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A "SIMPLEST POSSIBLE" PROPERTY OF THE GENERALIZED ROUTH–HURWITZ CONDITIONS*

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Abstract. To decide whether a prescribed complex polynomial has all its zeros with negative real parts, there are available many tests involving the checking of rational or polynomial inequalities in the coefficients. It is shown that the generalized Routh–Hurwitz conditions are from a certain point of view not replaceable by simpler conditions.

1. Introduction. The problem of deciding when a prescribed polynomial with real or complex coefficients is such that all its zeros have negative real parts has been studied in the early work of Hermite [1], if not earlier by Cauchy, who was interested in stating procedures for counting the number of roots of a polynomial in a half plane. Of course, much has been done since that time, and the majority of known results are collected in [2], [3] and [4].

Let x be a real vector whose entries are the coefficients of a real polynomial, or the real and imaginary parts of a complex polynomial. Most results are of the following form: a prescribed polynomial has all zeros with negative real parts (in brief, is Hurwitz) if and only if $p_j(x)>0, j=1,2,\cdots,J$, where the $p_j(\cdot)$ are either polynomial or rational in the components of x. For example, the Hermite test [1], generalized Routh–Hurwitz test [2] and Schwarz test [5] associated with a complex polynomial are all of this type.

Two comments on these stability conditions are relevant. First, it is possible to conceive a minor extension of this type of condition, which we illustrate by example. In lieu of the quantities $p_j(x)$, consider the quantities $(x_1-x_2)^2p_1(x)$, $p_2(x),\cdots,p_J(x)$. These have the property that they are nonnegative for all Hurwitz polynomials, and positive for almost all; with a suitable topology in the space of vectors x, a polynomial has the property that almost all polynomials in a small neighborhood of it satisfy the strict inequalities, and conversely if for almost all polynomials in a small neighborhood of a prescribed polynomial the inequalities hold strictly, the prescribed polynomial must be Hurwitz.

Stability conditions allowing this restricted nonnegativity replacement of pure positivity will be called "restricted nonnegativity" conditions, in contrast to the "pure positivity" condition of the second paragraph of the section. Of course, a "pure positivity" condition is a special "restricted nonnegativity" condition.

The second comment on the type of conditions considered is that one can replace rational conditions by polynomial ones: if $p_1(x)=q_1(x)/r_1(x)$, with q_1, r_1 relatively prime polynomials in the components of x, then $p_1>0$ if and only if $q_1r_1>0$. Any results applicable to the class of stability conditions involving only polynomials then, in fact, apply to stability conditions involving rational functions.

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In this paper, we examine the class of stability conditions involving polynomials, including conditions of the "restricted nonnegativity" type. Our main result is that the generalized Routh–Hurwitz conditions are the simplest set of conditions for a complex polynomial to be Hurwitz, in two respects: no other set has fewer inequalities, and no other set is such that the sum of the degrees of the inequalities is lower than the sum of the degrees of the Routh–Hurwitz inequalities.

In § 2 we review the statement of the generalized Routh–Hurwitz condition, and in § 3 we establish that one at least of the Routh–Hurwitz inequalities is, in a certain sense, contained in an arbitrary set of polynomials defining a stability condition. Section 4 is devoted to proving the main result, and § 5 contains concluding remarks.

2. Generalized Routh–Hurwitz conditions. Let $f(z)$ be an nth degree polynomial with complex coefficients and with

(1) $$f(jz)=b_0z^n+b_1z^{n-1}+\cdots+b_n+j(a_0z^n+a_1z^{n-1}+\cdots+a_n).$$

The a_i, b_j are real.

Define the $2n\times 2n$ matrix

$$H=\begin{bmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & a_{2n-1} \\ b_0 & b_1 & \cdot & \cdot & \cdot & b_{2n-1} \\ 0 & a_0 & a_1 & \cdot & \cdot & a_{2n-2} \\ 0 & b_0 & b_1 & \cdot & \cdot & b_{2n-2} \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \end{bmatrix}$$

where $a_m, b_m=0$ for $m>n$. Let $\Delta_1, \Delta_2, \cdots, \Delta_n$ denote the leading principal minors of H of dimension $2, 4, \cdots, 2n$. Then it is known [2], [3], [4] that $f(z)$ has all its zeros inside $\mathrm{Re}[z]<0$ if and only if $\Delta_1>0$, $\Delta_2>0, \cdots, \Delta_n>0$.

