(z_1, z_2) \neq 0 \quad |z_1| = 1 \quad |z_2| \leqslant 1$.

To show this one may follow procedure given in [9]. First the Schur-Cohn polynomial which should be negative definite for stability, is constructed. Let

$$S = [(b_{01} + b_{11} z_1 + b_{21} z_1^2)(b_{01} + b_{11} z_1^{-1} + b_{21} z_1^{-2}) - (b_{00} + b_{10} z_1 + b_{20} z_1^2)(b_{00} + b_{10} z_1^{-1} + b_{20} z_1^{-2})] = -f(z_1, z_1^{-1})$$

where it is noted that $f(z_1, z_1^{-1})$ is self inversive. Let

$$g(z_1) = z_1^2 f(z_1, z_1^{-1}) \quad \text{and}$$

$$g_{\text{new}}(z_1) = \left[\frac{dg(z_1)}{dz_1} \right]^*$$

where * represents conjugate polynomial. The polynomial $g_{\text{new}}(z_1)$, has to be tested for positivity on $|z_1| = 1$.

In this particular case

$$\begin{aligned} g_{\text{new}}(z_1) = {} & 4(b_{00} b_{20} - b_{01} b_{21}) \\ & + 3(b_{00} b_{10} + b_{10} b_{20} - b_{01} b_{11} - b_{11} b_{21}) z_1 \\ & + 2(b_{00}^2 + b_{10}^2 + b_{01}^2 - b_{11}^2 - b_{21}^2) z_1^2 \\ & + (b_{00} b_{10} + b_{10} b_{20} - b_{01} b_{11} - b_{11} b_{21}) z_1^3. \end{aligned}$$

As $b_{10} = b_{11} = 0$

$$\begin{aligned} g_{\text{new}}(1) &= g_{\text{new}}(-1) \\ &= 4(b_{00} b_{20} - b_{01} b_{21}) + 2(b_{00}^2 + b_{20}^2 - b_{01}^2 - b_{21}^2) \\ &= 2[(b_{00} + b_{20})^2 - (b_{01} + b_{21})^2] \\ &= 2(b_{00} + b_{20} + b_{01} + b_{21})(b_{00} + b_{20} - b_{01} - b_{21}) \\ &= 4a_{00} \left(\frac{A_{11}}{\Delta_1} + \frac{A_{12}}{\Delta_1} \right) \left(\frac{C_{11}}{\Delta_2} - \frac{C_{12}}{\Delta_2} \right) > 0 \end{aligned}$$

using (2.5) and (2.6). Also

$$\begin{aligned} &(\text{coefficient of } z_1^2) - (\text{coefficient of } z_1^0) \\ &= 2(b_{00}^2 + b_{20}^2 - b_{01}^2 - b_{21}^2) - 4(b_{00} b_{20} - b_{01} - b_{21}) \\ &= 2(b_{00} + b_{20} + b_{01} - b_{21})(b_{00} - b_{20} - b_{01} + b_{21}) \\ &= 4a_{00} \left(\frac{C_{11} + C_{12}}{\Delta_2} \right) \left(\frac{A_{11} - A_{12}}{\Delta_1} \right) > 0 \end{aligned}$$

using (2.5) and (2.6) once again. Hence $g_{\text{new}}(z_1)$ satisfies the required test for positivity. This concludes the proof.

It will be shown in the next part, that the proof of this part can be readily ascertained as a special case of certain low order polynomials. However, the proof exposed here indicates the direction and the difficulty in the proofs for higher order polynomials.

III. Proof of the Shanks' Conjecture for a Class of Polynomials of Higher Degrees in the Two Variables

This section involves two important subsections: First, proof of the positive-definiteness of the Toeplitz Matrix, which arises in the equation, of the approximate inverse of the *same order* of polynomials of higher degrees in two variables; and second, proof of the Conjecture for a class of polynomials of higher degrees in the two variables, utilizing the first subsection.

