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Linear-Quadratic Discrete-Time Control and Constant Directions*

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Constant directions can be defined for the most general form of linear-quadratic minimization problem. Many characterizations of them can be given, including algorithms for their construction.

Key Word Index—Linear systems; optimal control; discrete time systems; (Riccati equations).

Summary—The general linear-quadratic discrete-time minimization problem is studied, in which no restrictions are placed on the singularity or otherwise of certain matrices, or the appearance of cross-product terms in the performance indices. Constant directions are characterized in a number of ways, and an algorithm is presented for replacing a prescribed problem with constant directions by one of lower state and/or control dimension, lacking constant directions. The replacement is achieved using a series of state and input co-ordinate basis changes, and allows simplification of the calculations obtaining the optimal control law and performance index of the original problem.

1. INTRODUCTION

IN CONTINUOUS-time, the general linear-quadratic control problem is called singular if the weighting matrix of the controls in the integrand of the cost is singular anywhere within the time interval of interest. In such cases, the Riccati equation naturally associated with this problem is not well defined, so an alternative approach to solving the problem is required.

In [1] it is shown that a singular control problem can be solved in terms of another control problem, again of a linear-quadratic nature and possibly singular, but of lower state and/or lower control space dimension. This reduction procedure is continued until one of three possible terminating problems is obtained: a nonsingular problem, or one with zero state dimension, or one with zero control dimension. Each of these terminating problems can be solved in a straightforward manner; the solution of the originally given control problem can then be simply constructed.

Moreover, in carrying out any state dimension reduction, it becomes evident that part of the optimal cost depends on the coefficient matrices of the problem, excluding the terminal weighting matrix, in an identifiable way.

In contrast, it is well known that for any discrete-time linear-quadratic optimal control problem, regardless of the singularity or otherwise of any matrices in the cost, the associated Riccati equation is well defined and may be solved in a straightforward manner to yield a solution of the optimal control problem. As such, the concept of singularity is apparently meaningless for discrete-time problems.

As mentioned above, one possible, though not necessary, consequence of singularity in continuous-time is the identification of blocks of the matrix defining the performance index from knowledge of the coefficient matrices only. It is this idea that has been studied in the context of the discrete-time linear-quadratic control problem and the associated Riccati equation in [2], [3]. There the concept of a constant direction is introduced, being a direction in which the solution of the Riccati equation is, first, constant (except for a finite transient period), and second, independent of the terminal weighting matrix. In [2], the multi-input constant coefficient case is considered for a restricted class of coefficient matrices, where the nature of the restriction is described in Section III. There, the approach is to identify all constant directions of all orders and then to reduce the dynamic order of the Riccati equation away from the final transient period. Also introduced in [2] is the idea of a state being taken to zero optimally on an interval as a way of characterizing a constant direction. In this paper, this idea also proves to be useful.

Work related to that in [2] was reported in [4] where the dual of the control problem, that of covariance factorization, is studied. These authors considered only scalar covariances, which corresponds to assuming a scalar input in the control problem formulation. Nonstationary covariances are studied, and to handle the time-varying case, the idea of a constant direction is extended and called a degenerate direction.

The contributions of this paper are two-fold. Primarily, we study in detail the question of characterizing all constant directions for an arbitrary discrete-time linear-quadratic control problem.

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This generalizes corresponding results in [2] though our methods are somewhat different. Secondarily, we obtain results analogous to those for the general continuous-time singular linear-quadratic control problem. In particular, from the coefficient matrices we construct a matrix which plays the role of the control weighting matrix in continuous-time. That is, singularity of this matrix allows the originally given problem to be solved in terms of the solution of another control problem but with lower state and/or lower control dimension. Moreover, a reduction procedure which terminates in one of three control problems can be defined. Each of these terminating problems, a nonsingular problem, a zero state dimension problem and a zero control dimension problem can be solved simply; the solution to the originally given problem can be computed by tracing back through the reduction procedure.

An outline of the paper is as follows. Section II reviews the general discrete-time control problem. In Section III, we characterize j-constant directions completely in terms of being taken to zero optimally in j steps for some terminal weighting matrix and as vectors in the range of a certain matrix. In Section IV, state dimension reduction is demonstrated provided 1-constant directions exist. The results of the preceding sections are combined in Section V to derive an algorithm which solves any singular discrete-time control problem, making use of computational simplifications arising from constant direction existence. Section VI contains brief concluding remarks, including mention of connections with the Silverman structure algorithm [5, 6] and the recently developed 'fast' methods for Riccati equation solution of Kailath and his coworkers [7].

II. REVIEW OF LINEAR-QUADRATIC CONTROL

On the interval $[K, N]$ consider the dynamic system

$$x(i+1)=Ax(i)+Bu(i),\quad i=K,\dots,N-1 \quad (2.1)$$
$$x(K)=x_K$$

where $x(i)\in R^n$, $u(i)\in R^m$. Let U_K^{N-1} be a control sequence

$$[u(K),u(K+1),\dots,u(N-1)]$$

and assign to the initial state x_K, control sequence U_K^{N-1} and terminal weighting matrix S, the cost functional

$$V_{N-K}[x_K,U_K^{N-1},S]=x'(N)Sx(N)$$
$$+\sum_{i=K}^{N-1}\{x'(i)Qx(i)+2x'(i)Cu(i)+u'(i)Ru(i)\} \quad (2.2)$$

where $[x(K),x(K+1),\dots,x(N)]$ is the trajectory of (2.1) generated by U_K^{N-1} and x_K. The matrices Q, C, R and S are of appropriate dimension, with Q, R and S symmetric. (No definiteness assumptions are made.) Also define

$$V_{N-K}^*[x_K,S]=\inf_{U_K^{N-1}} V_{N-K}[x_K,U_K^{N-1},S] \quad (2.3)$$

The discrete-time linear-quadratic control problem is stated in two parts.