We remark that Δ_n is readily recognized as the resultant [6], [7] of the two polynomials $A(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n$ and $B(z)=b_0z^n+b_1z^{n-1}+\cdots+b_n$. In the sequel, we shall use two important properties of the resultant. First, viewed as a multivariable polynomial in $a_0, a_1, \cdots, a_n, b_0, \cdots, b_n$, it is prime; see [6, p. 87]. Second, when the coefficients of $A(z)$, $B(z)$ take particular numerical values with a_0, b_0 not both zero, the value taken by Δ_n is zero if and only if $A(z)$ and $B(z)$ have a nontrivial common factor. If $\Delta_{n-1}\neq 0$, the greatest common divisor of the two polynomials is of degree 1; if $\Delta_{n-1}=0$, $\Delta_{n-2}\neq 0$, it is of degree 2, and so on [7, p. 150].

If the two polynomials have a greatest common divisor of degree 1, it is of the form $z+\bar{\omega}_0$ for $\bar{\omega}_0$ real. It follows then that $f(jz)$ is zero when $z=-\bar{\omega}_0$, i.e. $-j\bar{\omega}_0$ is a zero of $f(z)$. Conversely, if $-j\bar{\omega}_0$ is a zero of $f(z)$, $z+\bar{\omega}_0$ is a common factor of $A(z)$ and $B(z)$ and $\Delta_n=0$.

3. A preliminary result. Notational convention to be used in this and the next section is as follows. The symbol x will be shorthand for the $(2n+2)$-vector $(a_0, a_1, \cdots, a_n, b_0, \cdots, b_n)$. An overbar on a coefficient or vector of coefficients will denote a particular value of that coefficient or vector; the symbol f, sometimes with superscripts or subscripts, will denote a polynomial in z with indeterminate coefficients, and when an overbar is used, a polynomial in z with coefficients taking particular values. Unfortunately at times we have to be slightly flexible in the use of this latter convention.

Let $q_k(x)$ for $k=1, 2, \cdots, K$ be a set of real multivariable polynomials with the following properties. If $q_k(\bar{x})>0$ for all k, then $\Delta_j(\bar{x})>0$ for $j=1, 2, \cdots, n$; if $\Delta_j(\bar{x})>0$ for all j, then $q_k(\bar{x})\geqq 0$ for all k, and for almost all x in a suitably small neighborhood of $\bar{x}$, $q_k(x)>0$. Evidently the q_k provide a set of polynomials constituting a tool for checking the Hurwitz character of a prescribed polynomial.

The main result of this section is as follows.

PROPOSITION 1. *With $q_k(\cdot)$ as described above, $\Delta_n(x)$ divides $\prod_{k=1}^{K} q_k(x)$ in $R[x]$.*

The proof will proceed with the aid of several lemmas. The overall strategy is to show first that if $\bar{x}$ is such that $\Delta_i(\bar{x})>0$ for $i\leqq n-1$ and $\Delta_n(\bar{x})=0$, then $\prod q_k(\bar{x})=0$. Then we show that for any $\bar{x}$ such that $\Delta_n(\bar{x})=0$, we must have $\prod q_k(\bar{x})=0$. The proposition is then a consequence of the fact that $\Delta_n(x)$ is prime.

