the actual state of nature deviates "mildly" from the environment nominally assumed in the design. We put the word mildly in quotation marks since it is a key word in the entire concept. Quantifying mildly is impossible without a good physical feel for the noise generating process and its inherent constraints. If, for example, the noise is generated directly from a well understood and accurately described process, such as thermal noise in a resistor, then tight bounds may be placed on the statistical description of the noise process, and a mild deviation is a very small change in the process statistics. If, on the other hand, the noise to be modeled is a process derived from a source which is not very well understood or quantified, then the nominal statistical description may be imprecise. Thus, mild deviations from the nominal description may represent large changes in the process statistics.

The hope of the engineer who adheres to the philosophy of robust design is that he may be able to find a policy performing within a few percent of optimality for the most likely environment while protecting against performance deterioration of much greater magnitude caused by "mild" deviations from the assumed environment. The most frequent formalization of the robust design problem is as a mathematical game between the engineer and nature. How well the robust design can be effected will depend on the situation at hand and the engineer's ability to pose the most appropriate game to be solved. A solution to the resulting minimax optimization problem consists of a most robust policy and a least favorable state for nature. If the least favorable state happens to correspond to the most likely or nominal state, the solution is particularly meaningful, since no penalty need be paid for robustness. As we shall show, this is the situation in many linear-quadratic estimation and control problems. First, however, we need some basic concepts from game theory.

An abstract game may be defined as a triple (A, B, V) where A and B are the sets of possible strategies for the engineer and for nature, respectively, and V is a function from $A \times B$ to the real line which measures the performance of a strategy pair (a,b). Nature chooses b to maximize V, and the engineer chooses a to minimize it. A pair (a_0,b_0) is called a saddlepoint pair for the game if $V(a_0,b_0)$, the value of the game, satisfies

$$V(a_0,b) \leq V(a_0,b_0), \qquad \forall b \in B \tag{1}$$

and

$$V(a_0,b_0) \leq V(a,b_0), \qquad \forall a \in A. \tag{2}$$

It is not always possible to find a pair satisfying (1) and (2), and in such cases more extensive considerations are necessary. For the problems considered here a saddlepoint will exist; thus, there is no need to discuss the possible additional complications (the interested reader should see Ferguson [4]).

This setup can be applied directly to many linear-quadratic estimation and control problems. Let A contain all functions, both linear and nonlinear, of the observations and let B contain all probability distributions satisfying the standard first and second moment constraints. Furthermore, let a_0 correspond to the best linear solution to the problem and let b_0 correspond to the Gaussian distribution function satisfying the moment constraints (with greatest covariance matrix, if the second moment constraint is an inequality). Then, inequality (1) requires that the linear solution perform no more poorly with respect to any other distribution satisfying the constraints than it does with respect to the Gaussian. Since the problem is linear and the performance measure quadratic, a linear solution yields a value for V that is only a function of at most the first two moments of the noise distribution. These are constrained by hypothesis, so that (1) holds with equality. Inequality (2) is a statement that the linear solution must be optimal with respect to Gaussian distributions for the noise. Thus, if any model admits the same solution to both the linear least-squares problem and the optimal Gaussian problem, then this same solution also solves the minimax-robust problem. A large class of linear-quadratic estimation and control problems have solutions that satisfy this condition. The ease with which this fact is obtained should not belittle its importance. The philosophical ramifications of this added property are profound for, with it, we can provide added justification for using linear policies even when the noise is suspected to be non-Gaussian. Also illuminated is the extreme importance of the constraint on the second moment of the noise. It is this constraint on which the efficacy of the linear solution rests and not on the Gaussian assumption. As we have seen, any arbitrary deviation from Gaussian noise, which continues to satisfy the second moment constraint, cannot increase the cost. However, there exist distributions looking, for all the world, like the Gaussian which can drive the cost arbitrarily high; for example, consider a mixture distribution with the Gaussian being the true distribution 99.9 percent of the time and a Cauchy the true distribution 0.1 percent of the time. To detect the fact that the variance of the distribution is very large (in fact infinite) would require many thousands of observations; however, any given sample function of the noise could drive the estimation error or control cost very high.

Although the above results have escaped the general knowledge of the engineering community, they are not entirely without precedent. In fact, for the Kalman filtering problem the robust property of the solution is almost as old as the filter itself, having been pointed out by Carlton [5]. In various specific problems [6], [7] other workers have used similar ideas in a limited context. We believe appreciation of the simple result reported in this note and an application of the ideas in a broader research context can be of major importance as a direction for research in the next few years.