1. Find necessary and sufficient conditions for $V_{N-K}[x_K,U_K^{N-1},S]$ to be bounded below, independently of U_K^{N-1}, for each x_K and each $K=0,1,\dots,N-1$.
2. If these conditions hold, determine $V_{N-K}^*[x_K,S]$ and, if it exists, a control sequence U_K^{*N-1} depending on x_K such that

$$V_{N-K}^*[x_K,S]=V_{N-K}[x_K,U_K^{*N-1},S],$$
$$K=0,1,\dots,N-1.$$

As a matter of terminology, we shall say that the control problem has a solution on $[0,N]$ for terminal weighting matrix S if 1 and 2 are solvable. The well-known solution to this problem is summarised in Theorem 2.1. The notations $\mathcal{N}(X)$, $X^{\#}$, $X\geqq 0(>0)$ denote respectively null space, Moore-Penrose pseudo-inverse, nonnegative (positive) definiteness of the matrix X.

Theorem 2.1. The control problem has a solution on $[0,N]$ if and only if the symmetric matrices $P(i+1)$, $i=N-1,N-2,\dots,0$ satisfy

$$B'P(i+1)B+R\geqq 0 \qquad i=0,\dots,N-1 \quad (2.4)$$
$$\mathcal{N}(B'P(i+1)B+R)\subseteq\mathcal{N}(A'P(i+1)B+C)$$

with $P(i)$, $i=N-1,\dots,0$ defined recursively by the Riccati equation

$$P(i)=A'P(i+1)A+Q$$
$$-[A'P(i+1)B+C][B'P(i+1)B+R]^{\#}$$
$$\times[A'P(i+1)B+C]',$$
$$i=0,\dots,N-1 \quad (2.5)$$
$$P(N)=S.$$

If $P(i)$ is so defined, then the control sequence U_K^{*N-1} defined by

$$u^*(i)=-[B'P(i+1)B+R]^{\#}$$
$$\times[A'P(i+1)B+C]'x(i)$$
$$i=K,\dots,N-1 \quad (2.6)$$

achieves the infimum for each $K=0,\ldots,N-1$. That is,

$$V^*_{N-K}[x_K,S]=V_{N-K}[x_K,U_K^{*,N-1},S]=x_K'P(K)x_K. \quad (2.7)$$

The necessary and sufficient conditions (2.4) which $P(i+1)$, $i=0,\ldots,N-1$ must satisfy motivate the following definition.

Definition 2.1. The set of allowable weighting matrices, denoted by $\mathscr{S}$, is the set of $n\times n$ symmetric matrices S such that $B'SB+R\geq 0$ and $\mathscr{N}(B'SB+R)\subseteq\mathscr{N}(A'SB+C)$.

Remark 2.1

(a) When we wish to make the dependence of $P(i)$ on the terminal weighting matrix $S\in\mathscr{S}$ explicit, we shall write $P(i,S)$ for $P(i)$.

(b) The set $\mathscr{S}$ is open upwards in the sense that $S_0\in\mathscr{S}$ and $S\geq S_0$ implies $S\in\mathscr{S}$, as the reader can easily verify.

(c) Should $P(i+1)\notin\mathscr{S}$ for some $i=0,\ldots,N-1$, then $V^*_{N-i}[x_i,S]=-\infty$ for at least one x_i. If $P(i+1)\in\mathscr{S}$, then the solution $P(i)$ of the Riccati equation (2.5) may also be characterized as the maximal symmetric solution of

$$\begin{bmatrix} A'P(i+1)A+Q-P(i) & A'P(i+1)B+C \\ B'P(i+1)A+C' & B'P(i+1)B+R \end{bmatrix}\geq 0. \quad (2.8)$$

This fact will be used in the development of the state-space reduction procedure in Section IV.

The preceding discussion is the standard approach for solving the linear-quadratic control problem in that $P(N-i,S)$ is determined by separate minimizations over each of the controls $u(N-1)$, $u(N-2)$, …, $u(N-i)$ respectively. However, it is possible to calculate $P(N-i,S)$ directly by minimizing with respect to an extended control vector

$$u'_{(i)}(N-i)\triangleq[u'(N-i)\ldots u'(N-1)]$$

provided we suitably define the dynamics and the cost.

In particular consider $i=2$ and the interval $[N-2,N]$. From (2.1) and (2.2) for $K=N-2$, we can write the dynamics as

$$x(N)=A_{(2)}x(N-2)+B_{(2)}u_{(2)}(N-2) \quad (2.9)$$

where

$$A_{(2)}\triangleq A^2 \quad\text{and}\quad B_{(2)}\triangleq[AB\ \ B]$$

and the cost as

$$\begin{aligned}V_{(2)}[x(N-2),U_{N-2}^{N-1},S]&=x'(N)Sx(N)\\&+\{x'(N-2)Q_{(2)}x(N-2)\\&+2x'(N-2)C_{(2)}u_{(2)}(N-2)\\&+u'_{(2)}(N-2)R_{(2)}u_{(2)}(N-2)\}\end{aligned} \quad (2.10)$$

where

$$Q_{(2)}\triangleq A'QA+Q \qquad C_{(2)}\triangleq[A'QB+C\ \ A'C]$$

$$R_{(2)}\triangleq\begin{bmatrix} B'QB+R & B'C \\ C'B & R\end{bmatrix}.$$

Consequently, if the control problem associated with (2.1) and (2.2) has a solution on $[0,N]$ for terminal weighting S then $P(N-2,S)$ is given by

$$\begin{aligned}P(N-2,S)&=A'_{(2)}SA_{(2)}+Q_{(2)}\\&-[A'_{(2)}SB_{(2)}+C_{(2)}]\\&\times[B'_{(2)}SB_{(2)}+R_{(2)}]^{\#}[A'_{(2)}SB_{(2)}+C_{(2)}]'.\end{aligned} \quad (2.11)$$

For convenience, we call the above the 2nd stage control problem, with terminal weighting S.

