Matrix Inequality Solution to Linear-Quadratic Singular Control Problems

DAVID J. CLEMENTS, MEMBER, IEEE,
BRIAN D. O. ANDERSON, FELLOW, IEEE, AND PETER J. MOYLAN, MEMBER, IEEE

Abstract—**The existence of a solution to a linear-quadratic singular control problem is equivalent to the existence of a solution to a certain matrix inequality. This paper studies an approach to solving the inequality, and identifies the maximal solution of the inequality as defining the performance index infimum for the control problem.**

I. INTRODUCTION

In a companion paper [1] we presented a transformation procedure for solving a general singular linear-quadratic control problem. There the approach was to work directly with the cost and dynamics and, via a series of transformations, to reduce the problem to solving a linear-quadratic problem of lower state and/or control space dimension, and, in addition, either being nonsingular or having zero state-space dimension or having zero control space dimension.

Here, we again derive this transformation procedure but the method differs from that in [1] in that we first find necessary and sufficient conditions, involving the existence of a matrix $P(\cdot)$ and the satisfaction of a matrix inequality by $P(\cdot)$ on the interval of interest, for the existence of a solution to the control problem. (These necessary and sufficient conditions involving the P matrix are the appropriate generalization of the well-known necessary and sufficient condition for the nonsingular control problem to have a solution, viz. that the associated Riccati equations have no escape times on the interval of interest.) In this paper, our approach is to look for a solution matrix $P(\cdot)$ of the matrix inequality as the first step in solving the control problem.

We consider the cost

$$V[x_0,u(\cdot)]=x'(T)Sx(T)+\int_{t_0}^{T}q(x(t),u(t),t)\,dt \tag{1}$$

where $q(x(t),u(t),t)=x'(t)Qx(t)+2x'(t)Hu(t)+u'(t)Ru(t)$, and the dynamics

$$\dot{x}(t)=Fx(t)+Gu(t), \qquad x(t_0)=x_0. \tag{2}$$

The state x has dimension n, the control u has dimension m, and all matrices are of consistent dimensions. The matrices F, G, Q, H, and R are assumed continuous, S is constant and, without loss of generality, S, Q, and R are assumed symmetric. The controls $u(\cdot)$ are piecewise continuous m-vector functions on $[t_0,T]$.

The linear-quadratic control problem is first, to find necessary and sufficient conditions for $V^*[x_0]=\inf_{u(\cdot)}V[x_0,u(\cdot)]$ to be finite for all x_0 and second, to determine a control $u^*(\cdot)$ which achieves $V^*[x_0]$.

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D. J. Clements was with the Department of Electrical Engineering, University of Newcastle, N.S.W., Australia. He is now with the School of Electrical Engineering, University of New South Wales, Kensington, N.S.W., Australia.

B. D. O. Anderson and P. J. Moylan are with the Department of Electrical Engineering, University of Newcastle, N.S.W., Australia.

To date, the search for necessary and sufficient conditions involving a matrix inequality has centered on the nonnegativity problem, i.e., $V[0,u(\cdot)]$ is required to be nonnegative for each $u(\cdot)$, rather than the control problem. Theorem 1 below summarizes the known results in this regard. First, however, it is convenient to introduce some new notation. Let $P(t)$ be an $n\times n$ matrix, symmetric and of bounded variation. Define the matrix differential

$$dM(P)=\begin{bmatrix} dP+(PF+F'P+Q)\,dt & (PG+H)\,dt \\ (PG+H)'\,dt & R\,dt\end{bmatrix}.$$

Theorem 1 [2]: Let F, G, H, Q, and R be continuous on $[t_0,T]$. Suppose (2) is controllable on $[t_0,\tau]$ for each $\tau\in(t_0,T]$; that is,

$$\int_{t_0}^{\tau}\Phi(\tau,\sigma)G(\sigma)G'(\sigma)\Phi'(\tau,\sigma)\,d\sigma>0, \qquad \text{for all } \tau\in(t_0,T] \tag{3}$$

where $\Phi(\cdot,\cdot)$ is the transition matrix associated with (2).

