Parametric Solution of the Stable Exact Model Matching Problem

B. D. O. ANDERSON AND R. W. SCOTT

Abstract—**The set of all stable solutions to the exact model matching problem is presented in parametric form. The result is obtained simply by means of a Smith–McMillan form constructed over the principal ideal domain of stable, proper, rational transfer functions.**

Manuscript received June 11, 1976. This work was supported by the Australian Research Grants Committee.

The authors are with the Department of Electrical Engineering, University of Newcastle, N.S.W., Australia.

I. Introduction

In recent years several workers have turned their attention to solving a problem which has become known as the minimal design problem [1]–[5]. The problem can be stated as follows: given a $p\times m$ rational transfer matrix $T_1(s)$ of rank p and a $p\times q$ rational transfer matrix $T_2(s)$, find an $m\times q$ proper rational transfer matrix $T(s)$ of minimal McMillan degree (or dynamic order) such that

$$T_1(s)T(s)=T_2(s). \tag{1.1}$$

This problem has been solved in various ways [1]–[5]. It is useful in that problems such as the minimal order inverse system problem [1], [2], the minimal order dynamic observer problem [2], [5], and the model following problem [5], [6], [8] can be reformulated as minimal design problems. It should be noted that if $p\geqslant m$ the minimal design problem either has no solution or a unique solution which can easily be found. Normally then, $p<m$.

Subsequently, attention has been turned to solving the minimal design problem with the further constraint that the solution be stable. Some progress has been made toward solving this problem [4], [5] but as yet a complete solution is not available.

If the minimality constraint is relaxed, then (1.1) represents the exact model matching problem [7]–[9]. It is of interest to search for a stable solution to this problem also [5].

This note deals with the latter problem. A parametric representation is obtained of the class of solutions to the stable exact model matching problem.

II. The Stable Exact Model Matching Problem

The set of rational, stable, proper functions form a principal ideal domain D with units $m(s)/n(s)$ where $\deg[m(s)]=\deg[n(s)]$ and $m(s),n(s)$ are stable polynomials [10], [11]. So a rational (but possibly unstable and improper) matrix $T_1(s)$ can be reduced to Smith–McMillan form $\Lambda_1(s)$ [10]. That is,

$$T_1(s)=A_1(s)\Lambda_1(s)B_1(s) \tag{2.1}$$

where $A_1(s),B_1(s)$ are unimodular matrices over D; thus, A_1, B_1, A_1^{-1}, and B_1^{-1} have rational, stable, proper entries. Also, $\Lambda_1(s)$ is a diagonal matrix which is rational but not necessarily stable or proper.

Now consider the exact model matching problem (1.1). Using (2.1), one has

$$\Lambda_1(s)\mathcal{T}(s)=A_1^{-1}(s)T_2(s) \tag{2.2}$$

where $\mathcal{T}(s)$ is defined as

$$\mathcal{T}(s)\triangleq B_1(s)T(s). \tag{2.3}$$

Note that since $B_1(s)$ and its inverse are stable and proper, $\mathcal{T}(s)$ is stable and proper if and only if $T(s)$ is stable and proper.

With $\Lambda_1(s)=[\operatorname{diag}\lambda_i(s)\;\vdots\;0]$ for known scalar λ_i, with k_{ij} the known entries of $A_1^{-1}(s)T_2(s)$, and with t_{ij} the unknown entries of $\mathcal{T}(s)$, (2.2) implies

$$t_{ij}=\frac{k_{ij}}{\lambda_i},\qquad 1\leqslant i\leqslant p,\quad 1\leqslant j\leqslant q. \tag{2.4}$$

That is, the entries in the upper p rows of $\mathcal{T}(s)$ [denoted by $\mathcal{T}_1(s)$] are fixed and hence a necessary condition that there exists a stable proper solution to (1.1) is that $\mathcal{T}_1(s)$ be stable and proper. Consider now the lower $m-p$ rows of $\mathcal{T}(s)$ [denoted by $\mathcal{T}_2(s)$]. In any solution of (1.1) $\mathcal{T}_2(s)$ is free to take on any value, so any stable proper matrix $\mathcal{T}_2(s)$ leads to a stable, proper solution of (1.1), and all stable, proper solutions of (1.1) will be obtained by letting $\mathcal{T}_2$ range over the set of stable proper matrices.

Partitioning $B_1^{-1}(s)$ as $[B_{11}(s)\;\vdots\;B_{12}(s)]$, where $B_{11}(s)$ is an $m\times p$ proper, stable, rational matrix and $B_{12}(s)$ is an $m\times(m-p)$ proper, stable rational matrix, leads to

$$T(s)=B_{11}(s)\mathcal{T}_1(s)+B_{12}(s)\mathcal{T}_2(s) \tag{2.5}$$

where the only unknown elements occur in $\mathcal{T}_2(s)$. Thus, the stable solutions of the exact model matching problem, if any exists, can be parameterized as

$$T(s)=K_1(s)+K_2(s)\mathcal{T}_2(s) \tag{2.6}$$

where the $K_i(s)$ are known, proper, stable, rational matrices and $\mathcal{T}_2(s)$ is any proper, stable rational matrix.

III. Conclusion

Having determined the set of all stable solutions to the exact model matching problem (2.6) the next natural step would be to find those values of the parameters which minimize the McMillan degree. We would then have a solution to the stable minimal design problem. The task of deciding whether there is a stable $\mathcal{T}_2$ causing T to have a certain McMillan degree is evidently one which, in formal terms at least, can be solved by decision algebra [12].

References

[1] S. H. Wang and E. J. Davidson, "A minimization algorithm for the design of linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 220–225, June 1973.

[2] ——, "A minimization algorithm for the design of linear multivariable systems," in *Proc.* 1973 *Joint Automatic Control Conf.*, pp. 269–277.

[3] A. S. Morse, "Minimal solutions to transfer matrix equations," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 131–133, Feb. 1976.

[4] W. A. Wolovich, "On the stability of the minimal design problem," in *Proc.* 1975 *Joint Automat. Control Conf.*, pp. 215–216.

[5] W. A. Wolovich, P. Antsaklis, and H. Elliott, "On the stability of solutions to the minimal and nonminimal design problems," to be published.

[6] A. S. Morse, "Structure and design of linear model following systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 346–354, Aug. 1973.

[7] W. A. Wolovich, "The use of state feedback for exact model matching," *SIAM J. Contr.*, vol. 10, pp. 512–523, Aug. 1972.

[8] ——, *Linear Multivariable Systems*. New York: Springer 1974.

[9] R. Rutman and Y. Shamash, "Design of multivariable systems via polynomial equations," *Int. J. Contr.*, vol. 22, pp. 729–737, Nov. 1975.

[10] G. D. Forney, "Convolutional codes—Part I: Algebraic structure," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 720–738, Nov. 1970.

[11] A. S. Morse, "System invariants under feedback and cascade control," to be published.

[12] B. D. O. Anderson, N. K. Bose, and E. I. Jury, "Output feedback stabilization and related problems—solution via decision methods," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 53–66, Feb. 1975.