Exponential Stability of Linear Equations Arising in Adaptive Identification

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Abstract—**The stability properties are examined of various time-varying linear differential equations which arise in model reference adaptive identification schemes. Necessary and sufficient conditions for exponential stability are presented.**

Manuscript received November 6, 1975; revised June 15, 1976. Paper recommended by G. L. Blankenship, Chairman of the IEEE S-CS Stability, Nonlinear, and Distributed Systems Committee. This work was supported by the Australian Research Grants Committee.

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I. INTRODUCTION

In the study of model reference adaptive identification of linear systems, time-varying differential equations arise, see e.g., [1]–[7]. It is important to be able to provide conditions for asymptotic stability of these equations, and even exponential asymptotic stability where possible, since asymptotic stability of the equations is equivalent to convergence of the identification algorithms.

In case the equation is periodically time-varying (which situation obtains when the input to the plant being identified is periodic), standard techniques of Lyapunov theory can be used to obtain convergence fairly easily [8]. As soon as the input is almost periodic—for example, a sum of two sinusoids with incommensurate frequencies—these techniques fail; indeed, the extension of Lyapunov results from the periodic case to the almost periodic case is recognized to be a significant problem [8, p. 67]. For inputs which are not even almost periodic, one would expect the difficulties to be greater again than in the almost periodic case.

Stimulated especially by the work of Narendra and his colleagues, e.g., [1]–[4], we examined the stability problem in a report [9], summarized in [7]. Most of the results were for the almost periodic case, though some applied to less restrictive situations, and we gave necessary conditions for exponential stability. These are derived below. Meanwhile, for one of the types of equation studied below, Morgan and Narendra have independently derived by quite different methods necessary conditions along lines allowing derivation also of a sufficiency result. This work, including many insightful examples, is contained in [10], and prompted us to modify our earlier necessity treatment to recover sufficiency results; these are described below.

The paper is structured as follows. In Section II, we present background technical material used in proving the stability results. The stability results themselves are established in Section III. This section can be read independently of the adaptive identification literature. though its understanding is enhanced by knowledge of the source of the differential equation and the significance (from the viewpoint of adaptive identification) of the side conditions required to ensure stability. A summary of some of the relevant adaptive identification background is therefore included in an Appendix and is cross-referenced in Section III. Section IV contains concluding remarks.

II. BACKGROUND

Let $F(\cdot): R_+ \to R^{n\times n}$ and $H(\cdot): R_+ \to R^{n\times r}$ be regulated matrix functions (i.e., one-sided limits exist for all $t\in R_+$). Let $\Phi(\cdot,\cdot)$ be the transition matrix associated with $F(\cdot)$. We say that the pair $[F,H]$ is uniformly completely observable[1] if [11] the following three conditions hold (any two implying the third): for some positive α_1, α_2, α_3, α_4, and δ, and for all $s,t\in R_+$,

$$\alpha_1 I \leqslant N(s,s+\delta) \leqslant \alpha_2 I \tag{2.1}$$

$$\alpha_3 I \leqslant \Phi'(s,s+\delta) N(s,s+\delta)\Phi(s,s+\delta) \leqslant \alpha_4 I \tag{2.2}$$

$$\|\Phi(t,s)\| \leqslant \alpha_5(|t-s|) \tag{2.3}$$

where

$$N(s,s+\delta) = \int_s^{s+\delta} \Phi'(t,s) H(t) H'(t) \Phi(t,s)\, dt \tag{2.4}$$

and $\alpha_5(\cdot): R_+ \to R$ is bounded on bounded intervals. We remark that if the above conditions hold for some δ, they hold for all $\delta' > \delta$.

We shall make use of the following properties.

Lemma 1: Let $K(\cdot): R_+ \to R^{n\times r}$ be regulated, and such that

$$\int_s^{s+\delta'} \|K(t)\|^2 dt \leqslant \alpha_6 \tag{2.5}$$

for some positive constants α_6 and δ' and all $s\in R_+$; then $[F,H]$ is uniformly completely observable if and only if $[F+KH', H]$ is uniformly completely observable.

Proof: This lemma is the dual of [12, theorem 4]. ∇∇

Remark: If the hypothesis of the lemma holds for one fixed δ', it

[1] The uniformity is with respect to time.

holds for all positive $\delta>0$ (although α_6 depends on δ). This fact will be used in several proofs in the next section.

Lemma 2: The following conditions are equivalent.

1) The equation $\dot{x}=Fx$ is exponentially asymptotically stable.

2) There exists a symmetric differentiable matrix $P(\cdot):R_+\to R^{n\times n}$, a regulated $H(\cdot):R_+\to R^{n\times r}$, and positive constants β_1, β_2 such that for all $t\in R_+$,

$$0<\beta_1 I\leqslant P(t)\leqslant \beta_2 I<\infty \tag{2.6}$$

$$-\dot{P}=PF+F'P+HH' \tag{2.7}$$

$$[F,H] \text{ is uniformly completely observable.} \tag{2.8}$$

Moreover, should $\dot{x}=Fx$ be exponentially asymptotically stable, and should (2.6) and (2.7) hold for some P and H, then $[F,H]$ is uniformly completely observable.