1) A necessary condition for $V[0,u(\cdot)]\geq 0$ for each $u(\cdot)$ is that there exists an $n\times n$ matrix $P(t)$, symmetric and of bounded variation on each interval $[t_1,t_2]\subset(t_0,T]$, such that $P(T)\leq S$ and such that

$$\int_{t_1}^{t_2}[x' \quad u']\,dM(P)\begin{bmatrix} x\\ u\end{bmatrix}\geq 0 \tag{4}$$

for each $[t_1,t_2]\subset(t_0,T]$, each $u(\cdot)$, and each $x(t_1)$. [In (4), $x(t)$ is defined from $\dot{x}=Fx+Gu$, and knowledge of $u(\cdot)$ and $x(t_1)$.]

2) A sufficient condition is that there exists an $n\times n$ matrix $P(t)$, symmetric and of bounded variation on $[t_0,T]$, with $P(T)\leq S$ and with (4) holding for all $[t_1,t_2]\subseteq[t_0,T]$. (The controllability assumption (3) is not required for sufficiency.)

Several points should be noted. First, this theorem is purely an existence theorem. It gives no insight into how any matrix P satisfying the theorem might be computed. Second, it is not clear what the connection is, if any, to the linear-quadratic control problem. Third, there is evidently a gap between the necessary and sufficient conditions, and there are examples where the necessary condition holds, but not the sufficient condition. (Interestingly, in studying the case of free $x(t_0)$, it will turn out that the gap between the necessary and sufficient conditions disappears.)

A number of proofs of Theorem 1, or closely related theorems, have appeared in the literature [2]–[7]. Basically, these proofs can be divided into two classes. In [3]–[5] the singular problem is considered as the limit of a sequence of nonsingular problems while in [2] the proof proceeds by establishing that the performance index is of a quadratic nature irrespective of the singularity or nonsingularity of the problem. The contribution of [6] is to extend an earlier proof available only for the nonsingular ($R>0$) and totally singular ($R=0$) cases to the partially singular case. In [7], a time-invariant version appears.

The problem of constructing the P matrix in (4) is central to the problems of covariance factorization and time-varying passive network synthesis. It was in the latter context that an algorithm suitable for the stationary case was developed [8], and then it was recognized that this algorithm with variations was also applicable to the time-varying synthesis problem [5], and with other variations to the covariance factorization problem [4]. An algorithm was in fact suggested in [4] for finding a P matrix satisfying (4) under additional differentiability and constancy of rank assumptions. In this paper, we show that the Anderson–Moylan algorithm is precisely Kelley's transformation [1] executed in a particular coordinate basis and in showing this, we derive the generalized Legendre–Clebsch conditions in a reasonably straightforward manner.

In connection with the optimal control problem, the Anderson–Moylan algorithm, considered in isolation from the Kelley transformation procedure, can be shown to yield the optimal performance index. Linking it with the Kelley transformation procedure yields the optimal controls as well.

An outline of the paper is as follows. In Section II we adjust the result of Theorem 1 to provide necessary and sufficient conditions for the control, rather than nonnegativity, problem. Further, we point out that although the matrix inequality normally has an infinite number of solutions, there is a maximal solution which defines the performance index infimum. In Section III the problem of solving the matrix inequality is tackled. When the inequality is associated with a singular problem, it proves possible to reduce the dimension of the inequality. A series of such reductions leads either to an inequality associated with a nonsingular or trivial singular problem, and in either case, the inequality is easily solved. Section IV contains concluding remarks.

