INT. J. CONTROL, 1976, VOL. 23, NO. 1, 1–16

Schwarz matrix properties for continuous and discrete time systems†

BRIAN D. O. ANDERSON‡, ELIAHU I. JURY§
and M. MANSOUR‖

New properties of the Schwarz matrix associated with a prescribed polynomial are derived. A matrix analogous to the Schwarz matrix is described which is useful in studying the root distribution of a prescribed polynomial relative to the unit circle.

1. Introduction

In this paper we study a certain matrix associated with a prescribed polynomial (real or complex) which has come to be known as the Schwarz matrix, following (Schwarz 1956). This matrix contains information about the root distribution of the polynomial, relative to the imaginary axis, i.e. on occasions, the matrix indicates the number of zeros of the polynomial in the region $\mathrm{Re}\,[s] < 0$ for example. We explore in more detail than before properties applicable to the case when the polynomial is real and all zeros lie in $\mathrm{Re}\,[s] < 0$, obtaining a simplified description of this case. We also consider the question of the relation between two Schwarz matrices associated with a complex polynomial and a related real polynomial of twice the degree; in this way, we are able to suggest a Routh table-type procedure for computing the Schwarz matrix associated with a complex polynomial and, at the same time, relate the connection to the simplified description via the Schwarz matrix of real polynomials with all zeros in $\mathrm{Re}\,[s] < 0$.

With the material referred to above as a guide, we address the question of constructing a Schwarz matrix analogue to provide information about the root distribution of a prescribed polynomial relative to the unit circle. The question at once arises as to what constitutes an analogue of the Schwarz matrix. This point is taken up in detail in the text; suffice it to say here that the analogy is seen in terms of the structure of certain Lyapunov functions and their derivatives. We derive various properties of the analogous matrix (which we choose to call a discrete-time Schwarz matrix). In particular, we study the connection between the matrix entries and other quentities used previously in studying root-distribution problems relative to the unit circle.

Received 19 June 1975.

† Work supported by Australian Research Grant Committee and the National Science Foundation under Grant GK–33377.

‡ Department of Electrical Engineering, University of Newcastle, New South Wales 2308, Australia.

§ Department of Electrical Engineering, University of California, Berkeley, California, U.S.A.

‖ Lehrstuhl für Automatik, ETH, Zürich, Switzerland.

2. The Schwarz matrix

In this section we review some properties of the Schwarz matrix and the entries therein. The articles by Schwarz (1956), Wall (1945) and Mansour (1965) cover much of this material.

We begin by restricting attention to real polynomials and matrices. Let

$$f(s)=s^n+\sum_{i=1}^{n} a_i s^{n-i} \tag{1}$$

be a real monic polynomial; from it we may construct the Hurwitz determinants, i.e. the leading principal minors of the $n\times n$ matrix

$$H=\begin{bmatrix} a_1 & a_3 & a_5 & \dots & 0 & 0 & \dots & 0 \\ 1 & a_2 & a_4 & \dots & 0 & 0 & \dots & 0 \\ 0 & a_1 & a_3 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & a_2 & & 0 & 0 & & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & & \cdot \end{bmatrix} \tag{2}$$

Call D_i the ith such minor. Assuming the D_i are non-zero, we may then define quantities

$$\left.\begin{aligned} &b_1=D_1, \quad b_2=D_2/D_1, \quad b_3=D_3/D_1D_2 \\ &b_4=D_1D_4/D_2D_3 \dots, \quad b_r=D_{r-3}D_r/D_{r-2}D_{r-1} \\ &b_n=D_{n-3}D_n/D_{n-2}D_{n-1} \end{aligned}\right\} \tag{3}$$

The b_i may also be found in terms of the leading coefficients of the Routh array (Gantmacher 1959) with leading rows

$$\begin{matrix} 1 & a_2 & a_4 & a_6 & \dots \\ a_1 & a_3 & a_5 & a_7 & \dots \end{matrix}$$