A. The Proving of Two Facts

Let the polynomial representing the original filter be of the form

$$A(z_1, z_2) \triangleq \sum_{j=0}^{m} \sum_{i=0}^{n} a_{ij} z_1^i z_2^j. \tag{3.1}$$

Let the corresponding PLSI of same order be of the form

$$B(z_1, z_2) \triangleq \sum_{j=0}^{m} \sum_{i=0}^{n} b_{ij} z_1^i z_2^j. \tag{3.2}$$

Let

$$C(z_1, z_2) \triangleq A(z_1, z_2) B(z_1, z_2) = \sum_{j=0}^{2m} \sum_{i=0}^{2n} c_{ij} z_1^i z_2^j \tag{3.3}$$

where

$$\left. \begin{aligned} c_{ij} &= \sum_{q=0}^{m} \sum_{p=0}^{n} a_{pq} b_{i-p, j-q} \quad \forall (i-p) \geqslant 0 \\ &= \sum_{q=0}^{m} \sum_{p=0}^{n} a_{i-p, j-q} b_{pq} \quad \forall (j-q) \geqslant 0 \end{aligned} \right\}. \tag{3.4}$$

Let

$$C \triangleq [c_{00} c_{01} \cdots c_{2n2m}]^T \tag{3.5}$$

$$e^1 = [1 \; 0 \cdots 0]^T. \tag{3.6}$$

The b_{ij}'s should be such that $(e^1 - C)^T (e^1 - C)$ is minimized with respect to b_{ij}. Define $\mathcal{A}_i \in \mathfrak{R}^{(2n+1) \times (m+1)}$ as follows:

$$\mathcal{A}_i = [\tilde{a}_{pq}] \triangleq \begin{bmatrix} a_{i0} & 0 & \cdots & \cdots & 0 \\ a_{i1} & a_{i0} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{im} & a_{i,m-1} & \cdots & \cdots & a_{i0} \\ 0 & a_{im} & \cdots & \cdots & a_{i1} \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & a_{im} \end{bmatrix} \tag{3.7}$$

or formally

$$\tilde{a}_{pq} = \begin{cases} 0, & q > p \\ a_{i,p-q}, & 0 \leqslant p - q \leqslant m \\ 0, & p - q > m. \end{cases}$$

Let

$$\mathcal{B} \triangleq [b_{00} \cdots\cdots b_{mn}]^T. \tag{3.8}$$

Then

$$\mathcal{C} = \mathcal{A}\mathcal{B} \tag{3.9}$$

where

$$\mathcal{A} \triangleq \begin{bmatrix} \mathcal{A}_0 & 0 & \cdots\cdots & 0 \\ \mathcal{A}_1 & \mathcal{A}_0 & \cdots\cdots & 0 \\ & & \ddots & \\ \mathcal{A}_n & \mathcal{A}_{n-1} & \cdots\cdots & \mathcal{A}_0 \\ 0 & \mathcal{A}_n & \cdots\cdots & \mathcal{A}_1 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots\cdots & \mathcal{A}_n \end{bmatrix}. \tag{3.10}$$

Let

$$Q = (e^1 - C)^T (e^1 - C).$$

For minimizing Q with regard to $\mathcal{B}$

$$\frac{dQ}{d\mathcal{B}} = 2(e^1 - C)^T \frac{dC}{d\mathcal{B}} = 0$$

$$= 2(e^1 - C)^T \mathcal{A} = 0$$

$$\Longleftrightarrow C^T\mathcal{A} = (e^1)^T\mathcal{A}$$

$$\Longleftrightarrow \mathcal{B}^T \mathcal{A}^T\mathcal{A} = (e^1)^T\mathcal{A}$$

or

$$\mathcal{A}^T\mathcal{A}\mathcal{B} = \mathcal{A}^T e^1.$$

Let

$$\Gamma \triangleq \mathcal{A}^T\mathcal{A} \tag{3.11}$$

$$\therefore \mathcal{A}^T\mathcal{A}\mathcal{B} = \Gamma\mathcal{B} = \mathcal{A}^T e^1 = \begin{bmatrix} a_{00} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.12}$$

Fact 1: $\mathcal{A}$ is of full rank (i.e., rank of $\mathcal{A} = (n+1)\cdot(m+1)$.

Proof: Assuming $a_{00} \neq 0$, we can construct $\mathcal{A}_i$ as defined in (3.7) and $\mathcal{A}$ as defined in (3.10).

From matrix $\mathcal{A}$, one can always pick a submatrix of dimension $(n+1)\cdot(m+1)$, having a_{00} on the principal diagonal. We observe that it is a lower triangular matrix and hence nonsingular, since $a_{00} \neq 0$.

Similar argument can be applied for any

$$a_{ij} \neq 0 \quad \begin{matrix} 0 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant m. \end{matrix}$$

So the largest nonsingular submatrix of $\mathcal{A}$ has the dimension equal to $(n+1)\cdot(m+1) \Rightarrow \mathcal{A}$ has rank $= (n+1)\cdot(m+1)$.