LEMMA 1. *Let $f(z)=\hat{f}(z)(z+j\omega_0)$ where ω_0 is a real indeterminate, and $\hat{f}(z)$ is an $(n-1)$-st degree polynomial with indeterminate (complex) coefficients. Let $\hat{x}$, $\hat{\Delta}$ refer to the coefficients and generalized Hurwitz determinants associated with $\hat{f}$. Then*

$$\Delta_i(x)=\hat{\Delta}_i(\hat{x}), \qquad i\leqq n-1,$$
$$\Delta_n(x)=0.$$

Proof. That $\Delta_n(x)=0$ was pointed out at the end of the previous section. To establish that $\Delta_i(x)=\hat{\Delta}_i(\hat{x})$ observe that with $\hat{f}(jz)=\hat{b}_0z^{n-1}+\cdots+\hat{b}_{n-1}+j(\hat{a}_0z^{n-1}+\cdots+\hat{a}_{n-1})$, one has

$$\begin{aligned}f(jz)= &-[\hat{a}_0z^n+(\hat{a}_1+\hat{a}_0\omega_0)z^{n-1}+\cdots+(\hat{a}_{n-1}+\hat{a}_{n-2}\omega_0)z+\hat{a}_{n-1}\omega_0]\\ &+j[\hat{b}_0z^n+(\hat{b}_1+\hat{b}_0\omega_0)z^{n-1}+\cdots+(\hat{b}_{n-1}+\hat{b}_{n-2}\omega_0)z+\hat{b}_{n-1}\omega_0].\end{aligned}$$

Then

$$\Delta_i(x)=$$

$$\begin{bmatrix} \hat{b}_0 & (\hat{b}_1+\hat{b}_0\omega_0) & (\hat{b}_2+\hat{b}_1\omega_0) & \cdots & (\hat{b}_{n-1}+\hat{b}_{n-2}\omega_0) & \hat{b}_{n-1}\omega_0 & 0 & \cdots \\ -\hat{a}_0 & -(\hat{a}_1+\hat{a}_0\omega_0) & -(\hat{a}_2+\hat{a}_1\omega_0) & \cdots & -(\hat{a}_{n-1}+\hat{a}_{n-2}\omega_0) & -\hat{a}_{n-1}\omega_0 & 0 & \cdots \\ 0 & \hat{b}_0 & (\hat{b}_1+\hat{b}_0\omega_0) & \cdots & (\hat{b}_{n-2}+\hat{b}_{n-3}\omega_0) & \cdot & \cdot & \cdot \\ 0 & -\hat{a}_0 & -(\hat{a}_1+\hat{a}_0\omega_0) & \cdots & -(\hat{a}_{n-2}+\hat{a}_{n-3}\omega_0) & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & & \vdots & & & \end{bmatrix}$$

The result follows by subtracting ω_0 times the first column from the second, ω_0 times the second column from the third, and so on, and then interchanging the first and second rows, third and fourth rows, and so on. □

The prime use of Lemma 1 is simply to establish Lemma 2, which brings us to the first major step in proving Proposition 1.

LEMMA 2. *Let $\bar{x}$ be such that $\Delta_n(\bar{x})=0$ and $\Delta_i(\bar{x})>0$ for $i \leqq n-1$. Then $\prod q_k(\bar{x})=0$.*

Proof. By the remarks at the end of § 2, $\Delta_n(\bar{x})=0$ and $\Delta_{n-1}(\bar{x})\neq 0$ imply that the polynomial $\bar{f}(z)$ with coefficients $\bar{x}$ can be written as $\bar{f}(z)=\bar{\bar{f}}(z)(z+j\bar{\omega}_0)$ for some real $\bar{\omega}_0$ and some $(n-1)$st degree polynomial $\bar{\bar{f}}(z)$. Since $\Delta_i(\bar{x})>0$ for $i \leqq n-1$, by Lemma 1 it follows that $\bar{\bar{f}}(z)$ is Hurwitz. Set $\bar{f}_m(z)=\bar{\bar{f}}(z)(z+m^{-1}+j\bar{\omega}_0)$ for $m=1,2,\cdots$. Then $\bar{x}_m \to \bar{x}$ as $m\to\infty$ and so $q_k(\bar{x}_m)\to q_k(\bar{x})$ as $m\to\infty$. Because $\bar{f}_m$ is Hurwitz, $0\leqq q_k(\bar{x}_m)$, so that $0 \leqq q_k(\bar{x})$. If $\prod q_k(\bar{x})\neq 0$, then $q_k(\bar{x})>0$, contradicting the non-Hurwitz nature of $\bar{f}$. Therefore $\prod q_k(\bar{x})=0$. □

Our next goal is to show that $\Delta_n(\bar{x})=0$ implies $\prod q_k(\bar{x})=0$, irrespective of the signs of $\Delta_i(\bar{x})$ for $i\leqq n-1$. To do this we shall make use of some algebraic geometry ideas; we also make crucial appeal to the primeness of $\Delta_n(x)$.