Denoting these leading entries by $\alpha_0=1$, $\alpha_1=a_1$, α_2, ..., it follows from the relation $D_i=\alpha_0\,\alpha_1 \dots \alpha_i$, see (Gantmacher 1959), that

$$b_1=\alpha_1, \quad b_2=\alpha_2, \quad b_3=\alpha_3/\alpha_1, \quad b_4=\alpha_4/\alpha_2 \dots, \quad b_n=\alpha_n/\alpha_{n-2} \tag{4}$$

or

$$\alpha_0=1, \quad \alpha_1=b_1, \quad \alpha_2=b_2, \quad \alpha_3=b_3b_1, \quad \alpha_4=b_4b_2 \dots \tag{5}$$

The b_i are shown by Wall (1945) to be coefficients arising in a certain continued fraction expansion associated with $[f(s)-f(-s)][f(s)+f(-s)]^{-1}$ or its inverse. It is also shown by Wall (1945) that the following two matrices are similar:

$$A=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{bmatrix} \tag{6}$$

$$S=\begin{bmatrix} -b_1 & 1 & 0 & 0 & . & . & . \\ -b_2 & 0 & 1 & 0 & . & . & . \\ 0 & -b_3 & 0 & 1 & . & . & . \\ 0 & 0 & -b_4 & 0 & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ . & . & . & . & . & . & 1 \\ . & . & . & . & . & -b_n & 0 \end{bmatrix} \tag{7}$$

This fact is also established by Schwarz (1956), where complex a_i in A are permitted, in which case S becomes more complicated, as noted before. Schwarz (1956) also describes a procedure for proceeding between A and S via a similarity transformation.

The matrix A will be recognized as a companion matrix having the characteristic polynomial $f(s)$ of (1). The matrix S, now known as the Schwarz matrix, naturally has the same characteristic polynomial. Being tridiagonal, its characteristic polynomial is easy to find. Let $F_k(s)$ be the characteristic polynomial of the matrix comprising the first k rows and columns of S. It follows that

$$\left.\begin{aligned} F_{-1}(s) &= 1 \\ F_0(s) &= 1 \\ F_1(s) &= s+b_1 \\ F_k(s) &= sF_{k-1}(s)+b_kF_{k-2}(s) \\ f(s) = F_n(s) &= s^n+a_1s^{n-1}+\ldots+a_n \end{aligned}\right\} \tag{8}$$

The definitions of $F_{-1}(s)$ and $F_0(s)$ are chosen simply to ensure that the recursive definition of $F_k(s)$ is valid for $k \geqslant 1$.

Equations (3) and (4) with the accompanying remarks provide a procedure for passing from the coefficients a_i of $f(s)$ to the interesting entries b_i of the Schwarz matrix. Equations (8) allow a reversal of the procedure. A more complicated relationship between the two sets of coefficients can be obtained by explicit construction of a similarity transformation linking A and S. Relevant material is provided by Wall (1945), Parks (1962), Puri and Weygandt (1963), Chen and Chu (1966, 1967), Barnett and Storey (1967), Loo (1968), Power (1969) and Chen (1974).

The signs of the b_i provide information about the root distribution of the polynomial $f(s)$, as one would expect from the connection of the b_i with the Hurwitz determinants, and the fact that the Hurwitz determinants provide root-distribution information (Gantmacher 1959). First, a necessary and sufficient condition for all zeros of $f(s)$ to lie in Re $[s]<0$ is that $b_i>0$ for all i.

There are several ways to see this. One can use the fact that $f(s)$ has all zeros in Re $[s]<0$ if and only if $D_i>0$ or $\alpha_i>0$ for all i, see Gantmacher (1959), and then (3) or (4) yields the result. Alternatively, one can use a Lyapunov function approach, as per, for example (Parks 1962, Puri and Weygandt 1963, Kalman and Bertram 1960): suppose $|sI-S|$ has all zeros in Re $[s]<0$. Then all b_i are non-zero, as the following reasoning shows. If $b_1=0$, trace $S=0$ and if $b_n=0$, det $S=0$ and, in either case, it is not possible for all eigenvalues of S to be in Re $[s]<0$; if $b_i=0$, $i\neq 1, n$, S becomes a block upper triangular, so that its eigenvalues are the eigenvalues of the two triangular matrices. The lower such matrix has zero trace, which means that its eigenvalues cannot lie in Re $[s]<0$. Accordingly, assuming any $b_i=0$ for $i=1, \ldots, n$ yields a contradiction.