Remark: In the context of this lemma, we require that although $\|x(t)\|$ decay exponentially fast, it decays no faster than exponentially. This means that there exist positive γ_1, γ_2, γ_3, γ_4 such that $\gamma_1\exp[-\gamma_2\cdot(t-s)]\leqslant\|\Phi(t,s)\|\leqslant\gamma_3\exp[-\gamma_4(t-s)]$ for all $t\geqslant s\geqslant 0$.

Proof: That 1) implies 2) follows as in [12, proof of theorem 5], taking $L(\cdot)$ of that theorem to be I; that 2) implies 1) is a restatement of part of the theorem. The remainder follows by reversal of part of the proof of the theorem. The details are as follows. Identify a Lyapunov function $V(x,t)=x'P(t)x$. In view of the bounds on $P(\cdot)$ exponential asymptotic stability is equivalent to the existence of some positive δ, β_3, β_4, β_5, and β_6 for which

$$-\beta_3\leqslant\frac{V(x(s+\delta),s+\delta)-V(x(s),s)}{V(x(s),s)}\leqslant-\beta_4$$

and

$$-\beta_5\leqslant\frac{V(x(s+\delta),s+\delta)-V(x(s),s)}{V(x(s+\delta),s+\delta)}\leqslant-\beta_6$$

for all $x(s)$ and s. Now use the fact that

$$\begin{aligned}V(x(s+\delta),s+\delta)-V(x(s),s)&=\int_s^{s+\delta}\dot{V}(x,t)\,dt\\&=-x'(s)\int_s^{s+\delta}\Phi'(t,s)H(t)H'(t)\Phi(t,s)\,dt\,x(s)\\&=-x'(s+\delta)\int_s^{s+\delta}\Phi'(t,s+\delta)H(t)H'(t)\Phi(t,s+\delta)x(s+\delta)\end{aligned}$$

together with the bounds on $P(\cdot)$ to conclude the uniform complete observability result. $\nabla\nabla\nabla$

III. Applications

Equation (3.1) in Theorem 1 below is representative of some equations arising in adaptive identification; its origin is summarized in the Appendix, Section A-I. Condition (3.2) below has an interpretation in adaptive identification as a "persistently exciting" condition, see Section A-II. One half of the theorem was established in [9]; the complete theorem is stated in [10]; save that in [9] and [10], boundedness of $V(\cdot)$ is assumed.

Theorem 1: Let $V(\cdot):R_+\to R^{n\times r}$ be regulated. Then

$$\dot{x}=-VV'x \tag{3.1}$$

is exponentially asymptotically stable if and only if for some positive δ, α_1, and α_2, and for all $s\in R_+$,

$$\alpha_1 I\leqslant\int_s^{s+\delta}V(t)V'(t)\,dt\leqslant\alpha_2 I. \tag{3.2}$$

Remark: Equation (3.2) is equivalent to uniform complete observability of $[0,V]$.

Proof: Suppose (3.2) holds. Then $[0,V]$ is uniformly completely observable, and so, by Lemma 1, $[-VV',V]$ is uniformly completely observable. (Identify K with $-V$ and observe that the right inequality of (3.2) guarantees that (2.5) holds.) Take $\frac{1}{2}x'x$ as a Lyapunov function for (3.1), i.e., $P(t)=I$ in Lemma 2. Then the conditions of Lemma 2-2) are satisfied with $F=-VV'$ and $H=V$.

Conversely, assume exponential asymptotic stability. Equations (2.6) and (2.7) hold with $P=\frac{1}{2}I$, $F=-VV'$ and $H=V$. Accordingly, $[-VV',V]$ is uniformly completely observable. Now provided

$$\int_s^{s+\delta}\|V(t)\|^2\,dt\leqslant\alpha_6 \tag{3.3}$$

holds for some positive α_6 and δ, we may again use Lemma 1 to conclude that $[0,V]$ is uniformly completely observable, i.e., that (3.2) holds. We see that (3.3) is a consequence of exponential stability as follows. Let $\Phi(\cdot,\cdot)$ be the transition matrix associated with $\dot{x}=-VV'x$. Then (2.3) and the uniform complete observability of $[-VV',V]$ imply that for any fixed δ and all s.

$$\|\Phi(s,s+\delta)\|\leqslant\alpha_5(\delta)$$

so that for some positive α_7

$$\det\Phi(s,s+\delta)\leqslant\alpha_7(\delta).$$

Now it is a standard result that the transition matrix $\Phi(\cdot,\cdot)$ of $\dot{x}=F(t)x$ satisfies [13, p. 82]

$$\det\Phi(s_1,s_2)=\exp\int_{s_2}^{s_1}\mathrm{tr}\,F(t)\,dt.$$

Applying this result with $F=-VV'$ yields

$$\exp\int_s^{s+\delta}\mathrm{tr}\,VV'\,dt=\det\Phi(s,s+\delta)\leqslant\alpha_7(\delta)$$

which implies (3.3). $\nabla\nabla$

The natural question arises as to how significant the uniform assumption is. If the lower bound in (3.2) fails, there may or may not be convergence, and if there is convergence, it will not be exponential. For example, $\dot{x}=-t^{-1}x$ for $t\geqslant 1$ has a solution $x(t)=Kt^{-1}$, which converges to zero, but not exponentially fast, while $\dot{x}=-e^{-t}x$ for $t\geqslant 0$ is not asymptotically convergent. In both cases, the integral in (3.2) is positive definite for all s and any $\delta>0$, but not uniformly so. On the other hand, if the upper bound fails, we should expect convergence at least as fast as exponential. In this case of course, $V(\cdot)$ would have to be unbounded.