In general, suppose the j^{th} stage control problem with terminal weighting S has been defined in terms of quantities $A_{(j)}$, $B_{(j)}$, $Q_{(j)}$, $C_{(j)}$, $R_{(j)}$ and $u_{(j)}(i)$; then the $(j+1)^{\text{th}}$ stage control problem with terminal weighting S is defined in terms of the quantities

$$\begin{aligned}A_{(j+1)}&\triangleq A_{(j)}A\\ B_{(j+1)}&\triangleq[A_{(j)}B\ \ B_{(j)}]\\ Q_{(j+1)}&\triangleq A'Q_{(j)}A+Q\\ C_{(j+1)}&\triangleq[A'Q_{(j)}B+C\ \ A'C_{(j)}]\\ R_{(j+1)}&\triangleq\begin{bmatrix} B'Q_{(j)}B+R & B'C_{(j)} \\ C'_{(j)}B & R_{(j)}\end{bmatrix}\\ u'_{(j+1)}(i)&\triangleq[u'_{(j)}(i)\ \ u'(i+j)].\end{aligned} \quad (2.12)$$

Now it is clear that if the control problem associated with (2.1) and (2.2) has a solution on $[0,N]$ then the j^{th} stage control problem with terminal weighting $P(i+j,S)$ defined by

$$x(i+j)=A_{(j)}x(i)+B_{(j)}u_{(j)}(i) \quad (2.13)$$

and

$$\begin{aligned}V_{(j)}[x(i),u_{(j)}(i),P(i+j,S)]&\\=x'(i+j)P(i+j,S)x(i+j)&\\+\{x'(i)Q_{(j)}x(i)+2x'(i)C_{(j)}u_{(j)}(i)&\\+u'_{(j)}(i)R_{(j)}u_{(j)}(i)\}&\end{aligned} \quad (2.14)$$

has a solution for all $[i, i+j]\subset[0,N]$ and conversely. Moreover,

$$\begin{aligned}P(i,S)=&A'_{(j)}P(i+j,S)A_{(j)}+Q_{(j)}\\&-[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}]\\&\times[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}]^{\#}\\&\times[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}]'\end{aligned}\tag{2.15}$$

with an optimal control $u^*_{(j)}(i)$ obviously equivalent to the optimal sequence U_i^{*i+j-1}. Moreover, existence of the optimal control implies the following generalization of (2.4):

$$\mathcal{N}[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}]\subset\mathcal{N}[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}].\tag{2.16}$$

Existence also implies that

$$B'_{(j)}P(i+j,S)B_j(S)+R_{(j)}\geqq 0,$$

but we shall make more use of (2.16).

By setting $i+j=N$, (2.15) becomes an equation for $P(N-j,S)$. Of course, calculation of $P(N-j,S)$ using the j^{th} stage formulation is highly inefficient. Nevertheless, it is of theoretical interest, in that, as will later be evident, the 1-constant directions of the j^{th} stage control problem are identical to the j-constant directions of the original control problem. (See the next section for a definition of j-constant directions.)

III. CONSTANT DIRECTIONS—BASIC PROPERTIES

As in [2], we formalize the notion of a constant direction.

Definition 3.1. Suppose $1\leqq j\leqq N-1$. The n-vector α is called a j-constant direction of (2.5) on $[0,N]$ if and only if $P(i,S)\alpha$ is independent of those S in $\mathcal{S}$ for which a solution to the optimal control problem exists and of i for $i=0,1,\ldots,N-j$.

Denote the set of j-constant directions by I_j. Clearly, each I_j is a finite-dimensional linear space and is therefore completely described by a finite set of linearly independent vectors. Moreover, for $1\leqq j\leqq N-2$, it is immediate that $I_{j+1}\supseteq I_j$.

It is of interest to have available various alternative characterizations of j-constant directions. One such characterization involves the notion of a state being taken to zero optimally in j steps, which we now define precisely.

For each j with $1\leqq j\leqq N-1$, consider the control problem with dynamics and cost as in (2.1) and (2.2), restricted to the interval $[N-j,N]$, with coefficient matrices A, B, Q, C and R and terminal weighting matrix $\bar{S}$. Suppose that this control problem has initial state $x(N-j)=\alpha$.

Definition 3.2[2]. We say that α can be taken to zero optimally in j steps for $S=\bar{S}$ if and only if there exists a control sequence U_{N-j}^{N-1} such that the corresponding trajectory of (2.1) satisfies $x(N-j)=\alpha$ and $x(N)=0$, and

$$V_j[\alpha,U_{N-j}^{N-1},\bar{S}]=V_j^*[\alpha,\bar{S}].$$

(An equivalent definition in terms of the j^{th} stage control problem of (2.13) and (2.14) is obviously possible).

In [2], it is shown that the j-constant directions are completely characterized by the property of being taken to zero optimally in j steps for zero terminal weighting matrix when $C=0$, $R=0$, $Q\geqq 0$, $S\geqq 0$ and A is invertible. Moreover, the 1-constant directions can not only be characterized in this form but also simply as the range of the matrix $A^{-1}B$, while the j-constant directions are related to the null space of a matrix W_j (defined in [2]). The remainder of this section is concerned with the extensions of these results to control problems with coefficient matrices which are arbitrary save that a solution to the control problem exists on $[0,N]$ for some terminal weighting matrix $S\in\mathcal{S}$. Theorem 3.1 below contains the main result; it is preceded by several lemmas.

Define the $(n+jm)\times(n+jm)$ dimensional matrix

$$\Lambda_{(j)}=\begin{bmatrix}A_{(j)} & B_{(j)}\\ C'_{(j)} & R_{(j)}\end{bmatrix}\tag{3.1}$$

Lemma 3.1. Suppose $w_{(j)}\in\mathcal{N}(\Lambda_{(j)})$ and partition $w'_{(j)}$ as $[\alpha'\ \beta'_{(j)}]$ where α is n-dimensional and $\beta^{(j)}$ is jm-dimensional. Then if the control problem has a solution on $[0,N]$ for terminal weighting matrix S, we have

$$P(i,S)\alpha=Q_{(j)}\alpha+C_{(j)}\beta_{(j)}\quad\text{for each}\quad i=0,1,\ldots,N-j.\tag{3.2}$$

Proof. Since $w_{(j)}\in\mathcal{N}(\Lambda_{(j)})$ we have

$$A_{(j)}\alpha+B_{(j)}\beta_{(j)}=0\quad\text{and}\quad C'_{(j)}\alpha+R_{(j)}\beta_{(j)}=0.\tag{3.3}$$

