notes null space as being necessary for the control problem to have a solution.

Finally, let us return to (6) and set

$$dY = \begin{bmatrix} dY_{11} & dY_{12} & dY_{13} \\ dY_{12}' & dY_{22} & dY_{23} \\ dY_{13}' & dY_{23}' & I \end{bmatrix}$$

where

$$dY_{11} = dP_{11} + (Q_{11} + P_{11}F_{11} + F_{11}'P_{11} + P_{12}F_{21} + F_{21}'P_{12}')dt$$

$$dY_{12} = dP_{12} + (Q_{12} + P_{11}F_{12} + P_{12}F_{22} + F_{11}'P_{12} + F_{21}'P_{22})dt$$

$$dY_{22} = dP_{22} + (Q_{22} + P_{22}F_{22} + F_{22}'P_{22} + P_{12}'F_{12} + F_{12}'P_{12})dt$$

$$dY_{13} = (P_{11}G_{11} + P_{12}G_{21} + H_{11})dt$$

$$dY_{23} = (P_{12}'G_{11} + P_{22}G_{21} + H_{21})dt.$$

If H_{12} and H_{22} are assumed to be differentiable, we can define hat quantities $\hat F$, $\hat G$, $\hat H$, $\hat Q$, and $\hat R$ in obvious fashion, similarly to [1], so that dY can be written as

$$dY = d\hat M(\hat P) = \begin{bmatrix} d\hat P + (\hat P\hat F + \hat F'\hat P + \hat Q)dt & (\hat P\hat G + \hat H)dt \\ (\hat P\hat G + \hat H)'dt & \hat R dt \end{bmatrix} \tag{9}$$

where $\hat P = P_{11}$.

We now observe that $d\hat M(\hat P)$ has the same form as $dM(P)$ in (4). We therefore seek a minimization problem of the same form as (1) and (2) corresponding to $d\hat M(\hat P)$. However, given the development of [1] it is clear that the required minimization problem is just that described in Section III of that paper. In summary we have Theorem 3.

Theorem 3: Suppose that R and G are in standard form and that the coefficient matrices are sufficiently differentiable and also satisfy certain constancy-of-rank requirements. Then there exists an $n \times n$ matrix $P(t)$, symmetric and of bounded variation on $[t_0, T]$, such that $P(T) \leqslant S$ and (4) holds if and only if there exists a matrix $\hat P(t)$ of appropriate dimension, symmetric, and of bounded variation on $[t_0, T]$ such that

1) $\hat P(T) \leqslant \hat S$;

2) $\int_{t_1}^{t_2} [\hat x' \hat u'] d\hat M(\hat P) \begin{bmatrix} \hat x \\ \hat u \end{bmatrix} \geqslant 0$, for all $\hat u(\cdot)$, all $[t_1, t_2] \subseteq [t_0, T]$ and all $\hat x(t_1)$;

3) $H_{22}(t)$ is symmetric on $[t_0, T]$;

4) $S_{22} + H_{22}(T) \geqslant 0$;

5) $N[S_{22} + H_{22}(T)] \subseteq N[S_{12} + H_{12}(T)]$.

Here $\hat x$ is assumed to satisfy an equation of the form $\dot{\hat x} = \hat F\hat x + \hat G\hat u$, the matrix $d\hat M(\hat P)$ depends on $\hat P$, $\hat F$, $\hat G$, $\hat H$, $\hat Q$, and $\hat R$, and these latter matrices have certain definitions in terms of P, F, G, H, Q, and R.

Again, as in [1], repeated applications of Theorem 3 and the reduction to standard form procedure can be made until one of three possibilities obtains. Either there arises a zero dimensional $\hat P$ in which case P would be completely and uniquely identified by a series of equalities such as (7) and (8), or one obtains $d\hat M(\hat P)$ with $\hat P$ of positive dimension with $\hat R$ nonsingular, or in transforming from nonstandard to standard form G and H become zero. For the second case we can show that inequalities 1) and 2) of Theorem 3 have a solution $\hat P$ if and only if the associated Riccati equation has no escape times on $[t_0, T]$, moreover, the solution of the Riccati equation is one of many possible solutions of inequalities 1) and 2) of Theorem 3, in fact being the maximal solution of the inequalities. Finally, $\hat P$ as calculated from the Riccati equation is connected to the optimal cost via the standard quadratic form. Tracing back to the original control problem, the solution P of (1.12) so generated defines the optimal cost for each $x(t_0)$ for problem (3.1). Similar procedures hold for each of the other two possible terminating problems.