In the following Theorem, we consider a more complicated equation, again arising in the study of model reference adaptive identification, see Section A-III of the Appendix. (Actually, we have made some trivial modifications to the equation as it appears in, for example, [4] and [9].) This equation is also the subject of Theorem 3. The side condition (3.5) ensuring stability of the equation is discussed in Section A-IV of the Appendix.

Theorem 2: Let $V(\cdot):R_+\to R^{n\times r}$ be regulated, and satisfy for some positive δ and α_6 and all $s\in R_+$

$$\int_s^{s+\delta}\|V(t)\|^2\,dt<\alpha_6. \tag{3.3}$$

Let A be a real constant $n\times n$ matrix with $A+A'=-I$ and B a real constant $n\times r$ matrix with rank r. Then

$$\dot{x}=\begin{bmatrix}0 & -VB'\\ BV' & A\end{bmatrix}x \tag{3.4}$$

is exponentially stable if and only if for some positive δ, α_1' and all $s\in R_+$,

$$\alpha_1' I\leqslant\int_s^{s+\delta}\int_s^t V(\tau)\,d\tau\int_s^t V'(\tau)\,d\tau\,dt. \tag{3.5}$$

Proof: Observe that the bound (3.3) implies the existence of a positive α_8 such that for all $s\in R_+$,

$$\int_s^{s+\delta}\|V(t)\|\,dt \le \alpha_8. \tag{3.6}$$

This follows from the Schwarz inequality

$$\int_s^{s+\delta}\|V(t)\|\,dt \le \left[\int_s^{s+\delta}\|V(t)\|^2dt\right]^{1/2}\left[\int_s^{s+\delta}1\,dt\right]^{1/2}.$$

Now identify $P(t)$ of Lemma 2-2) with I, and $F(t)$ with the system matrix in (3.4). Then (2.7) holds with

$$HH'=\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix}$$

or $H'=[0\quad I]$. The result of the theorem follows if and only if (3.5) implies and is implied by the uniform complete observability of $[F,H]$. Use Lemma 1 with

$$K=\begin{bmatrix}-VB'\\ A\end{bmatrix}$$

to note that uniform complete observability of $[F,H]$ is equivalent to uniform complete observability of

$$\begin{bmatrix}0 & 0\\ BV' & 0\end{bmatrix},\begin{bmatrix}0\\ I\end{bmatrix}.$$

The associated transition matrix is

$$\Phi(t,s)=\begin{bmatrix}I & 0\\ B\int_s^t V'(\tau)\,d\tau & I\end{bmatrix}$$

and the observability matrix $N(s,s+\delta)$ is accordingly

$$\int_s^{s+\delta}\begin{bmatrix}\int_s^t V(\tau)\,d\tau\, B'\\ I\end{bmatrix}\begin{bmatrix}B\int_s^t V'(\tau)\,d\tau & I\end{bmatrix}dt.$$

Now (3.6) shows that $\|\Phi(t,s)\|\le\alpha_5(|t-s|)$ for some $\alpha_5(\cdot)$ and all $s,t\in R_+$ as required in (2.3); the upper bound in (2.1) on $N(s,s+\delta)$ also exists because of (3.6). It remains therefore to show that (3.5) implies and is implied by the existence of some positive α_1,δ such that for all $s\in R_+$,

$$\alpha_1 I<\int_s^{s+\delta}\begin{bmatrix}\int_s^t V(\tau)\,d\tau\, B'\\ I\end{bmatrix}\begin{bmatrix}B\int_s^t V'(\tau)\,d\tau & I\end{bmatrix}dt. \tag{3.7}$$

That this implies (3.5) is trivial. We argue now that failure of (3.7) implies failure of (3.5).

Suppose then for arbitrary δ and an arbitrary monotone decreasing sequence of positive ϵ_i with $\epsilon_i\to 0$ as $i\to\infty$, there exists x with $\|x\|=1$ and corresponding s_i such that

$$x'\int_{s_i}^{s_i+\delta}\begin{bmatrix}\int_{s_i}^t V(\tau)\,dt\, B'\\ I\end{bmatrix}\begin{bmatrix}B\int_{s_i}^t V'(\tau)\,d\tau & I\end{bmatrix}dt\,x<\epsilon_i.$$