LEMMA 3. *Let $\mathcal{S}$ be the set of multivariable polynomials $s(\cdot)$ in x such that any $\bar{x}$ for which $\Delta_n(\bar{x})=0$ and $\Delta_i(\bar{x})>0$ for $i\leqq n-1$ causes $s(\bar{x})=0$. Then $\mathcal{S}$ is an ideal in $R[x]$.*

The proof is a trivial application of the definition of an ideal [6]. Note that $\prod q_k(\cdot)$ by Lemma 2 and $\Delta_n(\cdot)$ by definition lie in $\mathcal{S}$.

Recall also that any polynomial ideal has a finite basis [7, p. 142]. Let this basis be $g_1(\cdot),\cdots,g_r(\cdot)$. Associated with the ideal $\mathcal{S}$ is a variety S, which is the set of x for which $g_i(x)=0$, $i=1,\cdots,r$. Any variety may be decomposed as the union of a finite number of irreducible varieties[1] [8, p. 9]; thus $S=S_1\cup\cdots\cup S_r$. Each S_i is of a certain dimension d_i; the tangent space at any x_i on S_i has dimension $\leqq d_i$, and for almost all x_i has dimension d_i [8, pp. 84–88].

We may define a second variety V as simply the zero set of $\Delta_n(\cdot)$. Because Δ_n is prime, V is irreducible and of dimension $2n+1$, since x is a $(2n+2)$-vector [8, p. 25].

LEMMA 4. *With varieties S and V as defined above, $S=V$.*

Proof. Choose an $\bar{x}$ for which $\Delta_n(\bar{x})=0$ and $\Delta_i(\bar{x})>0$ for $i\leqq n-1$. It is clear that a neighborhood around $\bar{x}$ will intersect $\Delta_n(x)=0$ in a $(2n+1)$-dimensional submanifold; the continuity of the $\Delta_i(x)$ with x ensures that if the neighborhood is sufficiently small, $\Delta_i>0$ on the submanifold. By Lemma 3, any $s\in\mathcal{S}$ is zero on this submanifold, and so, by the definition of S, the points of the submanifold lie on S. Hence there exist points on S with a tangent space of dimension $2n+1$. Therefore, in the decomposition of S into irreducible varieties, at least one variety, say S_1, has dimension $2n+1$.

Now because $\Delta_n\in S$, we have $\Delta_n(x)=\sum a_i(x)g_i(x)$ for all x and accordingly, $x\in S$ implies $x\in V$, i.e., $S\subset V$. Hence $S_1\subset V$. Since S_1 and V have the same dimension and are both irreducible as noted above, it follows (see [8, p. 23]), that $S_1=V$. Hence $S=V$.

[1] The results in [8] which we appeal to, though generally stated for projective varieties, all extend to affine varieties by standard devices as outlined particularly well in [9].

The proof of the proposition is now almost immediate. As noted following Lemma 3, $\prod q_k$ lies in S, i.e., $\prod q_k$ vanishes on S. Since $S = V$, this means that $\prod q_k(\bar{x}) = 0$ whenever $\Delta_n(\bar{x}) = 0$. Because Δ_n is prime, Δ_n divides $\prod q_k$.

4. Minimality of the generalized Routh–Hurwitz conditions. Associated with an nth degree complex polynomial there are n generalized Routh–Hurwitz conditions of degree $2, 4, \cdots, 2n$ in the coefficients of the polynomial. The sum of these degrees is $n(n+1)$. Our aim in this section is to show that it is not possible to reduce these numbers of n and $n(n+1)$ by working with some alternative set of polynomial inequalities. In case $n = 1$, the claim is immediate. To establish the result for arbitrary n, we shall proceed by induction.