Postmultiply (2.15) by α, and use (3.3) to obtain

$$\begin{aligned}P(i,S)\alpha=&A'_{(j)}P(i+j,S)A_{(j)}\alpha+Q_{(j)}\alpha\\&-[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}]\\&\times[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}]^{\#}\\&\times[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}]'\alpha\\=&-A'_{(j)}P(i+j,S)B_{(j)}\beta_{(j)}+Q_{(j)}\alpha\\&+[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}]\\&\times[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}]^{\#}\\&\times[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}]\beta_{(j)}.\end{aligned}$$

By hypothesis, the control problem has a solution for terminal weighting S. Therefore conditions (2.16) hold for each $[i,i+j]\subset[0,N]$ and so consequently does the identity

$$\begin{aligned}A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}\\=[A'_{(j)}P(i+j,S)B_{(j)}+C_{(j)}]\\\times[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}]^{\#}\\\times[B'_{(j)}P(i+j,S)B_{(j)}+R_{(j)}].\end{aligned}\quad(3.4)$$

Substituting (3.4) in the above we obtain (3.2).

Remark 3.1. It is possible that there could exist two different vectors

$$w_{(j)}=[\alpha'\quad\beta'_{(j)}]'\quad\text{and}\quad\bar{w}_{(j)}=[\alpha'\quad\bar{B}'_{(j)}]'$$

both in $\mathcal{N}(\Lambda_{(j)})$. Then $\beta_{(j)}-\bar{\beta}_{(j)}\in\mathcal{N}(B'_{(j)}\ R'_{(j)}]')$; using this fact and (2.16), it follows that also $\beta_{(j)}-\bar{\beta}_{(j)}\in\mathcal{N}(C_{(j)})$. In such a circumstance, there are in a sense superfluous controls. Consider the case $j=1$. Then we can reduce the dimension of the control space by the nullity p of $[B'\ R]'$. For if $[B'\ R]'$ is not of full column rank, we can change the control space basis so that

$$\begin{bmatrix}B\\R\end{bmatrix}=\begin{bmatrix}B_1&\overbrace{0}^{p}\\R_{11}&0\\0&0\end{bmatrix}$$

where $[B'_2\ R'_{11}]'$ has full column rank; and since

$$\mathcal{N}(B'SB+R)\subseteq\mathcal{N}(A'SB+C)$$

we can conclude that C has the form $[C_1\ 0]$. Consequently, the final p components of the control play no part in the control problem.

If the control problem has a solution on $[0,N]$ for terminal weighting S_0 it also has a solution on $[0,N]$ for each terminal weighting S satisfying $S\geqq S_0$. Thus, from Lemma 3.1 and under the hypothesis of this lemma, we can assert that

$$V_k^*[\alpha,S]=V_k^*[\alpha,S_0]$$

for each $S\geqq S_0$ and any $k=j,\ldots,N$. The next lemma explores the consequences of such an equality of performance indices for a state α and some j.

Lemma 3.2. Suppose that $V_j^*[\alpha,S]=V_j^*[\alpha,S_0]$ for a state α and all $S\geqq S_0$, for some $S_0\in\mathcal{S}$. Then for $\bar{S}>S_0$, with the overbar indicating the terminal weighting matrix $\bar{S}$ is being considered, we have

(a) $\bar{x}^*(N)=0$ for *any* optimal control sequence $\bar{U}^{*N-1}_{N-j}$ associated with $\bar{S}$,

(b) $V_j^*[\alpha,S]=V_j[\alpha,\bar{U}^{*N-1}_{N-j},S]=V_j^*[\alpha,S_0]$ for all $S\geqq S_0$.

Proof. By hypothesis,

$$V_j^*[\alpha,\bar{S}]=V_j^*[\alpha,S_0]\leqq V_j[\alpha,U^{N-1}_{N-j},S_0]$$

for any control sequence U^{N-1}_{N-j}, and therefore for any optimizing control sequence $\bar{U}^{*N-1}_{N-j}$ associated with terminal weighting $\bar{S}$. Now,

$$V_j^*[\alpha,\bar{S}]=V_j[\alpha,\bar{U}^{*N-1}_{N-j},S_0]+\bar{x}^{*\prime}(N)(\bar{S}-S_0)\bar{x}^*(N).$$

Consequently, $\bar{x}^*(N)=0$ since $\bar{S}>S_0$. This proves (a). Part (b) follows trivially.

Parts (a) and (b) of the above lemma yield:

Corollary 3.1. Suppose that $V_j^*[\alpha,S]=V_j^*[\alpha,S_0]$ for a state α and all $S\geqq S_0$ for some $S_0\in\mathcal{S}$. Then α can be taken to zero optimally in j steps for all $S\geqq S_0$.

It is important to note that in Corollary 3.1 the minimum norm optimal control is not guaranteed to take α to zero optimally in j steps for $S\geqq S_0$ but $S\ngtr S_0$. This does not invalidate the result of Corollary 3.1, however, because Definition 3.2 only requires *some* optimal control to take α to zero in j steps, not necessarily the minimum norm control.

By way of a converse to this corollary, we have the following:

Lemma 3.3. Suppose that α can be taken to zero optimally in j steps for some $S_0\in\mathcal{S}$. Then α can be taken to zero optimally in j steps for all $S\geqq S_0$ and

$$V_j^*[\alpha,S]=V_j^*[\alpha,S_0].$$

Proof. It is easily checked that the control taking α to zero optimally for $S_0\in\mathcal{S}$ also is optimum for all $S\geqq S_0$.

We have shown that the null vectors of $\Lambda_{(j)}$ determine states which can be taken to zero optimally. Let us now check the reverse of this idea.