Partition x as $[x_1'\quad x_2']$ and set $W_i(t)=B\int_{s_i}^t V'(\tau)\,d\tau\,x_1$. Then

$$\int_{s_i}^{s_i+\delta}\|W_i(t)+x_2\|^2dt<\epsilon_i. \tag{3.8}$$

We can now argue that this implies $x_2=0$, as follows; for $t\in[s_i,s_i+\delta]$ the Schwarz inequality implies

$$\left|\int_{s_i}^t[\dot W_i(\tau)]'[W_i(\tau)+x_2]\,d\tau\right|$$
$$\le\left[\int_{s_i}^t \dot W_i'(\tau)\dot W_i(\tau)\,d\tau\right]^{1/2}\left[\int_{s_i}^t\|W_i(\tau)+x_2\|^2d\tau\right]^{1/2},$$

i.e.,

$$\left|\frac{1}{2}\|W_i(t)+x_2\|^2-\frac{1}{2}\|x_2\|^2\right|<\int_{s_i}^{s_i+\delta}x_1'V(t)B'BV'(t)x_1\,dt\,\sqrt{\epsilon_i}\le K\sqrt{\epsilon_i}$$

for some positive constant K, existing by (3.3) and independent of i. Then (3.8) implies that

$$\int_{s_i}^{s_i+\delta}\|x_2\|^2dt<2K\sqrt{\epsilon_i}\,\delta+\epsilon_i.$$

Since $\epsilon_i\to 0$ as $i\to\infty$, this shows that $x_2=0$. Then (3.8) reduces with $\|x_1\|=1$, to

$$x_1'\int_{s_i}^{s_i+\delta}\int_{s_i}^t V(\tau)\,d\tau\,(B'B)\int_{s_i}^t V'(\tau)\,d\tau\,dt\,x_1<\epsilon_i$$

and this is equivalent to failure of (3.5). $\nabla\nabla$

The natural question arises as to how the left inequality of (3.2) might be related to (3.5). (This question also presents itself in the adaptive identification problem context. See, e.g., [9] and Section A-IV of the Appendix.) Our main conclusions are as follows; under the assumption that the bound (3.3) holds:

1) Equation (3.2) is necessary for (3.5)
2) However, (3.2) is not sufficient for (3.5)
3) If $V(\cdot)$ is constrained appropriately, (3.2) is sufficient for (3.5).

We shall now establish these conclusions.

Proposition 1: Let $V(\cdot)$ be as in the hypothesis of Theorem 1. If there exists no positive α_1 such that (3.2) holds, there exists no positive α_1' such that (3.5) holds.

Proof: Suppose that for arbitrary δ and an arbitrary monotone decreasing sequence of positive ϵ_i with $\epsilon_i\to 0$ as $i\to\infty$, there exists x with $\|x\|=1$ and corresponding s_i such that

$$x'\int_{s_i}^{s_i+\delta}V(t)V'(t)\,dt\,x<\epsilon_i.$$

Set

$$W_i(t)=\int_{s_i}^t V'(\tau)\,d\tau\,x.$$

It is easy to check that for $t\in[s_i,s_i+\delta)$, $W_i(t)$ is bounded independently of i by virtue of (3.3), and in fact

$$\frac{1}{2}W_i'(t)W_i(t)=\int_{s_i}^t W_i'(\tau)\dot W_i(\tau)\,d\tau$$
$$<\left[\int_{s_i}^t W_i'(\tau)W_i(\tau)\,d\tau\right]^{1/2}\left[\int_{s_i}^t\dot W_i'(\tau)\dot W_i(\tau)\,d\tau\right]^{1/2}$$
$$<\left[\int_{s_i}^{s_i+\delta}W_i'(\tau)W_i(\tau)\,d\tau\right]^{1/2}\left[\int_{s_i}^{s_i+\delta}x'V(t)V'(t)x\,dt\right]^{1/2}$$
$$<\sqrt{\epsilon_i}\,K$$

for some K, independent of i. Therefore,

$$x'\int_{s_i}^{s_i+\delta}\int_{s_i}^t V(\tau)\,d\tau\int_{s_i}^t V'(\tau)\,d\tau\,dt\,x=\int_{s_i}^{s_i+\delta}W_i'(t)W_i(t)\,dt<\sqrt{\epsilon_i}\,K'$$

for some K' independent of i. Consequently, the left inequality of (3.5) must fail for the particular δ chosen; however, δ is arbitrary, and so the inequality fails for all δ. $\nabla\nabla$

We shall argue the second conclusion above in outline only. Suppose that (3.2) holds for some V with $\dot V$ bounded (as indeed it can). Replace V by $\hat V=V\,\mathrm{sgn}(\cos e^t)$. Thus, $\hat V$ is V switched in sign more and more rapidly. Since $VV'=\hat V\hat V'$, (3.2) still holds for $\hat V$. On the other hand, the switchings of $\hat V$ mean that the positive lower bound in (3.5) will fail for large enough s.

Finally, we shall discuss the situation when (3.2) and (3.5) are equivalent. We use an idea employed by Yuan and Wonham [14].

Let C_Δ be a set $\{t_i\}$ of points in $[0,\infty)$ for which there exists a Δ such that for any $t_i, t_j \in C_\Delta$ with $t_i \neq t_j$, one has $|t_i - t_j| \geq \Delta$. Thus, C_Δ comprises points spaced at least Δ apart. Denote by $\mathcal{V}$ the set of real functions $v(\cdot)$ on $[0,\infty)$ such that for each $v(\cdot) \in \mathcal{V}$ *there corresponds* some Δ and some C_Δ such that

1) $v(t)$ and $\dot{v}(t)$ are continuous and bounded on $\{[0,\infty) - C_\Delta\}$.