Now, since the b_i are non-zero, one can define the $n\times n$ matrix

$$P=\operatorname{diag}\{b_1, b_1/b_2, b_1/b_2b_3, \ldots\}$$

as a non-singular matrix, and verify that $PS+S'P=-Q$, where

$$Q=\operatorname{diag}\{2b_1^2, 0, 0, \ldots, 0\}$$

The zero distribution property of $|sI-S|$ guarantees that

$$P=\int_0^\infty \exp(S't)Q\exp(St)\,dt$$

Evidently $P\geqslant 0$, and since it is non-singular, $P>0$, whence $b_i>0$ for all i. Conversely, if $b_i>0$ for all i one can verify by use of the Lyapunov function $V=x'Px$ for which $\dot{V}=-x'Qx$ that $\dot{x}=Sx$ is stable; also one can check that Qx is not identically zero along any non-zero trajectory of $\dot{x}=Sx$, establishing asymptotic stability of this equation or equivalently, that $|sI-S|$ has all zeros in Re $[s]<0$.

In case $f(s)$ does not have all its zeros in Re $[s]<0$, then the b_i do still give information concerning the root distribution of $f(s)$, provided that $b_i\neq 0$ for all i. In this case, see Schwarz (1956, Theorem 5), with slight modification, the number of positive terms in the sequence $b_1, b_1b_2, \ldots$ is the number of zeros in Re $[s]<0$ of $f(s)$.

In the remainder of this section we shall consider the case of

$$\phi(s)=s^m+\sum_{i=1}^{m} a_i s^{m-i}=s^m+\sum_{i=1}^{m}(a_i'+ja_i'')s^{m-i} \tag{9}$$

with a_i', a_i'' real, $j=\sqrt{-1}$. The main results of Schwarz are as follows. Via a Euclidean algorithm procedure, the details of which are unimportant for the moment, one may construct a set of real quantities $\beta_1, \beta_2, \ldots, \beta_m$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ such that with n in (6) replaced by m, the matrix A in (6) is similar to the matrix

$$S=\begin{bmatrix} -\beta_1+j\gamma_1 & 1 & 0 & 0 & \cdots & . & . \\ -\beta_2 & j\gamma_2 & 1 & 0 & \cdots & . & . \\ 0 & -\beta_3 & j\gamma_3 & 1 & \cdots & . & . \\ 0 & 0 & -\beta_4 & j\gamma_4 & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & 1 \\ . & . & . & . & \cdots & -\beta_n & j\gamma_n \end{bmatrix} \tag{10}$$

Schwarz does not give an explicit formula for the transforming matrix T such that $A = TST^{-1}$, but one could presumably be obtained by extending the ideas of Chen and Chu (1966, 1967), Barnett and Storey (1967), Loo (1968) and Power (1969) to the complex case. One can readily pass from the β_i, γ_i set to the a_i via

$$\left.\begin{aligned} &\phi_0(s) = 1 \\ &\phi_1(s) = s + \beta_1 - j\gamma_1 \\ &\vdots \\ &\phi_i(s) = (s - j\gamma_i)\phi_{i-1}(s) + \beta_i\phi_{i-2}(s) \\ &\vdots \\ &\phi(s) = \phi_m(s) = s^m + \sum_{i=1}^{m} a_i s^{m-i} \end{aligned}\right\} \tag{11}$$

Here, $\phi_r(s)$ is the characteristic polynomial of the matrix formed from the first r rows and columns of S.