IV. Conclusions

The contributions of the paper are twofold. First, we have tied together the existence of a solution to a linear-quadratic control problem and the existence of a solution to a linear matrix inequality; because there are necessary and sufficient conditions linking the two existence problems, the result is perhaps tidier than that involving the linear-quadratic nonnegativity problem. Also, by involving the notion of finding a performance index infimum, we have exhibited a property of the class of solutions to the matrix inequality, in particular, the existence of a maximal solution.

Second, we have presented a procedure for solving the matrix inequality, or, what is equivalent, for computing the performance index infimum for the control problem. This procedure is computationally the same as that in [1], but the thinking giving rise to it and its justification are quite different.

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Transformational Solution of Singular Linear-Quadratic Control Problems

DAVID J. CLEMENTS, MEMBER, IEEE, AND BRIAN D. O. ANDERSON, FELLOW, IEEE

***Abstract*—General linear-quadratic singular variational problems with free end-point are studied. An algorithm is presented, involving the execution of a sequence of coordinate basis transformations in the state and control space; the algorithm establishes whether a prescribed problem has a solution, determining the optimal control and performance index in case the solution exists.**

I. Introduction

This paper studes the existence and computation of optimal controls in a general linear-quadratic control problem without end-point constraints. A subproblem is to find necessary and sufficient conditions for the nonnegativity of a quadratic cost functional, subject to linear differential equation constraints with zero initial condition; this problem is closely related to the second variation problem of optimal control.

Specifically, in the finite interval $[t_0, T]$ consider the set U of m-vector piecewise continuous functions $u(\cdot)$. Then for each n-vector x_0 and each $u(\cdot)$ in U, define the cost functional

$$V[x_0, u(\cdot)] = x'(T)Sx(T) + \int_{t_0}^{T} q(x(t), u(t), t)\, dt \tag{1}$$

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D. J. Clements is with the School of Electrical Engineering, University of New South Wales, Kensington, N. S. W., Australia.

B. D. O. Anderson is with the Department of Electrical Engineering, University of Newcastle, N. S. W., Australia.

where $x(\cdot)$ is the corresponding n-vector trajectory of

$$\dot{x}(t)=Fx(t)+Gu(t), \qquad x(t_0)=x_0 \tag{2}$$

and where $q(x(t),u(t),t)=x'(t)Qx(t)+2x'(t)Hu(t)+u'(t)Ru(t)$. The matrices Q, H, R, F, and G have dimensions consistent with x and u and are continuous, while S is a constant matrix. Without loss of generality we may also assume S, Q, and R are symmetric.

In the course of this paper, we also require the coefficient matrices and matrices constructed from these to satisfy two classes of assumptions. The first of these involves the degree of differentiability while the second is concerned with constancy of rank on $[t_0,T]$. The details of the assumptions required vary from problem to problem, depending on the number of transformations needed for the implementation of the algorithm to be described in this paper. We therefore make the general assumption that the coefficient matrices have properties sufficient for the carrying out of the algorithm.

We now formally state the two problems of interest. First, we define

$$V^*[x_0]=\inf_{u(\cdot)\in U} V[x_0,u(\cdot)].$$

Problem 1 (Linear-quadratic control): Find necessary and sufficient conditions for $V^*[x_0]$ to be finite for each x_0. Should these conditions hold, compute a control $u^*(\cdot)$ such that $V^*[x_0]$ is achieved with $u^*(\cdot)$.

Problem 2 (Nonnegativity): Find necessary and sufficient conditions for $V[0,u(\cdot)]$ to be nonnegative for each $u(\cdot)\in U$. (Because of the linear-quadratic nature of the problem and the zero initial condition, this is equivalent to demanding that $V^*[0]=0$.)