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Extremal Solutions of Riemann-Stieltjes Inequalities of Linear Optimal Control

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Abstract—**The maximum and minimum solution of an integral inequality are identified arising in linear-quadratic control existence theory. The set of terminal weighting matrices for which a control problem has a solution is illuminated.**

Introduction

In [1], the linear-quadratic problem of minimizing

$$V[x(t), (t), u(\cdot)] = \int_t^{t_f} [x' \quad u'] \begin{bmatrix} Q(t) & H(t) \\ H'(t) & R(t) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt + x'(t_f) S_f x(t_f) \tag{1}$$

is studied, subject to

$$\dot{x} = F(t)x + G(t)u. \tag{2}$$

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The main conclusion is that the problem has a solution for all $x(t)$ and all $t\in[t_0,t_f]$ if and only if there exists a symmetric matrix $P(t)$ of bounded variation on $[t_0,t_f]$ for which $P(t_f)\leq S_f$ and

$$\int_{t_1}^{t_2}[x' \quad u']\begin{bmatrix} dP+PF+F'P+Q & PG+H \\ (PG+H)' & R \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}\geq 0 \tag{3}$$

for all $[t_1,t_2]\subset[t_0,t_f]$ with t_1, t_2 arbitrary, for all $x(t_1)$ and for all piecewise continuous $u(\cdot)$. We shall write this second condition as $\mathcal{I}(P)\geq 0$.

Our purpose here is to characterize a maximum and minimum solution of $\mathcal{I}(P)\geq 0$ and to use the minimum solution to explain for what S_f the optimal control problem has a solution.

The Maximum Solution

Theorem 1: Using the notation as above, let $P(\cdot)$ satisfy $P(t_f)\leq S_f$ and $\mathcal{I}(P)\geq 0$. Then for all $t\in[t_0,t_f]$

$$P\big(t;P(t_f)\leq S_f\big)\leq P^*\big(t;P^*(t_f)=S_f\big) \tag{4}$$

where $\mathcal{I}(P^*)\geq 0$, $V^*[x(t),t]=\inf_{u(\cdot)}V[x(t),t,u(\cdot)]=x'(t)P^*(t)x(t)$.

Before proving the theorem, we note that the postulates of the theorem imply the existence of $V^*[x(t),t]$ for all $x(t)$ and $t\in[t_0,t_f]$ (see [1]) while the quadratic form of $V^*[x(t),t]$ is also proved in [1] and that $\mathcal{I}(P^*)\geq 0$ is also proved in [1]. Thus, (4) claims a maximum property for $P^*(\cdot)$ among the set of matrices $P(\cdot)$ for which $P(t_f)\leq S_f$, $\mathcal{I}(P)\geq 0$.

Proof: An easy calculation yields

$$\int_{t}^{t_f}[x' \quad u']\begin{bmatrix} dP+PF+F'P+Q & PG+H \\ (PG+H)' & R \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}$$
$$=x'(t_f)\big[P(t_f)-S_f\big]x(t_f)-x'(t)P(t)x(t)+V[x(t),t,u(\cdot)]$$

whence $V[x(t),t,u(\cdot)]\geq x'(t)P(t)x(t)$, using the constraints on $P(\cdot)$. The result is immediate.

The Minimum Solution

To construct a minimum solution we consider the problem of minimizing by choice of $u(\cdot)$ and $x(t_0)$ the index

$$W[x(t),t,u(\cdot)]=\int_{t_0}^{t}[x' \quad u']\begin{bmatrix} Q(t) & H(t) \\ H'(t) & R(t) \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}dt-x'(t_0)S_0x(t_0). \tag{5}$$

Here, $S_0=S_0'$ and $u(\cdot)$ is required to take $x(t_0)$ to $x(t)$ via (2). Time reversal will relate this to the earlier problem and we can prove the following variation on [1] and Theorem 1.