Before stating and proving the main results, we make some preliminary remarks and definitions. With notation as in the previous section, define polynomials $p_k(x)$ by $q_k(x) = [\Delta_n(x)]^{\alpha_k} p_k(x)$ where the integer α_k is maximal. By Proposition 1 and the primeness of $\Delta_n(x)$, at least one of the $q_k(x)$ is divisible by $\Delta_n(\cdot)$, and so at least one α_k is positive.

Now suppose that $f(z)$ is of the form $\hat{f}(z)(z + j\omega_0)$ where $\hat{f}(z)$ is of degree $n-1$ with indeterminate coefficients collected in a real $2n$-vector $\hat{x}$, and ω_0 is a real indeterminate. Then x is defined by $\hat{x}$ and ω_0, and $\Delta_n(\hat{x}, \omega_0) = 0$. However, $p_k(\hat{x}, \omega_0)$ cannot be the zero polynomial, for otherwise arguments along the lines of the last section would imply that $p_k(\cdot)$ is divisible by $\Delta_n(\cdot)$. Select $\bar{\omega}_0$ such that $p_k(\hat{x}, \bar{\omega}_0)$ is not identically zero, and define $\bar{p}_k(\hat{x}) = p_k(\hat{x}, \bar{\omega}_0)$. The definitions of $\bar{\omega}_0$, p_k and $\bar{p}_k$ will be used in the proofs of Theorem 1 and 2 below.

THEOREM 1. *Let $f(jz)$ be the n-th degree polynomial given in* (1), *and let $x = (a_0, a_1, \cdots, a_n, b_0, \cdots, b_n)$. Suppose that $q_k(\cdot)$, $k = 1, 2, \cdots, K$, are real polynomials such that $q_k(\bar{x}) > 0$ for all k implies $\bar{f}$ is Hurwitz, and such that $\bar{f}$ Hurwitz implies $q_k(\bar{x}) \geqq 0$ and $q_k(x) > 0$ for all k and almost all x in a sufficiently small neighborhood of $\bar{x}$. Then $\sum \delta[q_k] \geqq n(n+1)$. Here, $\delta[q_k]$ denotes the degree of $q_k(\cdot)$.*

To prove the result, we shall proceed via a sequence of intermediate lemmas, beginning with the following extension of Lemma 1.

LEMMA 5. *Let $f_{\pm m}(z) = \hat{f}(z)(z \pm m^{-1} + j\bar{\omega}_0)$ where $m = 1, 2, \cdots$, and $\hat{f}(z)$ is an $(n-1)$-st degree complex polynomial with indeterminate coefficients. Let $f(z) = \lim_{m\to\infty} f_{\pm m}(z)$, and let $x, x_{\pm m}, \hat{x}, \Delta_i$, etc., be obviously defined. Suppose that for some specialization $\bar{\hat{x}}$ of $\hat{x}$, $\hat{\Delta}_i(\bar{\hat{x}}) \neq 0$, $i \leqq n-1$. Then for suitably large m, $\Delta_n(\bar{x}_{+m})$ and $\Delta_n(\bar{x}_{-m})$ have opposite signs.*

Proof. Since $\hat{\Delta}_{n-1}(\bar{\hat{x}}) \neq 0$, $\bar{\hat{f}}(z)$ has no pure imaginary zeros. Therefore $\bar{f}_{\pm m}(z)$ has no pure imaginary zeros and the number of right half plane zeros is given by the variations in sign in the sequence $1, \Delta_1(\bar{x}_{\pm m}), \cdots, \Delta_n(\bar{x}_{\pm m})$; see [2, p. 249]. By Lemma 1 and continuity, for m sufficiently large and $i \leqq n-1$, $\Delta_i(\bar{x}_{\pm m})$ approximates and therefore has the same sign as $\hat{\Delta}_i(\bar{\hat{x}}) = \Delta_i(\bar{x})$. Accordingly, since $f_{-m}(z)$ has one more zero in $\mathrm{Re}[z] > 0$ than $f_{+m}(z)$, $\Delta_n(\bar{x}_{+m})$ and $\Delta_n(\bar{x}_{-m})$ must have different signs. □

We remark that, strictly, the above proof does not use the fact that $\Delta_i(\hat{x}) \neq 0$ for $i < n-1$, since procedures are available for modifying the variations in sign formula to cope with the vanishing of intermediate Hurwitz determinants [2].