A well-known necessary condition for the nonnegativity (and therefore also for the control) problem is $R(t)\geqslant 0$ for each $t\in[t_0,T]$, this being the classical Legendre–Clebsch condition for the second variation [1]. We can therefore classify the above problems in the following manner. In case $R(t)>0$ for all $t\in[t_0,T]$, the problem is termed nonsingular and it is easily solved. The interesting cases are those when $R(t)=0$ (the totally singular case) and $R(t)$ is nonzero and singular (the partially singular case).

Historically, it was the nonnegativity problem in terms of the second variation problem of optimal control that was initially studied. Stronger necessary conditions than the classical Legendre–Clebsch condition were needed to eliminate singular extremals from consideration as minimizing arcs for problems which arose in aerospace trajectory optimization. For more detailed information of the history of this problem see the surveys [2] and [3] and the references therein. Arising from these studies were the generalized Legendre–Clebsch conditions which in the totally singular case can be written on $[t_0,T]$ as

$$\frac{\partial}{\partial u}\frac{d}{dt}\left(\frac{\partial \mathcal{H}}{\partial u}\right)'=0, \quad -\frac{\partial}{\partial u}\frac{d^2}{dt^2}\left(\frac{\partial \mathcal{H}}{\partial u}\right)'\geqslant 0 \tag{3}$$

where $\mathcal{H}$ is the Hamiltonian defined by $\mathcal{H}(x,u,\lambda,t)=x'Qx+2x'Hu+\lambda'(Fx+Gu)$ with the costate vector λ satisfying $-\dot{\lambda}=\partial\mathcal{H}/\partial x$.

If the second relation in (3) holds with equality, the procedure leading to (3) can be extended to give further necessary conditions. In general, the necessary conditions then become

$$\frac{\partial}{\partial u}\frac{d^q}{dt^q}\left(\frac{\partial \mathcal{H}}{\partial u}\right)'=0, \quad (-1)^p\frac{\partial}{\partial u}\frac{d^{2p}}{dt^{2p}}\left(\frac{\partial \mathcal{H}}{\partial u}\right)'\geqslant 0 \tag{4}$$

on $[t_0,T]$ where $q=1,2,\cdots,2p-1$ and where $d^{2p}/dt^{2p}(\partial\mathcal{H}/\partial u)'$ is the lowest order time derivative of $(\partial\mathcal{H}/\partial u)'$ in which some component of the control u appears explicitly with a nonzero coefficient. The integer p is called the order of the singular arc for scalar u; for vector u an extension of this definition is needed [8].

These conditions (4) were initially derived by Kelley [4], [5] for scalar controls only, in which case it can be shown that for odd q, the first equality in (4) is automatically satisfied. The original derivation [4] used the classical method of constructing special variations and considering terms of comparable orders. In [5] and [6], a transformation technique for deriving (4) is described and it is this transformation which will be studied in this paper for the general case of vector controls.

Results for the vector control problem are known. A general form of the generalized Legendre–Clebsch conditions [(4) being inadequate to cover all possibilities as just noted] has been derived by Goh [7] and Robbins [8]. Robbins' method was essentially variational, whereas Goh used a transformation on the states and controls in a treatment which represents an application of work he had done on the singular Bolza problem in the calculus of variations [9].

Kelley's transformation procedure replaces the original performance index and linear system equation by one involving a state variable of lower dimension than the original. Goh retains the full state-space dimension and there arise as a result a number of extra constraint conditions over and above those which might fairly be termed generalized Legendre–Clebsch conditions. These extra constraint conditions have been examined at length in [10].

An outline of the remainder of the paper is as follows. In Section II we describe a transformation involving the rejection of the dependent singular controls which takes the given problem to a standard form and this then allows us to develop, in Section III a further transformation reducing the state dimension should the problem be singular. Finally, in Section IV we describe an algorithm based on the transformations of the previous two sections and which allows one to calculate minimizing controls and the corresponding minimum cost. Section V contains concluding remarks.

In a companion paper [11] we show that the transformations presented here are precisely those transformations obtained by dualizing the results of the algorithm described in [16] for covariance generation.

II. Control Dimension Reduction and Standard Form

Both the control and state-space dimension reduction procedures depend on changing the coordinate bases of the control and state-space. Provided that the time-varying matrix defining the coordinate basis change is sufficiently smooth, not only is the nature of the basic problem unchanged, but also the various smoothness conditions on the matrices associated with the problem are retained in the basis transformation.