II. The Matrix Inequality Applied to the Control Problem

In the transformation approach [1] to the solution of the linear-quadratic control problem, the reductions in state and control dimensions and the calculation of the optimal control appear in a reasonably straightforward manner, with computation of the optimal cost completing the solution of the problem. Here we present an alternative derivation of those results employing the Anderson–Moylan algorithm in conjunction with a variation of Theorem 1. By this method manipulations are made on the matrix measure involving P in (4) in order to compute a solution P which we can show to define the performance index; the calculations of the state transformation and optimal control are not part of the main algorithm.

The following theorem gives the connection between the linear-quadratic problem and the type of necessary and sufficient conditions stated in Theorem 1.

Theorem 2: Assume continuity of F, G, H, Q, and R on $[t_0,T]$. Then $V^*[x_0]$ is finite for each x_0 if and only if there exists an $n\times n$ matrix $P(t)$, symmetric and of bounded variation on $[t_0,T]$, such that $P(T)\leqslant S$ and (4) hold for each $[t_1,t_2]\subseteq[t_0,T]$, each $u(\cdot)$, and each $x(t_1)$.

Proof: Suppose first that $V^*[x_0]$ is finite for each x_0. Elementary algebra [9] shows $V^*[x_0]$ will be quadratic[1] if for all $\lambda\in R$, $\xi_1,\xi_2\in R^n$, $V^*[\lambda\xi_1]=\lambda^2V^*[\xi_1]$, $V^*[\xi_1+\xi_2]+V^*[\xi_1-\xi_2]=2V^*[\xi_1]+2V^*[\xi_2]$ and $V^*[\xi_1+\lambda\xi_2]-V^*[\xi_1-\lambda\xi_2]=\lambda V^*[\xi_1+\xi_2]-\lambda V^*[\xi_1-\xi_2]$. The first two equalities follow from the existence of V^* and the linear-quadratic nature of (1) and (2), see [2] and originally [10]. The third equality is trivial for $\lambda=-1$, 0, and $+1$, and if proved for $\lambda>0$, follows immediately for $\lambda<0$. So assume $\lambda>0$, $\lambda\neq 1$. For arbitrary $u_1(\cdot),u_2(\cdot)$ one can check that

$$V(\xi_1+\lambda\xi_2,u_1+\lambda u_2)+\lambda V(\xi_1-\xi_2,u_1-u_2)$$
$$=\lambda V(\xi_1+\xi_2,u_1+u_2)+V(\xi_1-\lambda\xi_2,u_1-\lambda u_2).$$

For arbitrary $\epsilon>0$, choosing u_1,u_2 so that $V(\xi_1+\xi_2,u_1+u_2)\leqslant V^*(\xi_1+\xi_2)+\epsilon$ and $V(\xi_1-\lambda\xi_2,u_1-\lambda\xi_2)\leqslant V^*(\xi_1-\lambda\xi_2)+\epsilon$ leads to

$$V^*(\xi_1+\lambda\xi_2)+V^*(\xi_1-\xi_2)\leqslant\lambda V^*(\xi_1+\xi_2)+V^*(\xi_1-\lambda\xi_2)+(1+\lambda)\epsilon.$$

A similar argument leads to an inequality going the other way, and since ϵ is arbitrary, one obtains $V^*(\xi_1+\lambda\xi_2)+\lambda V^*(\xi_1-\xi_2)=\lambda V^*(\xi_1+\xi_2)+V^*(\xi_1-\lambda\xi_2)$, as required.

It is easily seen that with $V^*[x_0]$ finite for all x_0, then $V^*[x_1,t_1]$ must be finite for all t_1. (Here $V^*[x_1,t_1]$ denotes the infimum of (1) with t_0 replaced by t_1 and $x(t_1)=x_1$.) Therefore, $V^*[x_1,t_1]$ is quadratic in x_1.

Finally, given this quadratic bound, a straightforward adaptation of the proofs of Propositions 2–4 and [2, theorem 2] gives the result.

The proof of the sufficiency is the same as that for Theorem 1-1).