2) $v(t)$ and $\dot{v}(t)$ have finite limits as $t \downarrow t_i$ and $t \uparrow t_i$, $t_i \in C_\Delta$.

Think of functions in $\mathcal{V}$ as being smooth enough to have bounded continuous derivatives, save that a countable number of finite-step switchings are allowed, which cannot occur too frequently. An important subclass of $\mathcal{V}$ is the class of linear combinations of a finite number of sinusoids. As preparation for the main result relating (3.2) and (3.5), we introduce a lemma and corollary.

Lemma 3: Let $f:[a,b] \to R$ be a C^2 function on $[a,b]$, with $|f| < d_0$ and $|\ddot{f}| < d_2$ on $[a,b]$. Suppose that $(2d_0/d_2)^{1/2} \leq \frac{1}{2}(b-a)$. Then

$$|\dot{f}| < 2(2d_0 d_2)^{1/2} \text{ on } [a,b].$$

Proof: Choose $t_1 < t_2 \in [a,b]$ and observe that for some $t_3 \in [t_1, t_2]$,

$$f(t_2) = f(t_1) + (t_2 - t_1)\dot{f}(t_3)$$

whence

$$|\dot{f}(t_3)| < \frac{2d_0}{t_2 - t_1}.$$

Also,

$$|\dot{f}(t_1) - \dot{f}(t_3)| \leq \int_{t_1}^{t_3} |\ddot{f}(t)|\, dt < (t_2 - t_1) d_2$$

so that

$$|\dot{f}(t_1)| < \frac{2d_0}{t_2 - t_1} + (t_2 - t_1) d_2.$$

Select $t_1 \in [a, (a+b)/2]$ but otherwise arbitrary and $t_2 - t_1 = (2d_0/d_2)^{1/2}$ to yield

$$|\dot{f}(t_1)| < 2(2d_0 d_2)^{1/2}.$$

A minor variation yields the same inequality for $|\dot{f}(t_2)|$ with $t_2 \in [(a+b)/2, b]$ and the result follows. $\nabla\nabla\nabla$

With the above definition of $\mathcal{V}$, we then have a simple extension to the lemma.

Corollary 1: Let $f \in \mathcal{V}$, let $|f| < d_0$ on $[a,b]$ and $|\ddot{f}| < d_2$ on $[a,b] - C_\Delta$. Suppose that $(2d_0/d_2)^{1/2} \leq \frac{1}{2}\Delta$. Then

$$|\dot{f}| < 2(2d_0 d_2)^{1/2} \qquad \text{on } [a,b] - C_\Delta.$$

Proof: Apply the lemma to intervals of length $\Delta \in [a,b]$ not containing as an interior point any point of C_Δ. $\nabla\nabla$

Proposition 2: With $V(\cdot)$ as defined in Theorem 1, suppose also that entries of $V(\cdot)$ lie in $\mathcal{V}$. Then (3.2) and (3.5) are equivalent.

Proof: In view of Proposition 1, we need only show that failure of (3.5) implies failure of (3.2). Suppose that for arbitrary δ and an arbitrary monotone decreasing sequence of positive ϵ_i with $\epsilon_i \to 0$ as $i \to \infty$, there exists x with $\|x\| = 1$ and corresponding s_i such that

$$x' \int_{s_i}^{s_i+\delta} \int_{s_i}^{t} V(\tau)\, d\tau \int_{s_i}^{t} V'(\tau)\, d\tau\, dt\, x < \epsilon_i.$$

Set

$$W_i(t) = \int_{s_i}^{t} V'(\tau)\, d\tau\, x.$$

Then, as in Proposition 1, for $t \in [s_i, s_i + \delta]$ we have that

$$\frac{1}{2} W_i'(t) W_i(t) \leq \left[\int_{s_i}^{s_i+\delta} W_i'(\tau) W_i(\tau)\, d\tau \right]^{1/2} \left[\int_{s_i}^{s_i+\delta} x' V(t) V'(t) x\, dt \right]^{1/2}.$$

Because entries of $V(\cdot)$ lie in $\mathcal{V}$, $V(\cdot)$ is bounded. Therefore

$$\frac{1}{2} W_i'(t) W_i(t) < \sqrt{\epsilon_i}\, K$$

for some K, independent of i. Now apply a trivial vector generalization of Corollary 1, choosing i large enough that the bound on $(2d_0/d_2)^{1/2}$ in the corollary is met. There results

$$\|V'(t)x\| = \|\dot{W}_i(t)\| < \epsilon_i^{1/8} K'$$

for some K', independent of i and as a consequence

$$x' \int_{s_i}^{s_i+\delta} V(t) V'(t)\, dt\, x < \epsilon_i^{1/4} K''$$

for some K'' independent of i. Therefore, (3.2) fails for the value of δ selected. However, δ is arbitrary, so that (3.2) fails for any δ. $\nabla\nabla$

Using Proposition 2 in conjunction with Theorem 2, we have the following theorem.