Suppose that the matrix $R(\cdot)$ is of constant rank $p<m$ on $[t_0,T]$. This is the first of our constancy-of-rank assumptions. In [16] and [17], it is shown (by appealing to a theorem of Dolezal [12], [13] which guarantees the existence of smooth transformations) that the control space coordinate basis can be changed so that after the change $R=I_p\dotplus 0_{m-p}$. Still following [16] and [17], it is assumed that in the new basis the submatrix of G defined by its last $m-p$ columns has constant rank $s\leqslant m-p$ on $[t_0,T]$ and has C^1 entries. Then one can change both the control and state space bases so that, with $r=m-p-s$ and obvious partitioning, there results

$$R=\begin{bmatrix}\hat{R} & 0\\ 0 & 0_r\end{bmatrix}, \qquad \text{where } \hat{R}=\begin{bmatrix}I_p & 0\\ 0 & 0_s\end{bmatrix}$$

$$G=[\hat{G}\quad 0], \qquad \text{where } \hat{G}=\begin{bmatrix}G_{11} & 0\\ G_{21} & I_s\end{bmatrix}$$

$$H=\begin{bmatrix}\hat{H} & H_3\end{bmatrix}, \qquad \text{where } \hat{H}=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}.$$

Now write $u'(t)=[\hat{u}'(t)\ \ u_3'(t)]$ with $\hat{u}$ of dimension $p+s$ so that (1) and (2) can be written as

$$V[x_0,u(\cdot)]=\hat{V}[x_0,\hat{u}(\cdot)]+2\int_{t_0}^{T}x'H_3u_3\,dt$$

$$\dot{x}(t)=Fx(t)+\hat{G}\hat{u}(t), \qquad x(t_0)=x_0$$

where $\hat{V}[x_0,\hat{u}(\cdot)]$ is a cost functional of the same form as (1) but with control space dimension $p+s$ (less than m when $r>0$), and coefficient matrices $\hat{R}$, $\hat{G}$, and $\hat{H}$ replace R, G, and H.

It is easily established that a necessary and sufficient condition for $V^*[x_0]>-\infty$ for all x_0 is that $\hat{V}^*[x_0]>-\infty$ for all x_0 and $H_3=0$, a.e., on $[t_0,T]$. Continuity of course implies that $H_3\equiv 0$. For $V[0,u(\cdot)]$ to be nonnegative for each $u(\cdot)\in U$, it is necessary and sufficient that $\hat{V}[0,\hat{u}(\cdot)]$ be nonnegative for all $\hat{u}(\cdot)\in\hat{U}$ and $H_3'x\equiv 0$ on $[t_0,T]$ for all trajectories $x(\cdot)$ with $x(t_0)=0$.

In summary, we have argued that if the original problem is singular and certain constancy-of-rank and differentiability assumptions hold, we may replace it by a problem in standard form, i.e., where

$$R=\begin{bmatrix} I_p & 0\\ 0 & 0_s\end{bmatrix},\quad G=\begin{bmatrix} G_{11} & 0\\ G_{21} & I_{m-p}\end{bmatrix}. \tag{5}$$

In case the replacement problem has lower control space dimension, it may be singular ($s>0$) or nonsingular ($s=0$).

III. State Dimension Reduction

The development of this section essentially follows that of [6] and [14], generalized to the vector case. Assume that we are given (1) and (2) with R and G given by (5). We are interested in finding necessary (and sufficient) conditions for the nonnegativity and control problems to have solutions. Construct the Mayer form of the problem by introducing the scalar variable W_0 defined by

$$\dot{W}_0=x'Qx+2x'Hu+u'Ru,\qquad W_0(t_0)=0. \tag{6}$$

Equations (2) and (6) now define a set of $(n+1)$ differential equations in the variables W_0 and x. Recalling the standard form of R and G, and the resultant partitioning of u and x, it is clear that (6) involves u_2 linearly but not quadratically. Moreover, from the partitioned form of (2), and with obvious definitions of F_{11}, etc.,

$$\dot{x}_1=F_{11}x_1+F_{12}x_2+G_{11}u_1,\quad \dot{x}_2=F_{21}x_1+F_{22}x_2+G_{21}u_1+u_2 \tag{7}$$