The polynomial $\phi(s)$ has all zeros in Re $[s] < 0$ if and only if $\beta_i > 0$ for all i. The stability of $\dot{x} = Sx$ can be studied via the Lyapunov function $V = x'Px$, where

$$P = \text{diag}\ \{\beta_1, \beta_1/\beta_2, \ldots, \beta_1/\beta_2\beta_3 \ldots \beta_m\}$$

with $\dot{V} = -x'Qx$ and

$$Q = \text{diag}\ \{2\beta_1^2, 0, \ldots, 0\}$$

3. Some Liénard–Chipart-type simplifications

The Liénard–Chipart criterion (1914) is a simplification of the Hurwitz criterion for the polynomial $f(s)$ of (1) to have all zeros in Re $[s] < 0$. According to the Hurwitz criterion, a necessary and sufficient condition is that $D_i > 0$ for all i; according to the Liénard–Chipart criterion, a necessary and sufficient condition is that all odd order *or* even order Hurwitz determinants be positive, and that all coefficients a_i with odd subscript *or* even subscript be positive, together with $a_n > 0$. For a modern treatment, see Gantmacher (1959).

The Hurwitz determinant conditions of the Liénard–Chipart criterion have their counterpart in terms of the b_i, as we now show.

Proposition 1

The set of inequalities $D_i > 0$, i odd is equivalent to the set $b_1 > 0$, $b_2b_3 > 0$, $b_4b_5 > 0$, ... The set of inequalities $D_i > 0$, i even, is equivalent to the set $b_1b_2 > 0$, $b_3b_4 > 0$, $b_5b_6 > 0$, ...

Proof

From (3), we see that $b_1 = D_1$, $b_2b_3 = D_3/D_1^2$, $b_4b_5 = D_1D_5/D_3^2$, etc., and the first result is immediate. The second result is established similarly.

Another insight into this result is obtainable by connecting certain real and complex Schwarz matrices. Let $f(s)$ be as in (1) with $n = 2m$, and write

$$f(s) = G(s^2) + sH(s^2) \tag{12}$$

Set

$$\phi(s) = (-j)^m[G(js) + jH(js)] \tag{13}$$

so that $\phi(s)$ is a monic complex polynomial. Now define the sequence $F_i(s)$ as in (8). It is not hard to show that the following recursion exists :

$$F_{2i}(s) = (s^2 + b_{2i-1} + b_{2i})F_{2i-2}(s) - b_{2i-1}b_{2i-2}F_{2i-4}(s)$$

initialized by $F_0(s) = 1$, $F_2(s) = s^2 + b_1 s + b_2$. It follows that if $G_{2i}(s^2)$ and $H_{2i}(s^2)$ are defined by $F_{2i}(s) = G_{2i}(s^2) + sH_{2i}(s^2)$, then these quantities satisfy the same recursion relation, with $G_0(s^2) = 1$, $G_1(s^2) = (s^2 + b_2)$, $H_0(s^2) = 0$, $H_1(s^2) = b_1$. From these recursions for $G_{2i}(s^2)$ and $H_{2i}(s^2)$, recursions for $G_{2i}(js)$ and $H_{2i}(js)$ are immediate, and these lead to a recursion for the quantity

$$\psi_i(s) = (-j)^i[G_{2i}(js) + jH_{2i}(js)] \tag{14}$$

This quantity is a monic complex polynomial. The recursion is

$$\left.\begin{aligned} &\psi_i(s) = [s - j(b_{2i} + b_{2i-1})]\psi_{i-1}(s) + b_{2i-1}b_{2i-2}\psi_{i-2}(s) \\ &\psi_0(s) = 1, \quad \psi_1(s) = s + b_1 - jb_2 \end{aligned}\right\} \tag{15}$$

A comparison of (15) and (11) shows that with the identifications

$$\left.\begin{aligned} &\gamma_i = b_{2i} + b_{2i+1} \\ &\beta_i = b_1, \quad \beta_i = b_{2i-1}b_{2i-2} \end{aligned}\right\} \tag{16}$$

the set $\psi_i(s)$ is the same as the set $\phi_i(s)$. In summary, we have shown the following :

Proposition 2

Consider the real even degree polynomial $f(s)$ of (12) and the related complex polynomial $\phi(s)$ of (13). With the quantities b_i appearing in the Schwarz matrix of $f(s)$ and the quantities β_i, γ_i in the Schwarz matrix of $\phi(s)$ as described earlier, relations (16) hold between these Schwarz matrix parameters.