Fact 2: Γ is positive definite.

Proof: $\Gamma \triangleq \mathcal{A}^T\mathcal{A}$.

For any vector $x \in \mathcal{R}^{(n+1)\cdot(m+1)}$ Such that $x \neq 0$ (the null vector)

$$x^T\Gamma x = x^T \mathcal{A}^T \mathcal{A} x$$

$$= (\mathcal{A}x)^T (\mathcal{A}x) > 0$$

since $\mathcal{A}$ is of full rank.

Hence Γ is positive definite.

The above two facts are valid for any arbitrary polynomial in the two variables and hence are general results. Those might eventually prove the conjecture under certain conditions or may be used to verify the conjecture for a large number of higher degree polynomials. The results are used to prove the facts presented in the next subsection.

B. Proving Two More Facts

Fact 3: For a filter polynomial of the form

$$A_k^2(z_1, z_2) \triangleq a_{00} + a_{01}z_2 + a_{k0}z_1^k + a_{k1}z_2z_1^k. \tag{3.13}$$

The polynomial representing the appropriate inverse is also of the form

$$B_k^2(z_1, z_2) = b_{00} + b_{01}z_2 + b_{k0}z_1^k + b_{k1}z_2z_1^k. \tag{3.14}$$

Proof: For the filter polynomial defined in (3.13), $\mathcal{A}_i$ and $\mathcal{A}$ can be constructed as defined in (3.7) and (3.10) respectively.

$$\therefore \mathcal{A} = \begin{bmatrix} \mathcal{A}_0 & 0 & & & 0 \\ 0 & \mathcal{A}_0 & & & \vdots \\ \vdots & & & & \vdots \\ \mathcal{A}_k & 0 & \cdots\cdots & 0 & \mathcal{A}_0 \\ \vdots & & & & \vdots \\ 0 & \mathcal{A}_k & & & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots\cdots & & & \mathcal{A}_k \end{bmatrix} \tag{3.15}$$

i.e., except for two diagonal strips shown in (3.15), all other blocks are zero.

The block Toeplitz matrix defined in (3.11), is given by

$$\Gamma \triangleq \mathcal{A}^T\mathcal{A} = \begin{bmatrix} \mathcal{A}' & 0 & & 0 & K \\ 0 & \mathcal{A}' & 0 & \cdots\cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ K^T & 0 & \cdots\cdots & & \mathcal{A}' \end{bmatrix} \tag{3.16}$$

where

$$\mathcal{A}' \triangleq \mathcal{A}_0^T \mathcal{A}_0 + \mathcal{A}_k^T \mathcal{A}_k$$
$$\mathcal{K} = \mathcal{A}_k^T \mathcal{A}_0.$$

Let

$$\mathcal{B} \triangleq [b_{00} b_{01} \cdots b_{k0} b_{k1}]^T = \begin{bmatrix} B_0 \\ \hline B_1 \\ \hline \vdots \\ \vdots \\ \hline B_k \end{bmatrix} \tag{3.17}$$

where

$$B_i = \begin{bmatrix} b_{i0} \\ b_{i1} \end{bmatrix} \quad \text{for } i = 0, 1, \cdots k.$$

Equation (3.12) may now be rewritten

$$\Gamma \mathcal{B} = \begin{bmatrix} a_{00} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} D_0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.18}$$

where

$$D_0 = \begin{bmatrix} a_{00} \\ 0 \end{bmatrix}.$$

In view of matrix equations (3.16) and (3.17), (3.18) can be written as

$$\begin{bmatrix} \mathcal{A}' B_0 + \mathcal{K} B_k \\ \hline \mathcal{A}' B_1 \\ \hline \mathcal{K}^T B_0 + \mathcal{A}' B_k \end{bmatrix} = \begin{bmatrix} D_0 \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.19}$$

$$\therefore \mathcal{A}' B_0 + \mathcal{K} B_k = D_0 \tag{3.20}$$
$$\mathcal{A}' B_1 = 0 \tag{3.21}$$
$$\mathcal{A}' B_2 = 0 \tag{3.22}$$
$$\mathcal{A}' B_{k-1} = 0 \tag{3.23}$$
$$\mathcal{K}^T B_0 + \mathcal{A}' B_k = 0. \tag{3.24}$$

Since $\mathcal{A}_0$ and $\mathcal{A}_k$ are of full rank, $\mathcal{A}_0^T \mathcal{A}_0$ and $\mathcal{A}_k^T \mathcal{A}_k$ are positive definite and hence $\mathcal{A}' = \mathcal{A}_0^T \mathcal{A}_0 + \mathcal{A}_k^T \mathcal{A}_k$ is also positive definite and hence nonsingular. So from (3.21) to (3.23)

$$B_1 = 0; \; B_2 = 0 \cdots B_{k-1} = 0. \tag{3.25}$$

This concludes the proof of Fact 3.