we see that $x_1(\cdot)$ is influenced by u_2 only indirectly via $x_2(\cdot)$. This suggests that if (6) did not contain a u_2 term at all, the original problem could be replaced by one with state x_1 (of lower dimension than x) and controls x_2 and u_1, since u_2 is essentially x_2 differentiated. With this in mind, we attempt to find a transformation to new variables z_0, z_1, and z_2, with z_0 scalar and $z=[z_1'\ z_2']'$ the corresponding partitioning of the n-dimensional vector z, such that the dynamics of z_0 and z_1 are independent of u_2. A derivation similar to that in [14] leads to an examination of the transformation

$$\begin{aligned} z_0&=W_0-2x_1'H_{12}x_2-x_2'H_{22}^Sx_2\\ z_1&=x_1\\ z_2&=x_2\end{aligned} \tag{8}$$

where $H_{22}^S=\frac{1}{2}(H_{22}+H_{22}')$. The transformation is nonsingular as the Jacobian determinant equals unity. The replacement of (6) is obtained as

$$\dot{z}_0=x_1'\hat{Q}x_1+2x_1'\hat{H}_1x_2+2x_1'\hat{H}_2u_1+x_2'\hat{R}_1x_2+2x_2'\hat{R}_2u_1+u_1'u_1+x_2'H_{22}^Au_2 \tag{9}$$

where (with obvious definitions of symbols on the right and with an appropriate differentiability assumption)

$$\begin{aligned}\hat{Q}&=Q_{11}-H_{12}F_{21}-F_{21}'H_{12}'\\ \hat{H}_1&=Q_{12}-F_{11}'H_{12}-H_{12}F_{22}-\dot{H}_{12}-F_{21}'H_{22}^S\\ \hat{H}_2&=H_{11}-H_{12}G_{21}\\ \hat{R}_1&=Q_{22}-F_{12}'H_{12}-H_{12}'F_{12}-F_{22}'H_{22}^S-H_{22}^SF_{22}-\dot{H}_{22}^S\\ \hat{R}_2&=H_{21}-H_{12}'G_{11}-H_{22}^SG_{21}\end{aligned}$$

H_{22}^A is the antisymmetric part of H_{22}. (10)

Although (9) does apparently involve u_2 explicitly, we shall show that we can take $H_{22}^A=0$. Now, with notation $\hat{x}=x_1$, $\hat{u}'=[\hat{u}_1'\ \hat{u}_2']=[x_2'\ u_1']$, $\hat{F}=F_{11}$, $\hat{G}=[F_{12}\ G_{11}]$, $\hat{H}=[\hat{H}_1\ \hat{H}_2]$, and $\hat{R}=\begin{bmatrix}\hat{R}_1 & \hat{R}_2\\ \hat{R}_2' & I\end{bmatrix}$, we can write (1) as

$$V[x_0,u(\cdot)]=x'(T)Sx(T)+[2x_1'H_{12}x_2+2x_2'H_{22}^Sx_2]_{t=t_0}^{t=T}+\int_{t_0}^{T}\hat{q}(\hat{x}(t),\hat{u}(t),t)\,dt+\int_{t_0}^{T}x_2'H_{22}^Au_2\,dt \tag{11}$$

where $\hat{q}(\cdot,\cdot,\cdot)$ is defined in an obvious manner. We then have Lemma 3.1.

Lemma 3.1: A necessary condition for $V^*[x_0]$ to be finite for an arbitrary but fixed x_0 is $H_{22}^A(t)\equiv 0$ on $[t_0,T]$.

Proof: The proof is by contradiction and, though quite long, is basically straightforward real analysis. We can show that if at time τ some component of $H_{22}^A(\tau)$ is nonzero, then $V[x_0,u(\cdot)]$ can be made arbitrarily negative for controls everywhere zero except for components corresponding to the nonzero component of $H_{22}^A(\tau)$ of the form $-\alpha\cos\alpha(t-\tau)$ and $-\alpha\sin\alpha(t-\tau)$ in a neighborhood of τ for α sufficiently large.

Note that this lemma highlights the additional complexity of the vector control problem as compared with the scalar control problem where the need to introduce H_{22}^A does not arise.

We can now state the main result.