Theorem 3: Assume the same hypotheses as in Theorem 2, and also that entries of $V(\cdot)$ lie in $\mathcal{V}$. A necessary and sufficient condition for the exponential stability of (3.4) is that (3.2) hold, i.e., $[0, V]$ be uniformly completely observable.

The last results we obtain concern the equation

$$\dot{x} = \begin{bmatrix} -D \otimes VV' & -C' \otimes V \\ B \otimes V' & A \end{bmatrix} x \tag{3.9}$$

which also finds application in identification problems, see Section A-V of the Appendix.

Theorem 4: Consider (3.9) in which $V: R_+ \to R^n$ is a regulated vector function satisfying (3.3) and $\{A, B, C, D\}$ is a quadruple of constant matrices defining a minimal realization of a transfer function matrix $Z(s) = D + C'(sI - A)^{-1}B$ with $Z(s - \sigma)$ positive real for some $\sigma > 0$, nonsingular almost everywhere, and with $D = D'$. Then (3.9) is exponentially stable if and only if for some positive α_1' and δ and for all $s \in R_+$,

$$\alpha_1' I \leq \int_{s_i}^{s_i+\delta} \left[2D \otimes VV' + B'B \otimes \int_s^t V\, d\tau \int_s^t V'\, d\tau \right] dt. \tag{3.10}$$

A necessary condition is that (3.2) hold, and if entries of V lie in $\mathcal{V}$, this condition is also sufficient.

Proof: The positive real condition implies [15] the existence of a positive definite symmetric P and a matrix L such that $PA + A'P = -2\sigma P - LL'$ and $PB = C - LD^{1/2}$. With the aid of an inessential coordinate basis change in (3.9) and of the state-space, we can assume without loss of generality that $P = I$. Then one can check that

$$\begin{bmatrix} -D \otimes VV' & -C' \otimes V \\ B \otimes V' & A \end{bmatrix} + \begin{bmatrix} -D \otimes VV' & -C' \otimes V \\ B \otimes V' & A \end{bmatrix}'$$
$$= -\begin{bmatrix} \sqrt{2}\, D^{1/2} \otimes V & 0 \\ L & \sigma^{1/2} I \end{bmatrix} \begin{bmatrix} \sqrt{2}\, D^{1/2} \otimes V & 0 \\ L & \sigma^{1/2} I \end{bmatrix}'.$$

Consequently, (3.9) is exponentially stable if and only if the pair

$$\begin{bmatrix} -D \otimes VV' & -C' \otimes V \\ B \otimes V' & A \end{bmatrix}, \begin{bmatrix} \sqrt{2}\, D^{1/2} \otimes V & 0 \\ L & \sigma^{1/2} I \end{bmatrix}$$

is uniformly completely observable. Now by a minor extension of Lemma 1, $[F, H]$ is uniformly completely observable if and only if $[F + KHR, HR]$ has this property where R is a constant nonsingular matrix and K is as in Lemma 1. The choices

$$R = \begin{bmatrix} I & 0 \\ -\sigma^{-1/2} L & \sigma^{-1/2} \end{bmatrix} \quad K = \begin{bmatrix} \frac{1}{\sqrt{2}} D^{1/2} \otimes V & C' \otimes V \\ 0 & -A \end{bmatrix}$$

lead to the conclusion that (3.9) is exponentially stable if and only if the following pair is uniformly completely observable:

$$\begin{bmatrix} 0 & 0 \\ B\otimes V' & 0 \end{bmatrix}, \begin{bmatrix} \sqrt{2}\,D^{1/2}\otimes V & 0 \\ 0 & I \end{bmatrix},$$

i.e., if and only if for some positive α_1 and δ and all s,

$$\alpha_1 I \leqslant \int_s^{s+\delta} \begin{bmatrix} \sqrt{2}\,D^{1/2}\otimes V & B'\otimes\int_s^t V\,d\tau \\ 0 & I \end{bmatrix}\begin{bmatrix} \sqrt{2}\,D^{1/2}\otimes V' & 0 \\ B\otimes\int_s^t V'\,d\tau & I \end{bmatrix} dt. \tag{3.11}$$

(Equation (3.3) holding by assumption, ensures that the other requirements for uniform complete observability hold.)

Suppose that for arbitrary positive δ and an arbitrary monotone decreasing sequence of positive ϵ_i with $\epsilon_i \to 0$ as $i\to\infty$, there exists $x=[x_1'\ x_2']'$ with $\|x\|=1$ and corresponding s_i such that

$$x'\int_{s_i}^{s_i+\delta} \begin{bmatrix} \sqrt{2}\,D^{1/2}\otimes V & B'\otimes\int_s^t V\,d\tau \\ 0 & I \end{bmatrix}\begin{bmatrix} \sqrt{2}\,D^{1/2}\otimes V' & 0 \\ B\otimes\int_s^t V'\,d\tau & I \end{bmatrix} dt\, x < \epsilon_i$$

whence

$$x_1'\int_{s_i}^{s_i+\delta} 2D\otimes VV'\,dt\, x_1 < \epsilon_i$$

and

$$[x_1'\ \ x_2']\int_{s_i}^{s_i+\delta} \begin{bmatrix} B'\otimes\int_s^t V\,d\tau \\ I \end{bmatrix}\begin{bmatrix} B\otimes\int_s^t V'\,d\tau & I \end{bmatrix} dt \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < \epsilon_i.$$