LEMMA 6. *Let* $f(z)=\hat{f}(z)(z+j\bar{\omega}_0)$ *and let the polynomial* $\tilde{p}_k(\hat{x})$ *be defined from* $q_k(x)$ *as described earlier. Then if* $\bar{\hat{f}}$ *is Hurwitz,* $\tilde{p}_k(\bar{\hat{x}})\geqq 0$ *for* $k=1,2,\cdots,K$ *with strict inequality for almost all* $\hat{x}$ *sufficiently close to* $\bar{\hat{x}}$. *Conversely, if* $\tilde{p}_k(\bar{\hat{x}})>0$, $\bar{\hat{f}}$ *is Hurwitz.*

Proof. Suppose $\bar{\hat{f}}$ is Hurwitz. Then $\bar{f}_m(z)=\bar{\hat{f}}(z)(z+m^{-1}+j\bar{\omega}_0)$ is Hurwitz, implying $q_k(\bar{x}_m)>0$ for $k=1,2,\cdots,K$ and $\Delta_n(\bar{x}_m)>0$. Therefore $p_k(\bar{x}_m)>0$ for $k=1,2,\cdots,K$. Letting $m\to\infty$ yields $\tilde{p}_k(\bar{\hat{x}})=p_k(\bar{\hat{x}},\bar{\omega}_0)\geqq 0$ for $k=1,2,\cdots,K$. Since no $\tilde{p}_k(\cdot)$ is identically zero, we must have strict inequality for almost all $\hat{x}$ sufficiently close to $\bar{\hat{x}}$.

Conversely, suppose $\tilde{p}_k(\bar{\hat{x}})>0$ for $k=1,2,\cdots,K$. Then for sufficiently large m, $p_k(\bar{x}_m)>0$. If $\Delta_n(\bar{x}_m)>0$, then $q_k(\bar{x}_m)>0$ for $k=1,2,\cdots,K$, implying $\bar{f}_m$ and therefore $\bar{\hat{f}}$ are Hurwitz. It remains therefore to rule out the possibilities that $\Delta_n(\bar{x}_m)<0$ or $\Delta_n(\bar{x}_m)=0$. Assuming the former, we see from Lemma 5 that $\Delta_n(\bar{x}_{-m})>0$ while also $p_k(\bar{x}_{-m})>0$ for m sufficiently large. Then $q_k(\bar{x}_{-m})>0$ for $k=1,2,\cdots,K$, which contradicts the non-Hurwitz character of $\bar{f}_{-m}(z)=\bar{\hat{f}}(z)(z-m^{-1}+j\bar{\omega}_0)$. If $\Delta_n(\bar{x}_m)=0$, then $\hat{\Delta}_{n-1}(\bar{\hat{x}})=0$. However, for almost all $\hat{x}$ in a neighborhood of $\bar{\hat{x}}$, we still have $\tilde{p}_k(\hat{x})>0$ while $\hat{\Delta}_{n-1}(\hat{x})\neq 0$ and thus $\Delta_n(x_m)\neq 0$. Then $\hat{f}$ is Hurwitz, and it follows easily that $\bar{\hat{f}}$ must be Hurwitz. □

Lemma 6 and the earlier definitions show how to pass from a set of stability conditions for nth degree complex polynomials to a set for $(n-1)$st degree complex polynomials. This is the key to establishing Theorem 1.

Proof of Theorem 1. By the induction hypothesis, $\sum_k \delta[\tilde{p}_k(\hat{x})]\geqq(n-1)n$. Now it is easily established that $\sum_k \delta[p_k(x)]=\sum \delta[\tilde{p}_k(\hat{x})]$ from the definitions, while also $\sum_k \delta[q_k(x)]=\sum_k \alpha_k\,\delta[\Delta_n(x)]+\sum_k \delta[p_k(x)]$. Since $\alpha_k\geqq 0$ with at least one α_k positive and $\delta[\Delta_n(x)]=2n$, this gives $\sum_k \delta[q_k(x)]\geqq 2n+(n-1)n=(n+1)n$. This completes the induction.