Lemma 3.4. Suppose that α can be taken to zero optimally in j steps for all $S\geqq S_0$ where S_0 is some element of $\mathcal{S}$. Then there exists a vector $\beta_{(j)}$ such that $w_{(j)}\in\mathcal{N}(\Lambda_{(j)})$ for

$$w'_{(j)}=[\alpha'\quad\beta'_{(j)}].$$

Proof. With terminal weighting matrix $\bar{S}>S_0$, Lemma 3.2 implies that α can be taken to zero optimally in j steps with *any* optimizing control U^{*N-1}_{N-j} or equivalently $u^*_{(j)}(N-j)$. In particular, one such optimizing control is the minimum norm control

$$\bar{u}^*_{(j)}(N-j)=-[B'_{(j)}\bar{S}B_{(j)}+R_{(j)}]^{\#}[A'_{(j)}\bar{S}B_{(j)}+C_{(j)}]'\alpha\quad(3.5)$$

and therefore since $\bar{x}^*(N)=0$ we have

$$A_{(j)}\alpha - B_{(j)}[B'_{(j)}\bar{S}B_{(j)}+R_{(j)}]^{\#}[A'_{(j)}\bar{S}B_{(j)}+C_{(j)}]'\alpha=0. \tag{3.6}$$

Now, $\bar{S}\in\mathscr{S}$ and so the identity (3.4) holds for $P(i+j, S)=\bar{S}$, which together with (3.6) implies

$$C'_{(j)}\alpha - R_{(j)}[B'_{(j)}\bar{S}B_{(j)}+R_{(j)}]^{\#}[A'_{(j)}\bar{S}B_{(j)}+C_{(j)}]'\alpha=0. \tag{3.7}$$

Hence, with $\beta_{(j)}=\bar{u}^*_{(j)}(N-j)$, (3.6) and (3.7) become

$$A_{(j)}\alpha + B_{(j)}\beta_{(j)}=0$$

and

$$C'_{(j)}\alpha + R_{(j)}\beta_{(j)}=0$$

which completes the proof of the lemma.

We can summarize the preceding results as follows:

Theorem 3.1. Suppose that the solution to the control problem exists on $[0, N]$ for some terminal weighting S. Then the following statements are equivalent.

(a) α is a j-constant direction of (2.5) on $[0, N]$.

(b) α can be taken to zero optimally in j steps for all $S\in\mathscr{S}$.

(c) There exists $w_{(j)}\in\mathscr{N}(\Lambda_{(j)})$ with $w'_{(j)}=[\alpha'\ \beta'_{(j)}]$.

(d) [restricted form of (a)]. For some $S_0\in\mathscr{S}$ and all $S\geq S_0$, $V_j^*[\alpha, S]=V_j^*[\alpha, S_0]$.

(e) [restricted form of (b)]. α can be taken to zero optimally in j steps for some $S_0\in\mathscr{S}$.

Moreover, should any of the above hold, *any* optimal control associated with an $S\in\mathscr{S}$ such that $S-\eta I\in\mathscr{S}$ for some $\eta>0$ takes α to zero.

Proof. The implications (a)$\Rightarrow$(b), (e)$\Rightarrow$(d)$\Rightarrow$(c)$\Rightarrow$(a) follow from Corollary 3.1, Lemma 3.3, Lemma 3.4 with Corollary 3.1, and Lemma 3.1 respectively. Finally, (b)$\Rightarrow$(e) is trivial. The final part of the theorem is a consequence of Lemma 3.2. This completes the proof.

We also have the following simple consequence of parts (a) and (c) of Theorem 3.1 and Remark 3.1.

Theorem 3.2. Suppose the control problem has a solution on $[0, N]$ for some terminal weighting S. Then the space of j-constant directions I_j equals the range of W_{j1} where

$$W_j=[W'_{j1}\quad W'_{j2}]'$$

is a basis matrix for $\mathscr{N}(\Lambda_{(j)})$. Moreover, the dimension of I_j equals s_j-p_j where s_j is the nullity of $\Lambda_{(j)}$ and p_j is the nullity of $[B'_{(j)}\ R'_{(j)}]'$.

In this section, we have not singled out 1-constant directions for special attention. This we shall do in the next section since it is the 1-constant directions that are most easily found [$\mathscr{N}(\Lambda_{(1)})$ is easier to compute that $\mathscr{N}(\Lambda_{(j)})$], and, as it turns out, j-constant directions can be found by computing 1-constant directions for a collection of problems.

IV. STATE SPACE DIMENSION REDUCTION

In this section, we assume that the matrix $[B'\ R]'$ has full column rank or, equivalently, all superfluous controls have been eliminated. Then we know that the dimension of I_1 equals the nullity of $\Lambda=\Lambda_{(1)}$; let this number be l. We will now show that if there are nontrivial 1-constant directions, i.e. $l>0$, then the state space dimension can be reduced by l, and a related control problem can be defined on the interval $[0, N-1]$ rather than $[0, N]$. Moreover, the solution of the Riccati equation and the optimal controls on $[0, N]$ are simply related to those for the reduced state dimension problem on $[0, N-1]$.

The first stage in the procedure is to choose a basis of the state space to display the constant parts of the matrices $P(i, S)$, $i=0,\ldots,N-1$. Choose as a basis of the state space $\{\alpha_1,\ldots,\alpha_n\}$ arbitrarily, save that $\{\alpha_{n-l+1},\ldots\alpha_n\}$ spans I_1. With this state space basis, we have

$$P(i, S)=\begin{bmatrix} P_{11}(i, S) & P_{12} \\ P'_{12} & P_{22}\end{bmatrix}\!\!\}l \quad i=1,\ldots,N-1 \tag{4.1}$$

where P_{12} and P_{22} are constant matrices independent of S and $i=1,\ldots,N-1$.