Fact 4: $B_k^2(z_1, z_2)$ given in (3.14) represents a stable two-dimensional recursive digital filter.

Proof: Consider

$$A_k^2(z_1^{1/k}, z_2) = a_{00} + a_{01} z_2 + a_{k0} z_1 + a_{k1} z_1 z_2 \triangleq \bar{A}_k^2(z_1, z_2). \tag{3.26}$$

Let $\bar{B}_k^2(z_1, z_2)$ represent PLSI of $\bar{A}_k^2(z_1, z_2)$.

$$\therefore \bar{C}_k^2(z_1, z_2) \triangleq \bar{A}_k^2(z_1, z_2) \bar{B}_k^2(z_1, z_2) \triangleq C_k^2(z_1^{1/k}, z_2) \tag{3.27}$$

$\therefore B_k^2(z_1, z_2)$ is PLSI of $A_k^2(z_1, z_2)$

$\Longleftrightarrow (1 - c_{k\,00}^2)^2 + \Sigma (c_{k\,ij}^2)^2$ is minimized with regard to $b_{k\,ij}^2$

$\Longleftrightarrow (1 - c_{k\,00}^{-2})^2 + \Sigma (c_{k\,ij}^{-2})^2$ is minimized with respect to $b_{k\,ij}^{-2}$

$\Longleftrightarrow \bar{B}_k^2(z_1, z_2)$ is PLSI of $\bar{A}_k^2(z_1, z_2)$

$\bar{B}_k^2(z_1, z_2)$ is known to be stable (see [5])

$\therefore \bar{B}_k^2(z_1, z_2) \neq 0$ inside unit bidisc

$\Longleftrightarrow B_k^2(z_1^{1/k}, z_2) \neq 0$ inside unit bidisc

$\Longleftrightarrow B_k^2(z_1, z_2) \neq 0$ inside unit bidisc.

This concludes the proof of Fact 4.

Remark 1: As an immediate implication, the work presented in Section II can be seen to be a special case of Facts 3 and 4, when $k = 2$. This is also discussed in [8]. But the proof presented in Section II is an independent one, which indicates the general approach to the problem, if one is interested in proving the conjecture in general.

Remark 2: If the polynomial representing the filter is of the form

$$A_k^1(z_1, z_2) \triangleq a_{00} + a_{10} z_1 + a_{0k} z_2^k + a_{1k} z_1 z_2^k \tag{3.28}$$

then the corresponding PLSI of same order is also of the form

$$B_k^1(z_1, z_2) = b_{00} + b_{10} z_1 + b_{0k} z_2^k + b_{1k} z_1 z_2^k \tag{3.29}$$

and represents a stable recursive two-dimensional digital filter.

By interchanging variables z_1 and z_2 in Facts 3 and 4, the above observation is seen to be trivial.

Remark 3: All those polynomials $A(z_1, z_2)$ of arbitrary degrees in the two variables, which belong to the class defined by the following equation, have stable PLSI of the same order and hence can be stabilized by taking double PLSI, as conjectured. Here $A(z_1, z_2)$ [as defined in (3.1)], should be such that

$$\frac{1}{A(z_1, z_2)} = \left(\sum_{k=1}^{p} \frac{R_k}{A_k^1(z_1, z_2)} \right) + \left(\sum_{l=2}^{q} \frac{S_l}{A_l^2(z_1, z_2)} \right)$$

where: (i) R_k and S_l are constants;

(ii) $\dfrac{p(p+1)}{2} = m; \quad \dfrac{q(q-1)}{2} = n;$

(iii) $A_k^1(z_1, z_2) \triangleq a_{00}^1 + a_{10}^1 z_1 + a_{0k}^1 z_2^k + a_{1k}^1 z_1 z_2^k$; and