Remarks

1) A proof of this theorem could also be obtained by the same regularization procedures as can be used to prove Theorem 1 (see the discussion following the statement of that theorem). Prior establishment of the quadratic form of $V^*[x_0]$ when this quantity is guaranteed finite is still required.

2) The gap between the necessary conditions and the sufficient conditions of Theorem 1 has been removed in Theorem 2 by assuming the existence of the infimum of $V[x_0,u(\cdot)]$ for all x_0 rather than just $x_0=0$ as in Theorem 1.

[1]Strictly, [9] demands the bilinearity in ξ_1,ξ_2 of $V^*[\xi_1+\xi_2]-V^*[\xi_1-\xi_2]$. The requirement that $\{V^*[\xi_1+\xi_2]-V^*[\xi_1-\xi_2]\}+\{V^*[\xi_1+\xi_3]-V^*[\xi_1-\xi_3]\}=V^*[\xi_1+(\xi_2+\xi_3)]-V^*[\xi_1-(\xi_2-\xi_3)]$, however, follows easily from the equalities stated.

3) In contrast to Theorem 1, no controllability assumption is needed here.

4) The linkage between Theorems 1 and 2 is really very close. For example, it follows easily that if $V[0,u(\cdot)]\geqslant 0$ for all $u(\cdot)$, then $V^*[x_1,t_1]>-\infty$ for all $t_1\in(t_0,T]$ and it follows with some difficulty that if $V^*[x_0]>-\infty$, then there exists $t_{-1}<t_0$ and definitions of $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, $Q(\cdot)$, $R(\cdot)$ on $[t_{-1},t_0]$ linking smoothly at t_0 with the known values of these quantities such that $V[x_{-1}=0,t_{-1},u(\cdot)]\geqslant 0$ for all $u(\cdot)$, and the pair $F(\cdot),G(\cdot)$ is completely controllable on $[t_{-1},\tau]$ for all $\tau>t_{-1}$. (A proof of this latter "extendability" result will be given elsewhere [11].)

In general, there can be many matrices $P(\cdot)$ satisfying (4) with $[t_1,t_2]\subseteq[t_0,T]$. One particular one, however, is readily picked out.

Lemma 1: Suppose that F, G, H, Q, and R are continuous on $[t_0,T]$ and that with notation as defined in the proof of Theorem 2, $V^*[x_0]$ exists for all x_0. Then $V^*[x_1,t_1]=x_1'P^*(t_1)x_1$ for some symmetric $P^*(\cdot)$ of bounded variation on $[t_0,T]$ with $P^*(T)=S$ and satisfying (4) for arbitrary $[t_1,t_2]\subseteq[t_0,T]$, and such that for any other $P(t)$ meeting these conditions we have $P(t)\leqslant P^*(t)$ for each $t\in[t_0,T]$.

Proof: There are three things to prove, the quadratic nature of $V^*[x_1,t_1]$, the satisfaction of (4) by $P^*(\cdot)$, and the maximality property of $P^*(\cdot)$. The first two points are established in [2]. To prove the maximality, we note that (4) implies by an argument set out in [2] that for any $P(\cdot)$ as above, $x_1'P(t_1)x_1\leqslant V[x_1,t_1,u(\cdot)]$ for any $u(\cdot)$, and so $x_1'P(t_1)x_1\leqslant\inf_{u(\cdot)}V[x_1,t_1,u(\cdot)]=V^*[x_1,t_1]=x_1'P^*(t_1)x_1$. Since x_1 is arbitrary, the maximality follows. This completes the proof.