By our assumption that $[B'\ R]'$ has full column rank, we know that for each α_i, $i=n-l+1,\ldots,n$ there exists a unique β_i such that

$$w_i=[\alpha'_i\quad \beta'_i]'\in\mathscr{N}(\Lambda).$$

Let Z be the matrix $[\beta_{n-l+1}\ldots\beta_n]$. Partition Q and C conformably with $P(i, S)$, i.e. set

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q'_{12} & Q_{22}\end{bmatrix} \quad\text{and}\quad C=\begin{bmatrix} C_1 \\ C_2\end{bmatrix}. \tag{4.2}$$

In this state space basis the result of Lemma 3.1 can now be restated as

$$P_{12}=Q_{12}+C_1Z \quad\text{and}\quad P_{22}=Q_{22}+C_2Z. \tag{4.3}$$

Thus, we have completely identified P_{12} and P_{22} as constant parts of $P(i, S)$, $i=0,\ldots,N-1$. Moreover, the theory developed in Section III says that no part of $P_{11}(i, S)$ is independent of $i=0,\ldots,N-1$ and $S\in\mathscr{S}$, though some part may be for $i=0,\ldots,N-2$. (If some part of $P_{11}(i, S)$ were independent for i

$=0,\ldots,N-1$ and $S\in\mathscr{S}$, there would exist a vector in $\mathscr{N}(\Lambda)$ not in I_1 and this would be a contradiction).

By virtue of (4.3), the evaluation of $P(i,S)$ via the Riccati equation (2.5) on $[0,N]$ clearly involves a substantial amount of unnecessary calculation. It would be of interest if we could show that

$$P_{11}(i,S),\quad i=0,\ldots,N-1$$

could be computed via a Riccati equation for P_{11} rather that P, presumably involving different A, B, Q, C and R. Recall that the solution to the Riccati equation (2.5) is the maximal symmetric matrix $P(i,S)$ such that

$$\begin{bmatrix} A'P(i+1,S)A+Q-P(i,S) & A'P(i+1,S)B+C \\ B'P(i+1,S)A+C' & B'P(i+1,S)B+R \end{bmatrix}\geq 0 \tag{4.4}$$

for each $i=0,\ldots,N-1$ with $P(N,S)=S$. With A partitioned as

$$A=[A_1\,A_2]$$

this can then be written as

$$\begin{bmatrix} A_1'P(i+1,S)A_1+Q_{11}-P_{11}(i,S) & A_1'P(i+1,S)A_2+Q_{12}-P_{12} & A_1'P(i+1,S)B+C_1 \\ A_2'P(i+1,S)A_1+Q_{12}'-P_{12}' & A_2'P(i+1,S)A_2+Q_{22}-P_{22} & A_2'P(i+1,S)B+C_2 \\ B'P(i+1,S)A_1+C_1' & B'P(i+1,S)A_2+C_2' & B'P(i+1,S)B+R \end{bmatrix}\geq 0.$$

Since the vectors w_i form a basis matrix of $\mathscr{N}(\Lambda)$ we also have the relations

$$A_2+BZ=0 \quad\text{and}\quad C_2'+RZ=0. \tag{4.5}$$

Taking note of these relations and (4.3), it is easy to show that (4.4) is equivalent to

$$\begin{bmatrix} A_1'P(i+1,S)A_1+Q_{11}-P_{11}(i,S) & A_1'P(i+1,S)B+C_1 \\ B'P(i+1,S)A_1+C_1' & B'P(i+1,S)B+R \end{bmatrix}\geq 0 \tag{4.6}$$

and so it is clear that $P_{11}(i,S)$ is the maximal solution of the inequality (4.6). However we know that an equivalent definition is provided by a matrix Riccati equation

$$\begin{aligned}\hat{P}(i,S)=&\hat{A}'\hat{P}(i+1,S)\hat{A}+\hat{Q}\\ &-[\hat{A}'\hat{P}(i+1,S)\hat{B}+\hat{C}]\\ &\times[\hat{B}'\hat{P}(i+1,S)\hat{B}+\hat{R}]^{\#}\\ &\times[\hat{A}'\hat{P}(i+1,S)\hat{B}+\hat{C}]',\\ &\qquad i=0,\ldots,N-2\end{aligned} \tag{4.7}$$

where $\hat{P}(i,S)=P_{11}(i,S)$, $\hat{A}=A_{11}$ (supposing A_1 is partitioned as $[A_{11}'\,A_{21}']'$ and the other hat quantities are defined in terms of P_{12}, P_{22} and the original coefficient matrices A, B, Q, C and R. (The precise definitions are contained in the Appendix.) Finally, we initialize (4.7) with

$$\hat{S}=\hat{P}(N-1,S)=P_{11}(N-1,S).$$

Let us summarize what we have shown so far. If the Riccati equation (2.5) has a solution on $[0,N]$ for terminal weighting matrix S, and if the null space of Λ has dimension equal to l, then, modulo state and control space basis changes, the solution of (2.5) on $[0,N]$ with $P(N)=S$ is equivalent to the solution of (4.7) on $[0,N-1]$ with $\hat{P}(N-1,S)=P_{11}(N-1,S)$ and with P_{12} and P_{22} defined by (4.3) for $i=0,\ldots,N-1$.

To complete this section, we point out that (4.7) can be associated with a control problem, closely related to and of the same form as that originally given, but now involving hat quantities and defined on $[0,N-1]$ rather than on $[0,N]$. (This observation allows the relation of optimal controls for the two problems). Suppose that the state and control bases are chosen as described at the start of this section, and partition the state variable x as $[x_1'\,x_2']'$ with x_2 of dimension l. Define new state and control variables

$$\begin{aligned}\hat{x}&=x_1\\ \hat{u}&=u-Zx_2.\end{aligned} \tag{4.8}$$

With this notation, it follows from the dynamics of the original system (2.1), from the definitions of $\hat{A}$, $\hat{B}$ and from (4.3) and (4.5) that

$$\begin{aligned}\hat{x}(i+1)&=\hat{A}\hat{x}(i)+\hat{B}\hat{u}(i),\quad i=0,\ldots,N-2\\ \hat{x}(0)&=x_1(0)=\hat{x}_0.\end{aligned} \tag{4.9}$$

Equation (4.8) constitutes the dynamics of a reduced system on $[0,N-1]$.

Similarly, on the interval $[0,N-1]$ we can write the cost in terms of hat quantities only. Altogether then we can solve the control problem on the interval $[0,N]$ in terms of another control problem of lower state space dimension on the interval $[0, N-1]$, whenever there exist nontrivial 1-constant directions. [Knowing $\hat{u}(0)$, $x_1(0)$, $x_2(0)$, we obtain $u(0)$ from (4.8) and $x_1(1)$, $x_2(1)$ from $Ax(0)+Bu(0)$; then knowing $\hat{u}(1)$, $x_1(1)$, $x_2(1)$, we obtain $u(1)$ from (4.8), etc].