The second main result relates to the number of inequalities.

THEOREM 2. *With the same hypothesis as Theorem* 1, $K\geqq n$.

The proof will again proceed via a number of lemmas.

LEMMA 7. *With the integer* α_k *as defined earlier, at least one* α_k *is odd for some* k.

Proof. Let $\bar{f}_{-m}(z)=\bar{\hat{f}}(z)(z-m^{-1}+j\bar{\omega}_0)$, with $\bar{\hat{f}}(z)$ Hurwitz and such that $\tilde{p}_k(\bar{\hat{x}})>0$ for all k. Then since $\lim_{m\to\infty} p_k(\bar{x}_{-m})=\tilde{p}_k(\bar{\hat{x}})$, for sufficiently large m, $p_k(\bar{x}_{-m})>0$ for all k, while also $\Delta_n(\bar{x}_{-m})\neq 0$. If the α_k are all even, this implies that $q_k(\bar{x}_{-m})>0$ for all k, a contradiction of the fact that $\bar{f}_{-m}(z)$ is not Hurwitz.

LEMMA 8. *With the* q_k *reordered so that* $\alpha_1,\cdots,\alpha_s$ *are odd and* $\alpha_{s+1},\cdots,\alpha_k$ *are even and possibly zero,*

$$\{q_k(x)>0 \text{ for } k=1,2,\cdots,K\}$$
$$\leftrightarrow\{\Delta_n p_1>0,\,p_1p_2>0,\cdots,p_1p_s>0,\,p_{s+1}>0,\cdots,p_K>0\}$$

Proof. Both inequality sets are clearly equivalent to $\Delta_n p_1>0,\cdots,\Delta_n p_s>0,\,p_{s+1}>0,\cdots,p_K>0$. □

Now suppose that the $\tilde{p}_k(\hat{x})$ are as defined earlier. Also define $\tilde{q}_1(\hat{x})=\tilde{p}_1(\hat{x})\tilde{p}_2(\hat{x}),\cdots,\tilde{q}_{s-1}(\hat{x})=\tilde{p}_1(\hat{x})\tilde{p}_s(\hat{x}),\ \tilde{q}_s(\hat{x})=\tilde{p}_{s+1}(\hat{x}),\cdots,\tilde{q}_{K-1}(\hat{x})=\tilde{p}_K(\hat{x})$.

LEMMA 9. *Let* $f(z)=\hat{f}(z)(z+j\bar{\omega}_0)$, *with the* $\tilde{q}_k(\hat{x})$ *defined as above. Then if* $\bar{\hat{f}}$ *is Hurwitz,* $\tilde{q}_k(\bar{\hat{x}})\geqq 0$ *for* $k=1,2,\cdots,K-1$ *with strict inequality for almost all* $\hat{x}$ *sufficiently close to* $\bar{\hat{x}}$. *Conversely, if* $\tilde{q}_k(\bar{\hat{x}})>0$, *for* $k=1,2,\cdots,K-1$, $\bar{\hat{f}}$ *is Hurwitz.*

Proof. Suppose $\bar{\bar{f}}$ is Hurwitz. Then $\bar{f}_m(z)=\bar{\bar{f}}(z)(z+m^{-1}+j\bar{\omega}_0)$ is Hurwitz. Using Lemma 8, we see this implies that $p_1(\bar{x}_m)p_2(\bar{x}_m)>0,\cdots,p_1(\bar{x}_m)p_s(\bar{x}_m)>0$, $p_{s+1}(\bar{x}_m)>0,\cdots,p_K(\bar{x}_m)>0$. Letting $m\to\infty$ and using the definition of the $\tilde{q}_k(\hat{x})$ establishes that $\tilde{q}_k(\bar{\hat{x}})\geqq 0$. Since no $\tilde{p}_k(\hat{x})$ is identically zero, no $\tilde{q}_k(\hat{x})$ can be; therefore strict inequality holds for almost all $\hat{x}$ sufficiently close to $\bar{\hat{x}}$.