This result has added significance in the light of Proposition 1 and of Anderson *et al.* (1975). Suppose $\phi(s)$ has all its zeros in $\mathrm{Re}\,[s] < 0$; then $\beta_i > 0$ for all i and by (16), $b_1 > 0$, $b_2 b_3 > 0$, $b_4 b_5 > 0$. If the odd subscript or even subscript coefficients of $f(s)$ are positive, together with a_n, Proposition 1 and the remarks preceding it show that $f(s)$ has all its zeros in $\mathrm{Re}\,[s] < 0$. This is also the content of one of the results of Anderson *et al.* (1975).

A second consequence of Proposition 2 is that it gives an orderly way of computing the quantities β_i, γ_i associated with an arbitrary prescribed complex $\phi(s)$. For suppose that $\phi(s) = s^m + \sum_{i=1}^{m} (a_i' + ja_i'')s^{m-i}$ with a_i', a_i'' real. Then

$$f(s) = s^{2m} + a_1's^{2m-1} - a_1''s^{2m-2} - a_2''s^{2m-3} - a_2's^{2m-4} - a_3's^{2m-5} + a_3''s^{2m-6} + a_4''s^{2m-7} + \dots \tag{17}$$

(The sign pattern has four minus signs followed by four plus signs, two imaginary parts of coefficients of $\phi(s)$ followed by two real parts). Using the Routh table for $f(s)$ started by

$$\left.\begin{array}{cccccc} 1 & -a_1'' & -a_2' & a_3'' & a_4' & \dots \\ a_1' & -a_1'' & -a_3' & a_4'' & . & \dots \end{array}\right\} \tag{18}$$

one can obtain the quantities b_i in terms of the leading Routh coefficients α_i, as in (4), and then the β_i, γ_i follow from (16). Taken together, this yields

$$\left.\begin{array}{lll} \gamma_1=\alpha_2+\dfrac{\alpha_3}{\alpha_1}, & \gamma_2=\dfrac{\alpha_4}{\alpha_2}+\dfrac{\alpha_5}{\alpha_3}, & \gamma_3=\dfrac{\alpha_6}{\alpha_4}+\dfrac{\alpha_7}{\alpha_5}\dots \\ \beta_1=\alpha_1, & \beta_2=\dfrac{\alpha_3}{\alpha_1}\cdot\alpha_2, & \beta_3=\dfrac{\alpha_5}{\alpha_3}\cdot\dfrac{\alpha_4}{\alpha_2}\dots \end{array}\right\} \tag{19}$$

The idea of forming a Routh array for a polynomial with complex coefficients is certainly not new, see, e.g. Brown (1965).

As it stands, Proposition 2 is restricted to the relation between a real polynomial of even degree and complex polynomial of half the degree. Actually, as discussed by Anderson *et al.* (1975), there are two differing complex polynomials which can be associated with a polynomial of even degree, and likewise with a polynomial of odd degree (though here the complex polynomials have degree $n \pm 1/2$). Proposition 2 therefore considers only one of four possible relationships. Derivation of one of the results for odd polynomials (in terms of Schwarz matrix entries) seems straightforward, but derivation of the other two results does not. This is perhaps partly the case because the complex polynomials in question are not monic, at least when their coefficients are expressed as integral functions of the coefficients of the associated real polynomial, while it is intrinsic in computing the characteristic polynomial of the Schwarz matrix (indeed any matrix) that the polynomial should be monic while at the same time its coefficients are integral in the matrix entries. Incidentally, one would presume that the missing two relations would correspond to the second set of Liénard–Chipart inequalities.