Theorem 3.1: Assume continuity of F, G, Q, and R, continuous differentiability of H and that the nonnegativity or control problem is in standard form. Furthermore, with quantities $\hat{x}$, $\hat{u}$, $\hat{F}$, $\hat{G}$, $\hat{Q}$, $\hat{H}$, $\hat{R}$ as defined earlier, set

$$\hat{V}[\hat{x}_0,\hat{u}(\cdot)]=\hat{x}'(T)\hat{S}\hat{x}(T)+\int_{t_0}^{T}\hat{q}(\hat{x}(t),\hat{u}(t),t)\,dt \tag{12}$$

where $\hat{S}=S_{11}-(S_{12}+H_{12})(S_{22}+H_{22})^{\#}(S_{12}+H_{12})'|_{t=T}$ (# denotes pseudoinverse). The control (nonnegativity) problem corresponding to cost (1) and dynamics (2) has a solution if and only if 1) the control (nonnegativity) problem corresponding to cost (12) and dynamics $\dot{\hat{x}}(t)=\hat{F}\hat{x}(t)+\hat{G}\hat{u}(t)$, $\hat{x}(t_0)=\hat{x}_0$ has a solution, 2) $H_{22}(t)$ is symmetric on $[t_0,T]$, 3) $S_{22}+H_{22}(T)\geqslant 0$, and 4) the null space of $S_{22}+H_{22}(T)$ is contained in the null space of $S_{12}+H_{12}(T)$.

Remarks: 1) $\hat{V}^*[\hat{x}_0]$ and $V^*[x_0]$ are related by $V^*[x_0]=\hat{V}^*[\hat{x}_0]-2\hat{x}_0H_{12}(t_0)x_2(t_0)-x_2'(t_0)H_{22}(t_0)x_2(t_0)$ so that if $\hat{V}^*[\hat{x}_0]$ is quadratic in $\hat{x}_0$ so is $V^*[x_0]$ in x_0.

2) The proof of the theorem is straight forward and depends on Lemma 3.1 and the fact that the minimization of the right side of (11) can be carried out by two separate minimizations, one involving the terminal costs and the other involving $\hat{V}[\cdot,\cdot]$.

3) A necessary condition for $\hat{V}^*[\hat{x}_0]$ to be finite for each $\hat{x}_0$ is $\hat{R}(t)\geqslant 0$ on $[t_0,T]$. This condition is the generalized Legendre–Clebsch condition for the original problem and corresponds to the second condition in (3). The first condition in (3) corresponds to the symmetry condition on H_{22} proved in Lemma 3.1 and is again a generalized Legendre–Clebsch condition.

IV. Linear-Quadratic Control

The discussion in Section II considers a singular (control or nonnegativity) problem, and shows how to replace it by a problem in standard form, which can be nonsingular only if there is a control space dimension reduction. This replacement requires differentiability and constancy-of-rank assumptions, as well as a side condition on $H(\cdot)$.

Should the standard form problem be singular we can apply the transformation of Section III to replace the problem with one of lower state-space dimension as well as further side conditions.

Should this reduced state-space dimension problem be singular we again apply the transformation in Section II and so on. Eventually, we must obtain one of three possible terminal problems together with a number of side constraints. The terminal problem must either have zero control space dimension (in which case the problem is vacuous), or zero state-space dimension (in which case nonnegativity of the R matrix in the cost is a necessary and sufficient condition) or a nonsingular problem (see below). It can easily be shown that if the initial problem has $[F,G]$ completely reachable, this property is preserved by the transformations in both Sections II and III and so the zero control space dimension problem is ruled out in this case as a possible terminal problem. Of course the above transformations all rely on the coefficient matrices being sufficiently differentiable and having various constancy-of-rank properties.

If we terminate with a nonsingular control problem, with both control and state space dimensions nonzero, we have a well-known necessary and sufficient condition for this control problem to have a solution, this being that the associated Riccati equation have no escape times on the interval $[t_0, T]$. However for the nonnegativity problem this is sufficient but not necessary, a necessary condition being that there be no escape times on $(t_0, T]$ only. In summary, the transformation procedures always produce necessary and sufficient conditions for the control problem but may only produce separate necessary conditions and sufficient conditions for the nonnegativity problem.