III. Solving the Matrix Inequality

As in [1], we transform the matrices R and G to standard form via a series of coordinate basis changes of the state and control spaces. To do this, we first need to assume various constancy-of-rank and differentiability assumptions to ensure the existence of the required coordinate basis changes. Then, substitution of the various matrices defined with respect to these new bases into (4) will imply that a certain part of the H matrix is zero provided that the control problem has a solution. Finally, with redefinition of R, G, H, and u we can then define an equivalent control problem of lower control space dimension but with R and G in the standard form

$$R=\begin{bmatrix} I_p & 0\\ 0 & 0_{m-p}\end{bmatrix},\quad G=\begin{bmatrix} G_{11} & 0\\ G_{21} & I_{m-p}\end{bmatrix} \tag{5}$$

where p is the rank of R in the original problem. See [1] for more detail of this procedure.

Assume now that the problem is in standard form (5), so that any reduction in the control space dimension has already been carried out. Partition all vectors and matrices consistently with (5) and substitute into (4). We then obtain from (4), with $w'=[x_1'\quad x_2'\quad u_1']$,

$$\int_{t_1}^{t_2}w'dYw+2\int_{t_1}^{t_2}x_1'(P_{12}+H_{12})u_2dt+2\int_{t_1}^{t_2}x_2'(P_{22}+H_{22})u_2dt\geqslant 0 \tag{6}$$

where dY is dM with the last $m-p$ rows and columns deleted. With some straightforward real analysis, we can conclude that

$$P_{12}(t)+H_{12}(t)=0 \qquad \text{on } (t_0,T) \tag{7}$$
$$P_{22}(t)+H_{22}(t)=0 \qquad \text{on } (t_0,T). \tag{8}$$

We have now identified the blocks P_{12} and P_{22} of P uniquely for *any* P satisfying (4); any nonuniqueness can only occur in the P_{11} block. Moreover, P is symmetric on (t_0,T) implying by (8) the symmetry of H_{22} as a necessary condition. Indeed this is a generalized Legendre–Clebsch necessary condition.

It is a simple matter to show that the equalities (7) and (8) extend to the point t_0 in case $P(t)=P^*(t)$, with $P^*(\cdot)$ as defined above. This result depends on the maximality of $P^*(\cdot)$ and the fact that all jumps in $P(\cdot)$ are nonnegative, i.e., $P(t_-)\leqslant P(t)\leqslant P(t_+)$.

A similar study of the right hand end-point T leads to the conditions $S_{22}+H_{22}(T)\geqslant 0$ and $N[S_{22}+H_{22}(T)]\subseteq N[S_{12}+H_{12}(T)]$, where N denotes null space as being necessary for the control problem to have a solution.

Finally, let us return to (6) and set

$$dY=\begin{bmatrix} dY_{11} & dY_{12} & dY_{13} \\ dY_{12}' & dY_{22} & dY_{23} \\ dY_{13}' & dY_{23}' & I \end{bmatrix}$$

where

$$dY_{11}=dP_{11}+(Q_{11}+P_{11}F_{11}+F_{11}'P_{11}+P_{12}F_{21}+F_{21}'P_{12}')\,dt$$
$$dY_{12}=dP_{12}+(Q_{12}+P_{11}F_{12}+P_{12}F_{22}+F_{11}'P_{12}+F_{21}'P_{22})\,dt$$
$$dY_{22}=dP_{22}+(Q_{22}+P_{22}F_{22}+F_{22}'P_{22}+P_{12}'F_{12}+F_{12}'P_{12})\,dt$$
$$dY_{13}=(P_{11}G_{11}+P_{12}G_{21}+H_{11})\,dt$$
$$dY_{23}=(P_{12}'G_{11}+P_{22}G_{21}+H_{21})\,dt.$$

If H_{12} and H_{22} are assumed to be differentiable, we can define hat quantities $\hat{F}$, $\hat{G}$, $\hat{H}$, $\hat{Q}$, and $\hat{R}$ in obvious fashion, similarly to [1], so that dY can be written as

$$dY=d\hat{M}(\hat{P})=\begin{bmatrix} d\hat{P}+(\hat{P}\hat{F}+\hat{F}'\hat{P}+\hat{Q})\,dt & (\hat{P}\hat{G}+\hat{H})\,dt \\ (\hat{P}\hat{G}+\hat{H})'\,dt & \hat{R}\,dt \end{bmatrix} \tag{9}$$

where $\hat{P}=P_{11}$.