Theorem 2: Using the notation as above. $W^*[x(t),t]=\inf_{u(\cdot),x(t_0)}W[x(t),t,u(\cdot)]$ exists for all $x(t)$ and all $t\in[t_0,t_f]$ if and only if there exists a symmetric $P(\cdot)$ of bounded variation on $[t_0,t_f]$ such that $P(t_0)\geq S_0$ and $\mathcal{I}(P)\geq 0$. If $W^*[x(t),t]$ exists for all $x(t)$ and $t\in[t_0,t_f]$, it has the form $-x'(t)P_*(t)x(t)$ where $\mathcal{I}(P_*)\geq 0$ and $P_*(t_0)=S_0$. Moreover, any $P(\cdot)$ such that $\mathcal{I}(P)\geq 0$ and $P(t_0)\geq S_0$ is such that for all $t\in[t_0,t_f]$

$$P\big(t;P(t_0)\geq S_0\big)\geq P_*\big(t;P_*(t_0)=S_0\big). \tag{6}$$

Allowable Terminal Weighting Matrices S_f

If we combine Theorems 1 and 2, noting particularly the inequalities (4) and (6), one obtains with minor manipulations results such as the following theorem.

Theorem 3: Suppose $P_*(\cdot)$ exists on $[t_0,t_f]$ for some S_0. Then it exists for all $\bar{S}_0\leq S_0$ and $P^*(\cdot)$ exists on $[t_0,t_f]$ if and only if

$$S_f\geq P_*\big(t_f;P_*(t_0)=\bar{S}_0\big) \tag{7}$$

for some $\bar{S}_0\leq S_0$.

Taking $S_0=-\infty I$ or $S_f=\infty I$ is like considering control problems with $x(t_0)=0$ or $x(t_f)=0$, respectively. Given reachability and controllability, such problems have a solution if the problems with finite S_0, S_f have a solution, and there exist $P_*(t;x(t_0)=0)$ defined on $(t_0,t_f]$ and $P^*(t;x(t_f)=0)$ defined on $[t_0,t_f)$. Here, $x'(t)P_*(t;x(t_0)=0)x(t)=\inf_{u(\cdot)}W[x(t),t,u(\cdot)]$ subject to $x(t_0)=0$, and similarly for $P^*(t;x(t_f)=0)$.

One then has Theorem 4.

Theorem 4: Suppose that for all t, all $x(t)$ are reachable from $x(t_0)=0$ and controllable to $x(t_f)=0$. Suppose there exists a symmetric $P(\cdot)$ on $[t_0,t_f]$ satisfying $\mathcal{I}(P)\geq 0$. Then $P_*(t;x(t_0)=0)$ and $P^*(t;x(t_f)=0)$ exist on $(t_0,t_f]$ and $[t_0,t_f)$, respectively, with

$$P_*(t;x(t_0)=0)\leq P(t)\leq P^*\big(t;x(t_f)=0\big) \tag{8}$$

for all $t\in(t_0,t_f]$ for the left inequality and $[t_0,t_f)$ for the right inequality.

Equation (8) is reminiscent of certain inequalities in [2] for time-invariant, linear-quadratic problems over semiinfinite intervals.

Conclusion

We have exhibited a maximum solution of an integral inequality arising in linear-quadratic control and by introducing a time-reversed problem of little or no intrinsic interest, have also obtained a minimum solution. The minimum solution can be use to define the class of terminal weighting matrices for which the maximum solution exists, and conversely. A number of results are obtainable for problems with end-point contraints with inequality (8) perhaps constituting the most interesting.

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Eigenvalue-Generalized Eigenvector Assignment with State Feedback

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***Abstract*—In a recent paper [1], a characterization has been given for the class of all closed-loop eigenvector sets which can be obtained with a given set of distinct closed-loop eigenvalues. This note extends these results to characterize the class of generalized eigenvector chains which can be obtained with a given set of nondistinct eigenvalues. Included is an algorithm for computing a feedback matrix which gives the selected closed-loop eigenvalues and generalized eigenvector chains. Although there are limitations on the Jordan structure of the closed-loop system, this algorithm allows one to realize any "allowable" closed-loop Jordan configuration.**

The purpose of the note is to generalize a recent eigenvalue-eigenvector assignment result to include the case of multiple closed-loop eigenvalues. Moore [1] has shown that for a distinct self-conjugate set of closed-loop eigenvalues, the additional freedom offered by state feedback, beyond pole placement, is in the selection of a set of eigenvectors from an allowable class. This class was characterized and an algorithm was given to compute a feedback matrix which gives the selected eigenvalue-eigenvector sets.

Consider the system

$$\dot{x}(t)=(A+BF)x(t)$$

obtained by applying feedback $u(t)=Fx(t)$ to the system

$$\dot{x}(t)=Ax(t)+Bu(t)$$

where $x(t)\in R^n$, $u(t)\in R^m$.

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