By an argument like that used in proving Theorem 2, it follows that $x_2=0$ and

$$x_1'\int_{s_i}^{s_i+\delta}\left[2D\otimes VV' + B'B\otimes\int_s^t V\,d\tau\int_s^t V'\,d\tau\right] dt\, x_1 < \epsilon_i,$$

i.e., there does not exist positive α_1' and δ such that for all $s\in R_+$

$$\alpha_1' I \leqslant \int_{s_i}^{s_i+\delta}\left[2D\otimes VV' + B'B\otimes\int_s^t V\,d\tau\int_s^t V'\,d\tau\right] dt. \tag{3.10}$$

In summary, failure of (3.11) implies failure of (3.10). It is trivial that (3.11) implies (3.10). Therefore, (3.10) and (3.11) are equivalent, and the first part of the theorem is proved.

A straightforward extension of the argument used to prove Proposition 1 establishes that (3.2) is necessary for (3.10) to hold.

To show that (3.2) is sufficient in case entries of V lie in $\mathcal{V}$, we note that a straightforward extension of the argument used to prove Proposition 2 will show that if (3.10) fails, then the following inequality fails, for α_1'' an arbitrary positive constant and all $s\in R_+$:

$$\alpha_1'' I \leqslant \int_s^{s+\delta}[2D\otimes VV' + KB'B\otimes VV']\,dt$$

for a certain constant $K>0$. Now because $Z(s)$ is nonsingular almost everywhere, $2D+KB'B$ is nonsingular, and underbounded by βI for some positive β. This implies that $2D\otimes VV' + KB'B\otimes VV' \geqslant \beta I\otimes VV'$, and so the following inequality fails for arbitrary positive α_1 and all $s\in R_+$:

$$\alpha_1 I \leqslant \int_s^{s+\delta}[I\otimes VV']\,dt.$$

Equivalently, (3.2) fails. Therefore, (3.2) implies (3.10). ∇∇

IV. Conclusions

We have shown how the concept of uniform complete observability, including various consequences thereof, can be used to demonstrate the exponential stability of certain linear time-varying equations. Of interest is the simplification of all sets of necessary and sufficient conditions for exponential stability made possible by constraining a function to "not switch too often;" functions which are linear combinations of a finite number of sinusoids are included within the constraint class. In the identification context, such constraints are normally physically reasonable.

Appendix
Background Material on Adaptive Identification

In this Appendix, we attempt to put into perspective the differential equations and uniform observability conditions arising in the body of the paper. Derivations are truncated or omitted.

A-I Origin of (3.1)

Let a stable plant transfer function be $\Pi(s)=\sum_{i=1}^n b_i s^{i-1}(s^n+\sum_{i=1}^n a_i s^{i-1})^{-1}$ in which n is assumed known, the a_i and b_i unknown, and there are no pole zero cancellations. Measurements of the plant input and output are available, and the task is to find the a_i, b_i. One procedure for doing this, which encompasses the ideas of, e.g., [5], is as follows. Let F be a fixed an $n\times n$ matrix, arbitrary save that $\mathrm{Re}[\lambda_i(F)]<0$ for all i, and g a fixed n-vector, arbitrary save that $[F,g]$ is completely controllable. One seeks n-vectors k_1,k_2 such that $\Pi(s)=k_2'(sI-F)^{-1}g[1-k_1'(sI-F)^{-1}g]$; this is precisely equivalent to determining the a_i (though it might not seem so). Let $u_p(\cdot),y_p(\cdot)$ be the plant input and output. Define a "model" [driven by $u_p(\cdot)$ and $y_p(\cdot)$ and with output $y_m(\cdot)$] by

$$\begin{aligned} \dot v_1 &= Fv_1 + gy_p \\ \dot v_2 &= Fv_2 + gu_p \\ y_m &= l_1'(t)v_1 + l_2'(t)v_2. \end{aligned} \tag{A1}$$

Here, $l_1(t)$ and $l_2(t)$ are time-varying gains. If $l_1(t)=k_1$, $l_2(t)=k_2$ for all t, then one can show (it is not obvious) that $y_m(t)=y_p(t)$ for all t and $u_p(\cdot)$, after initial condition effects die away. The aim of adaptive identification is to cause $l_1(t)\to k_1$, $l_2(t)\to k_2$ as $t\to\infty$. To this end, one forms an adjustment law for the $l_2(t)$ based on the error $y_m(t)-y_p(t)$:

$$\begin{bmatrix} \dot l_1(t) \\ \dot l_2(t) \end{bmatrix} = -\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}[y_m(t)-y_p(t)]. \tag{A2}$$

(Scaling gains are permitted, but omitted for clarity.)