V. TOTAL REDUCTION OF THE PROBLEM

Motivated by the results of the previous two sections and results obtained for continuous-time singular linear-quadratic control problems[1] we make the following definition.

Definition 5.1. The optimal control problem is called singular whenever Λ is singular. Otherwise, it is called nonsingular.

The results of Sections III and IV will now be tied together to show how a given singular control problem can be solved by constructing one of three possible terminating problems, a nonsingular problem, a zero state dimension problem or a zero control dimension problem.

We assume that if any controls are redundant they have been eliminated. We further assume that $I_1 \neq \{0\}$ so that the state dimension reduction procedure described in Section IV holds.

Lemma 5.1. Suppose that the solution to the control problem exists on $[0, N]$ for some terminal weighting S. Assume that the state coordinate base is chosen such that the reduction procedure of Section IV may be applied. Then with $j \geqq 2$, α is a j-constant direction of the Riccati equation (2.5) if and only if α_1 is a $(j-1)$-constant direction of the Riccati equation (4.7), where $\alpha' = [\alpha_1' \ \alpha_2']$, α_2 having dimension l.

Proof. Suppose that α_1 is a $(j-1)$-constant direction of the Riccati equation (4.7) for $j \geqq 2$. Then, from the definition of a constant direction, and noting that (4.7) is defined on $[0, N-1]$, we have

$$\hat{P}(N-i, \hat{S})\alpha_1 = \text{constant} \tag{5.1}$$

independent of $\hat{S} \in \hat{\mathscr{S}}$, the set of weighting matrices for which a solution to the reduced problem exists, and all $i \geqq j$. Then for $\alpha' = [\alpha_1' \ \alpha_2']$ for any α_2 and any $S \in \mathscr{S}$,

$$P(N-i, S)\alpha = \begin{bmatrix} P_{11}(N-i, S) & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \tag{5.2}$$
$$= \begin{bmatrix} \hat{P}(N-i, \hat{S}) & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

where $\hat{S}$ is of the form $P_{11}(N-1, S)$. Therefore from (5.1), which applies to all $\hat{S} \in \hat{\mathscr{S}}$ and *a fortiori* to those of the form $P_{11}(N-1, S)$,

$$P(N-i, S)\alpha = \text{constant} \tag{5.3}$$

for all $S \in \mathscr{S}$ and $i \geqq j$. Thus, α is a j-constant direction of (2.5).

Conversely, suppose that α is a j-constant direction of (2.5) for $j \geqq 2$. Write $\alpha' = [\alpha_1' \ \alpha_2']$. Then (5.3) holds for any $i \geqq j$ and $S \in \mathscr{S}$. Therefore, (5.1) holds for any $i \geqq j$ and $\hat{S} \in \hat{\mathscr{S}}$ of the form $P_{11}(N-1, S)$. By Theorem 3.1, it is sufficient to show that (5.1) holds for all $i \geqq j$ and all $\hat{S} \geqq \hat{S}_0$ for some $\hat{S}_0 \in \hat{\mathscr{S}}$.

First, we show that if $S > S_0 \in \mathscr{S}$, then

$$P_{11}(N-1, S) > P_{11}(N-1, S_0).$$

We argue by contradiction. Suppose that there exists an $x \neq 0$ such that

$$P_{11}(N-1, S)x = P_{11}(N-1, S_0)x. \tag{5.4}$$

For any $\bar{S}$ satisfying $S_0 \leqq \bar{S} \leqq S$, we have

$$P_{11}(N-1, S_0) \leqq P_{11}(N-1, \bar{S}) \leqq P_{11}(N-1, S)$$

and therefore

$$0 \leqq x'[P_{11}(N-1, \bar{S}) - P_{11}(N-1, S_0)]x$$
$$\leqq x'[P_{11}(N-1, S) - P_{11}(N-1, S_0)]x = 0.$$

Hence,

$$P_{11}(N-1, \bar{S})x = P_{11}(N-1, S_0)x$$

for all $\bar{S}$ such that $S_0 \leqq \bar{S} \leqq S$. Now since $S > S_0$, an argument similar to that in Lemmas 3.2 and 3.3 shows that

$$P_{11}(N-1, \bar{S})x = P_{11}(N-1, S_0)x$$

for any $\bar{S} \geqq S_0$, even if $\bar{S} \leqq S$ does not hold. Therefore (5.2) for $i=1$ and Theorem 3.1 imply that $[x' \alpha_2']'$ is a 1-constant direction, which is a contradiction. (In view of the basis chosen, all 1-constant directions have the form $\alpha = [0 \ \alpha_2']'$).

Thus, (5.1) holds for

$$\hat{S}_0 = P_{11}(N-1, S_0) \quad \text{and} \quad \hat{S} = P_{11}(N-1, S)$$

with $\hat{S} > \hat{S}_0$. Again, an argument as in Lemmas 3.2 and 3.3 implies that (5.1) holds for all $\hat{S} \geqq \hat{S}_0$. Therefore, by Theorem 3.3, α_1 is a $(j-1)$-constant direction of (4.7) for $j \geqq 2$. This completes the proof of the lemma.

Hence, if the space of 1-constant directions is nonzero, the state space reduction procedure holds and *the j-constant directions, $j \geqq 2$, of the originally given problem become $(j-1)$-constant directions of the reduced state dimension problem.* Consider now the repeated application of the idea of the above lemma. Suppose that a reduced problem is obtained via the procedures of the last two sections.

Now if this new problem is singular, we first eliminate any unnecessary controls by Remark 3.1. Then if $\hat{I}_1 \neq \{0\}$, with $\hat{I}_1$ the space of 1-constant directions for the reduced state dimension problem, we can again reduce according to Section IV. Clearly, if at some stage in the above procedure we obtain $I_1 = \{0\}$ for one of the reduced problems, we cannot proceed any further. Now by Lemma 5.1 this is equivalent to having $I_l = I_{l+1}$ for some l in the original problem. We will now show that $I_l = I_{l+1}$ for some l implies that [illegible] for every $k \geq 2$ in the original problem. This means that there is no way, possibly using some other algorithm than that presented, of eliminating further constant directions.