We now observe that $d\hat{M}(\hat{P})$ has the same form as $dM(P)$ in (4). We therefore seek a minimization problem of the same form as (1) and (2) corresponding to $d\hat{M}(\hat{P})$. However, given the development of [1] it is clear that the required minimization problem is just that described in Section III of that paper. In summary we have Theorem 3.

Theorem 3: Suppose that R and G are in standard form and that the coefficient matrices are sufficiently differentiable and also satisfy certain constancy-of-rank requirements. Then there exists an $n\times n$ matrix $P(t)$, symmetric and of bounded variation on $[t_0,T]$, such that $P(T)\leqslant S$ and (4) holds if and only if there exists a matrix $\hat{P}(t)$ of appropriate dimension, symmetric, and of bounded variation on $[t_0,T]$ such that

1) $\hat{P}(T)\leqslant\hat{S}$;

2) $\int_{t_1}^{t_2}[\hat{x}'\hat{u}']\,d\hat{M}(\hat{P})\begin{bmatrix}\hat{x}\\ \hat{u}\end{bmatrix}\geqslant 0$, for all $\hat{u}(\cdot)$, all $[t_1,t_2]\subseteq[t_0,T]$ and all $\hat{x}(t_1)$;

3) $H_{22}(t)$ is symmetric on $[t_0,T]$;

4) $S_{22}+H_{22}(T)\geqslant 0$;

5) $N[S_{22}+H_{22}(T)]\subseteq N[S_{12}+H_{12}(T)]$.

Here $\hat{x}$ is assumed to satisfy an equation of the form $\dot{\hat{x}}=\hat{F}\hat{x}+\hat{G}\hat{u}$, the matrix $d\hat{M}(\hat{P})$ depends on $\hat{P}$, $\hat{F}$, $\hat{G}$, $\hat{H}$, $\hat{Q}$, and $\hat{R}$, and these latter matrices have certain definitions in terms of P, F, G, H, Q, and R.

Again, as in [1], repeated applications of Theorem 3 and the reduction to standard form procedure can be made until one of three possibilities obtains. Either there arises a zero dimensional $\hat{P}$ in which case P would be completely and uniquely identified by a series of equalities such as (7) and (8), or one obtains $d\hat{M}(\hat{P})$ with $\hat{P}$ of positive dimension with $\hat{R}$ nonsingular, or in transforming from nonstandard to standard form G and H become zero. For the second case we can show that inequalities 1) and 2) of Theorem 3 have a solution $\hat{P}$ if and only if the associated Riccati equation has no escape times on $[t_0,T]$, moreover, the solution of the Riccati equation is one of many possible solutions of inequalities 1) and 2) of Theorem 3, in fact being the maximal solution of the inequalities. Finally, $\hat{P}$ as calculated from the Riccati equation is connected to the optimal cost via the standard quadratic form. Tracing back to the original control problem, the solution P of (1.12) so generated defines the optimal cost for each $x(t_0)$ for problem (3.1). Similar procedures hold for each of the other two possible terminating problems.

IV. Conclusions

The contributions of the paper are twofold. First, we have tied together the existence of a solution to a linear-quadratic control problem and the existence of a solution to a linear matrix inequality; because there are necessary and sufficient conditions linking the two existence problems, the result is perhaps tidier than that involving the linear-quadratic nonnegativity problem. Also, by involving the notion of finding a performance index infimum, we have exhibited a property of the class of solutions to the matrix inequality, in particular, the existence of a maximal solution.

Second, we have presented a procedure for solving the matrix inequality, or, what is equivalent, for computing the performance index infimum for the control problem. This procedure is computationally the same as that in [1], but the thinking giving rise to it and its justification are quite different.

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