One can show that this equation may be rewritten as

$$\begin{bmatrix} \dot l_1(t) \\ \dot l_2(t) \end{bmatrix} = -\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}[v_1'(t)\ \ v_2'(t)]\begin{bmatrix} l_1(t)-k_1 \\ l_2(t)-k_2 \end{bmatrix} + w(t) \tag{A3}$$

where $w(t)\to 0$ exponentially fast. Letting $x_1(t)=l_1(t)-k_1$, $x_2(t)=l_2(t)-k_2$ we see that the equation has the form

$$\dot x = -VV'x + w. \tag{A4}$$

Given exponential stability of $\dot x = -VV'x$, exponential stability of this equation is immediate. A scalar plant leads to vector $V(\cdot)$ and a multivariable plant (not discussed here) to matrix $V(\cdot)$.

A-II The Observability Condition (3.2)

In the context of the adaptive identification problem, the observability condition (3.2) is termed a *persistently exciting* condition. In order that adaptive identification of the k_i be possible or equivalently that $x(t)\to 0$, there must be no pole zero cancellation (else the k_i are not unique), and the input $u_p(\cdot)$ must have sufficient complexity. (If it were a pure sine wave for example, one could hope to identify only the magnitude and phase of the plant transfer function at a single frequency and not all its parameters.) Also, the input must retain this property for all time. If these conditions, intuitively reasonable for adaptive identification, are fulfilled, then the lower bound in (3.2) holds, while the upper bound reflects boundedness of $u_p(\cdot)$.

A common procedure to ensure fulfillment of those requirements is to take $u_p(\cdot)$ to be a finite sum of sinusoids or periodic signals. In this way, $u_p(\cdot)$ is periodic, or almost periodic, and if there are sufficient different frequencies within $u_p(\cdot)$, the persistently exciting condition holds for $V(\cdot)$.

A-III Origin of (3.4)

An alternative approach to the above (useful because, as it turns out, integrators are saved) is developed in, e.g., [4] and [9]. The model, this time, partly in Laplace transform notation and neglecting the transform of exponentially decaying quantities, is

$$V_1(s)=\frac{1}{s^{n-1}+\sum_{i=1}^{n-1}\beta_i s^{i-1}}\begin{bmatrix}1\\ s\\ \cdot\\ \cdot\\ s^{n-1}\end{bmatrix}Y_p(s)$$

$$V_2(s)=\frac{1}{s^{n-1}+\sum_{i=1}^{n-1}\beta_i s^{i-1}}\begin{bmatrix}1\\ s\\ \cdot\\ \cdot\\ s^{n-1}\end{bmatrix}U_p(s) \tag{A5}$$

$$w_m(t)=l_1'(t)v_1(t)+l_2(t)v_2(t)$$

$$Y_m(s)=B'(sI-A)^{-1}BW_m(s), \qquad A+A'=-I.$$

One can show that $Y_m(s)=Y_p(s)$ if and only if $l_1(t)=\tilde{k}_1$, $l_2(t)=\tilde{k}_2$ for two constant n-vectors $\tilde{k}_1,\tilde{k}_2$ determined by and determining the plant transfer function. The task therefore is to ensure that $l_i(t)\to k_i$ as $t\to\infty$. One still adjusts $l_1(t)$, $l_2(t)$ using the error $y_m(t)-y_p(t)$:

$$\begin{bmatrix}\dot{l}_1\\ \dot{l}_2\end{bmatrix}=-\begin{bmatrix}v_1(t)\\ v_2(t)\end{bmatrix}\left[y_m(t)-y_p(t)\right] \tag{A6}$$

although the error $y_m(\cdot)-y_p(\cdot)$ is not formed in the same way as before. By taking

$$x=\begin{bmatrix}l_1(t)-\tilde{k}_1\\ l_2(t)-\tilde{k}_2\\ x_2\end{bmatrix}$$

where $\dot{x}_2=Ax_2+B[w_m(t)-\tilde{k}_1'v_1(t)-\tilde{k}_2'v_2(t)]$, (3.4) follows, other than for an additive, exponentially decaying term.

A-IV The Observability Condition (3.5)

The remarks concerning (3.2) apply of course, but there is additional intuition regarding the need for the integral in (3.5). In forming the error $y_m(t)-y_p(t)$ which is used for adjusting the $l_i(\cdot)$ in (A6), the $v_i(t)$ are integrated [see last equation in (A5)] in the second scheme, while no integration occurs in the first scheme [see last equation in (A1)]. The persistently exciting condition is required of the integrated $v_i(\cdot)$.

Proposition 1 states that if a persistently exciting condition is absent, it cannot be regained by integration, while Proposition 2 states that if it is present, and if $V(\cdot)\in\mathcal{V}$, then it is retained by integration. Requiring $V(\cdot)\in\mathcal{V}$ is equivalent to not allowing $V(\cdot)$, as time evolves, to contain less and less low frequency content. Since the effect of integration is to cut down high frequency content, taking $V(\cdot)\in\mathcal{V}$ therefore ensures that the integral of $V(\cdot)$ does not die away as $t\to\infty$.

A-V Origin of (3.9)

The thinking is much as for the origin of (3.4), save that instead of having $Y_m(s)=B'(sI-A)^{-1}BW_m(s)$ where the constraint $A+A'=-I$ forces $B'(sI-A)^{-1}B$ to be positive real one allows $Y_m(s)=Z(s)W_m(s)$ where $Z(s)$ is positive real (in a strict sense described in Theorem 4). Equation (3.9) is thus a generalization of (3.4).

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