V. Conclusions

We summarize briefly the contributions of the paper. First, we have given an algorithmic procedure for checking the existence of a solution to the general singular linear-quadratic control problem. Indirectly, this gives a procedure for checking the nonnegativity of the functional. Second, this algorithm can also be used in computing the optimal performance index and optimal control (the latter possibly not being unique) for linear-quadratic singular optimal control problems. Important properties of the algorithm are its capacity to handle vector control problems and its disadvantage, viz., a requirement that the ranks of certain matrices remain constant over the interval of interest, and that certain matrices enjoy differentiability properties.

Extension to the case of fixed (or partly fixed) end-point problems should not be particularly difficult.

There are other possible approaches to the optimal control problem which we have not mentioned to this point. By a standard completion of the square device, one can characterise the optimal control, if it exists, in open-loop form as the solution of a linear Fredholm integral equation which is only of the second kind in case the optimal control problem is nonsingular. A solution procedure for the dual singular problem (arising in detection theory) is studied in [15] and could presumably be modified to deal with the control problem. Again, in a companion paper [11], we start with a characterization of the solvability of a Riemann–Stieltjes inequality. We show how the inequality can be solved, and use this solution to define a solution to the control problem.

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Inherent Errors in Asynchronous, Redundant Digital Flight Controls

VINCENT J. DARCY AND CHARLES SLIVINSKY, MEMBER, IEEE

***Abstract*—In current aerospace research, redundant digital computers, operating asynchronously for low cost and high reliability, are being considered for the control of high-performance aircraft. Asynchronous sampling in the control loops leads to inherent errors even under normal, failure-free operation. Such errors are defined and characterized by a useful approximation, and state equations are derived for their description in a dual-redundant, closed-loop system. A covariance analysis of the inherent errors is developed. The state equations and the covariance analysis are applied to a simulation based on a digitized version of the U. S. Air Force A-7D pitch-axis control system, with the aid of a specially-prepared, packaged computer program.**

Introduction

In seeking solutions to the costly problem of proliferation and nonstandardization of aircraft avionics, the U. S. Air Force is sponsoring an advanced development program called DAIS (Digital Avionics Information System). DAIS uses modular digital hardware and software in studying solutions, and is a combined effort of several Air Force organizations. One of these organizations, the Air Force Flight Dynamics Laboratory, has the responsibility to insure that the use of digital, modular hardware does not jeapordize safety-of-flight integrity, nor measurably degrade the performance of either the pilot or the flight control systems.

To achieve the required overall flight-control-system reliability, redundancy is required, including redundant sensors, redundant flight-control processors, and redundant secondary actuators. Each of these sets of redundant components has either voter/monitors or monitors, which are implemented in either software or hardware and may themselves be redundant [1]. The voter/monitors and monitors examine redundant information to detect and isolate certain malfunctions.

In the DAIS hardware configuration there are four identical digital flight-control processors that are programmed to perform identical flight-control calculations. The four separate processors operate on the same physical variables and have common sampling rates. However, their sampling operations are not synchronized with each other by either software or hardware linkages; that is, the processors operate asynchronously.

There are two separate effects present in asynchronous operation. First, hardware failures may occur, which, if severe, may result in erronous control calculations. Second, under normal, failure-free operation, the asynchronous computers will not produce identical calculations because they are operating on physical variables that are sampled at slightly different times, as the sampling is under the control of processor crystals which produce frequencies that are close to each other but not identical. This second effect gives rise to what are designated as *inherent errors* between any two redundant computers. Hardware elements (voter/monitors) within the control system examine the differences between redundant computer outputs to determine which computers are functioning properly. Thus, it is important to be able to characterize inherent errors quantitatively, so that malfunctions and normal operation may be distinguished.

In this paper the analysis of inherent errors is undertaken for a simplified closed-loop configuration in which there is no redundancy in either the sensors or the secondary actuators. There are only two identical, asynchronous digital flight-control processors instead of four, as in the DAIS hardware. The voting scheme is to always select the same processor output as the command to the secondary actuator. Normal,

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V. J. Darcy is a Major in the U. S. Air Force; his work was performed while assigned to the Air Force Flight Dynamics Laboratory, Air Force Systems Command, Wright-Patterson Air Force Base, OH 45433.

C. Slivinsky is with the Department of Electrical Engineering, University of Missouri-Columbia, Columbia, MO 65201.