Since $I_{l+1} \subset I_{l+k}$ for all $k \geq 2$ we could only have $I_{l+1} \neq I_{l+k}$ if there were $(k+l)$-constant directions which were not $(l+1)$-constant directions for the original problem, or k-constant directions which were not 1-constant directions for the reduced problem. Since the reduced problem has no 1-constant direction, the result will follow from the following lemma:

Lemma 5.2. If $I_1 = \{0\}$, then $I_k = \{0\}$ for all $k > 1$.

Proof. We argue by contradiction. Let j be the least value of $k > 1$ for which $I_j \neq \{0\}$. Let

$$w_{(j)} = [\alpha' \quad \beta'_{(j)}]' \in \mathcal{N}(\Lambda_{(j)})$$

with $\alpha \neq 0$. By (2.12),

$$\begin{bmatrix} A_{(j-1)}A & A_{(j-1)}B & B_{(j-1)} \\ B'Q_{(j-1)}A + C' & B'Q_{(j-1)}B + R & B'C_{(j-1)} \\ C'_{(j-1)}A & C'_{(j-1)}B & R_{(j-1)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \qquad (5.5)$$

where $\beta_{(j)} = [\beta' \, \gamma']'$. Immediately, we see that

$$\Lambda_{(j-1)} \begin{bmatrix} A\alpha + B\beta \\ \gamma \end{bmatrix} = 0.$$

If $A\alpha + B\beta \neq 0$, there exists a $(j-1)$-constant direction, which is a contradiction. So $A\alpha + B\beta = 0$. Then

$$\gamma \in \mathcal{N}([B'_{(j-1)} R'_{(j-1)}]')$$

and so by Remark 3.1, $\gamma \in \mathcal{N}(C_{(j-1)})$. From the middle block row of (5.5), we have then

$$0 = B'Q_{(j-1)}[A\alpha + B\beta] + C'\alpha + R\beta + B'C_{(j-1)}\gamma = C'\alpha + R\beta.$$

This shows that $[\alpha' \; \beta']' \in \mathcal{N}(\Lambda)$, again a contradiction.

We now have enough information to describe a procedure for solving any singular linear-quadratic control problem.

1. Determine the nullity of Λ, say s. If $[B'R]'$ has full rank proceed to 2. If not, eliminate any unnecessary controls.

2. Let $l = \dim I_1$. If $l = 0$, we have a nonsingular problem. If $l > 0$, reduce the state dimension by l as described in Section IV. If $l = n$, we have a zero state dimension problem. If $l < n$, return to 1.

3. Cycle through 1 and 2 until the procedure terminates. This is guaranteed by the above theory, and moreover, it is guaranteed that *all* of the constant directions of the original problem are determined in at most n applications of 1 and 2.

4. Determine the solution of the terminating control problem and trace back through the reduction procedure to construct the solution of the originally given problem.

In the commonly occurring case of $[A, B]$ completely reachable, we cannot terminate with a zero control dimension problem. This follows from the fact that complete reachability is preserved under our reduction procedure.

VI. CONCLUSIONS

First, we comment on relations with other work. In [5, 6], a 'structure algorithm' is presented, pertinent to the case when

$$S \geq 0, \begin{bmatrix} Q & C \\ C' & R \end{bmatrix} \geq 0.$$

The manipulations for eliminating superfluous controls, characterizing and eliminating 1-constant directions, and identifying them with states which can be taken to zero optimally in one step are equivalent to manipulations in [5, 6]. The full extent of the parallels, and the point at which they break down because of the nonnegativity requirement of [5, 6] has yet to be explained. In [7], Chandrasekhar type algorithms are suggested for solving discrete-time control problems. These algorithms hinge on updating square roots of $P(i-1, S) - P(i)$. To the extent that the presence of constant directions will reduce the rank of these squares roots, there would seem profit in marrying the two sections. This however will be intricate, since singular $B'P(i)B + R$ (as may arise given constant directions) may force a need for adjusting formulas of [7]. In [1] results for the time-varying linear-quadratic control problem are presented in detail. However, for the derivation of these results, certain constancy of rank assumptions are required on the interval of interest. These assumptions might be thought of as a constant structure requirement. A similar idea applies to the extension of the constant coefficient results obtained in Sections II–V to the time-varying case; see [4] where such an idea has been used in connection with the generation of time-varying scalar covariances. Also, though we have confined our discussions to control problems, we could equally well have worked within a framework of filtering and covariance factorization, as in [4]. Combination of the ideas of this paper and of [4] will readily yield the results.

The main contribution of this paper is, we believe, to have discussed properties of constant directions in the context of the most general linear-quadratic control problem. Constant directions are characterized via the nullspace of a certain matrix, in terms of optimal controls yielding trajectories which terminate in the zero state and in terms of constant directions of a lesser index. Secondly, we have shown how the existence of 1-constant directions may be exploited in solving an optimal control problem. They may be eliminated to yield a lower dimension problem, the solution of which determines the solution of the original problem, with the adjunction of certain quantities computed in the construction of the lower dimension problem.

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APPENDIX

Definitions of quantities $\hat{A}$ etc. in reduced dimension problem of Section 4

$$A=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$B=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q'_{12} & Q_{22} \end{bmatrix}$$

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P'_{12} & P_{22} \end{bmatrix}$$

$$\hat{A}=A_{11}$$

$$\hat{B}=B_1$$

$$\hat{C}=C_1+A'_{11}P_{12}B_2+A'_{21}P'_{12}B_1+A'_{21}P_{22}B_2$$

$$\hat{Q}=Q_{11}+A'_{11}P_{12}A_{21}+A'_{21}P'_{12}A_{11}+A'_{21}P_{22}A_{21}$$

$$\hat{R}=R+B'_2P_{22}B_2+B'_1P_{12}B_2+B'_2P'_{12}B_1.$$