Conversely, let $\tilde{q}_k(\bar{\hat{x}})>0$. For suitably large m, $p_1(\bar{x}_{\pm m})p_2(\bar{x}_{\pm m})>0,\cdots,p_1(\bar{x}_{\pm m})p_s(\bar{x}_{\pm m})>0$, $p_{s+1}(\bar{x}_{\pm m})>0,\cdots,p_K(\bar{x}_{\pm m})>0$. Assume temporarily that $\tilde{p}_1(\bar{\hat{x}})\hat{\Delta}_{n-1}(\bar{\hat{x}})$ is not zero. [If $s>1$, $\tilde{p}_1(\bar{\hat{x}})$ is guaranteed to be nonzero, since $0\neq\tilde{q}_1(\bar{\hat{x}})=\tilde{p}_1(\bar{\hat{x}})p_2(\bar{\hat{x}})$.] Then $p_1(\bar{x}_{\pm m})$ have the same sign and, by Lemma 5, $\Delta_n(\bar{x}_{\pm m})$ have opposite signs. Therefore $\Delta_n(\bar{x}_{\pm m})p_1(\bar{x}_{\pm m})$ have opposite signs. If $\Delta_n(\bar{x}_{-m})p_1(\bar{x}_{-m})>0$, this, in conjunction with the inequalities $p_1(\bar{x}_{\pm m})p_2(\bar{x}_{\pm m})>0$, etc., implies by Lemma 8 that $\bar{f}_{-m}$ is Hurwitz, which is impossible. Therefore $\Delta_n(\bar{x}_m)p_1(\bar{x}_m)>0$ and using Lemma 8, $\bar{f}_{+m}$ is seen to be Hurwitz. Therefore $\hat{f}$ is Hurwitz.

It remains to consider the case where one or both of $\tilde{p}_1(\bar{\hat{x}})$ and $\hat{\Delta}_{n-1}(\bar{\hat{x}})$ are zero. For almost all $\hat{x}$ in a sufficiently small neighborhood of $\bar{\hat{x}}$, $\tilde{p}_1(\hat{x})\hat{\Delta}_{n-1}(\hat{x})$ must be nonzero while $\tilde{q}_k(\hat{x})>0$. Therefore $\hat{f}$ is Hurwitz by the argument of the preceding paragraph. Then $\bar{\bar{f}}$ must be Hurwitz. □

It is now easy to complete the proof of Theorem 2. Applying the induction hypothesis to the inequality set $\tilde{q}_k(\hat{x})>0$ associated with $\hat{f}$ yields $K-1\geqq n-1$. Therefore $K\geqq n$, as required.

We remark that there seems no direct way of combining the proofs of Theorems 1 and 2. Both theorems are proved by deriving from an inequality set associated with an nth degree polynomial a second set associated with an $(n-1)$st degree polynomial. The second set differs between the two theorems, as that set appropriate for proving the degree property is inappropriate for proving the number-of-inequalities property, and vice versa.

5. Conclusions and remarks. We have shown that the generalized Routh–Hurwitz conditions are the simplest set of polynomial inequalities defining the Hurwitz property of a complex polynomial, in the sense that no other set can contain fewer inequalities nor have a "total" degree smaller than that of the generalized Routh–Hurwitz conditions.

The question of what are the simplest set of polynomial inequalities defining the Hurwitz property of a *real* polynomial has not been tackled. Though the set of all real polynomials is obtainable by specializing certain coefficients in (1) to be zero, it does not of course follow that by making corresponding specializations in the generalized Routh–Hurwitz conditions, one obtains a set of "simplest possible" inequalities for real Hurwitz polynomials. Indeed this is demonstrably not the case, because this procedure recovers the standard Hurwitz test, and the Liénard–Chipart test is certainly simpler in terms of degree [2]. Work by the authors has come close to establishing that the Liénard–Chipart criteria are the simplest set of conditions for real polynomials to be Hurwitz, as might be expected; a full proof however is still lacking.

The same sort of results as those obtained in this paper appear to follow for "unit-circle" stability. More precisely, the Schur–Cohn inequalities as set out in [4; see pp. 28, 29] would appear to be the simplest possible in the two senses dealt with above.

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