(iv) $A_l^2(z_1, z_2) \triangleq a_{00}^2 + a_{01}^2 z_2 + a_{k0}^2 z_1^k + a_{k1}^2 z_1^k z_2.$ (3.30)

Proof: Since $A(z_1, z_2)$ belongs to the class defined by (3.30), Facts 3 and 4 together with Remark 2 guarantee a stable PLSI for each of the polynomials $A_k^1(z_1, z_2)$ and $A_l^2(z_1, z_2)$ and hence each one of them can be replaced by its stable double PLSI $\tilde{A}_k^1(z_1, z_2)$ and $\tilde{A}_l^2(z_1, z_2)$ respectively. Hence, $A(z_1, z_2)$, the stable version of $\tilde{A}(z_1, z_2)$ can be constructed as defined in (3.30) as follows:

$$\frac{1}{\tilde{A}(z_1, z_2)} = \left(\sum_{k=1}^{p} \frac{R_k}{\tilde{A}_k^1(z_1, z_2)}\right) + \left(\sum_{l=2}^{q} \frac{S_l}{\tilde{A}^2(z_1, z_2)}\right)$$

which represents the DPLSI of the original filter.

Remark 4: Physically, Remark 3 implies that all those two-dimensional recursive digital filters that can be split up into a parallel combination of filters having polynomials of the form defined in (3.30), can be stabilized by replacing each one of the parallel filters by its DPLSI, as conjectured.

Remark 5: Testing of reducibility of polynomials $A(z_1, z_2)$ into the form defined in (3.30) is cumbersome. But the guidelines for testing and factorization, based on the following theorems, are briefly presented here.

Theorem 1: The necessary condition for the polynomial to be reducible is that its form (i.e., the set of variables $\{1, z_1, z_2, z_1, z_2, \cdots, z_1^k z_2^l, \cdots, z_1^n z_2^m\}$ is reducible [10].

Theorem 2: A polynomial in two variables which is not identically zero can be resolved into the product of irreducible factors no one of which is constant in one, and essentially in only one, way.

Theorem 3: If two polynomials ψ and ϕ are relatively prime, there are only a finite number of pairs of values of (z_1, z_2) for which ψ and ϕ both vanish [11].

The proofs of the above theorems are omitted here for brevity. Interested reader may refer the references cited.

The outline of the algorithm for testing and factorizations in the form of (3.30) is as follows.

1) If the condition in Theorem 1 is satisfied by the polynomial $A(z_1, z_2)$, its reducibility can be easily checked by inspection, in many cases.

2) Once $A(z_1, z_2)$ is known to be reducible, the irreducible factors of $A(z_1, z_2)$, (which are guaranteed by Theorem 2) can be found.

3) The factors may be rearranged, as far as possible, to conform to the conditions in (3.30).

4) In general it may not always be feasible to write $1/A(z_1, z_2)$ in the form mentioned in (3.30). In this situation one can always replace it by its least square error approximate of that form.

The above algorithm can be directly utilized, in simple cases, to test and factorize $A(z_1, z_2)$ into the form mentioned in (3.30) and in more complex cases, a computer program may be developed using the guidelines mentioned above.

IV. Conclusions

In the present work, Shanks' conjecture has been verified for a class of filter polynomials of higher degrees in the two variables. But the testing and factorization of the polynomial into the form defined in (3.30) is very difficult, as mentioned in Remark 5 of Section III. An independent general approach is presented in Section II for the verification of the conjecture for a class of polynomials that are linear in one variable and quadratic in the other, which also brings into light the difficulties involved in proving the conjecture for the general case. Bednar also showed the inherent difficulties in proving the conjecture in general [12].

It is pertinent to mention the following future work on this problem.

1) The proof given in Section II suggests an approach that might be used to verify the conjecture for a wider class of polynomials than those considered in the present work.

2) Results of Section III suggest that, with some more knowledge about separability of filters [i.e., the conditions under which parallel decomposition defined in (3.30) is possible], the conjecture can be verified to a much wider class of polynomials.

3) It will be worthwhile to study the changes in the present work, required when considering PLSI of lower order and higher order rather than of same order, so that the conditions under which the conjecture fails and the conditions under which the conjecture is valid may hopefully become clear.

4) It will be beneficial to study the relationships between Toeplitz matrices and tridiagonal matrices etc., which arise in this approach as well as that of Genin–Kamp which will lead hopefully, to the solution of the problem of proving under certain conditions, the conjecture in general.

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