4. Unit circle results involving a Schwarz form

As an examination of any modern text, e.g. Jury (1974) and Barnett (1971) dealing with stability questions will show, there are numerous parallel results for polynomials $f(s)$ with all zeros inside $\mathrm{Re}\,[s]<0$ and inside $|s|<1$. Our aim in this section is to describe results akin to those involving the Schwarz matrix for the unit circle type problem.

This section develops along the following lines. Using a table analogous to the Routh table, we determine from $f(s)$ quantities Δ_i analogous to the b_i. We also show how to recover $f(s)$ from the Δ_i. Next, we exhibit a similarity transformation between a companion matrix A with characteristic polynomial $f(s)$ and a matrix Σ with entries determined from the Δ_i. This latter matrix is not nearly as sparse as the Schwarz matrix S encountered earlier, and the question arises as to why it should be thought of as analogous to S.

Justification for the analogy is provided by establishing a Lyapunov function $V=x'Px$ for $x(k+1)=\Sigma x(k)$ with P diagonal and with $\Delta V=-x'Qx$ with Q diagonal and of rank 1. To round off this part of the material, we note the connection between the Δ_i and the Schur–Cohn determinants.

Some of these results for the case of real $f(s)$ are obtained by Mansour (1965); here we present the complex version. Simplifications analogous to those provided by the Liénard–Chipart criterion are however possible in the real case, and we note these here for the first time. In this case we establish the connection between Δ_j and the reduced Schur–Cohn determinants.

Let $f(s)=s^n+\sum_{i=1}^{n} a_i s^{n-i}$ be an nth degree complex polynomial. We shall define a sequence of polynomials $f(s)=F_n(s)$, $F_{n-1}(s)$, $F_{n-2}(s)$, ... of degree, $n, n-1, n-2, \ldots$ via a table of long standing (Cohn 1922, Jury and Blanchard 1961, Jury 1964) which parallels the Routh table. Let

$$F_j(s)=\sum_{i=0}^{j} a_{ij}s^{j-i}, \quad a_{0j}=1 \tag{20}$$

with the table

$$\left.\begin{array}{cccccc}
1 & a_{1_n}=a_1 & a_{2_n}=a_2 & \cdots & . & a_{nn=a_n} \\
a^*_{nn} & a^*_{n-1,\,n} & a^*_{n-2,\,n} & \cdots & . & 1 \\
1 & a_{1,\,n-1} & a_{2,\,n-1} & \cdots & a_{n-1,\,n-1} & \\
a^*_{n-1,\,n-1} & a^*_{n-2,\,n-1} & a^*_{n-3,\,n-1} & \cdots & 1 & \\
\vdots & & & & &
\end{array}\right\} \tag{21}$$

The formula giving entries of any odd numbered row in terms of entries of the previous two rows is

$$a_{i,\,j-1}=\frac{a_{i,\,j}-a_{jj}a^*_{j-1,\,j}}{1-|a_{jj}|^2} \tag{22}$$

and this is equivalent to

$$F_{j-1}(s)=\frac{F_j(s)-a_{jj}s^n F^*_j(s^{-1})}{s(1-|a_{jj}|^2)} \tag{23}$$

where the superscript star denotes a complex conjugation of coefficients but not variables.

If any one a_{jj} has magnitude 1, the table cannot be directly formed. Otherwise, it can. We define

$$\Delta_j=a_{jj}, \quad j=1, 2, \ldots, n$$

The Δ_j will be used to define an analogue to the Schwarz matrix shortly. First however, note that given the Δ_j, it is possible to recursively obtain the $F_j(s)$. Indeed, manipulation of (23) will show that

$$\left.\begin{array}{l}
F_j(s)=sF_{j-1}(s)+\Delta_j s^{j-1}F^*_{j-1}(s^{-1}) \\
F_0(s)=1
\end{array}\right\} \tag{24}$$

In terms of the coefficients a_{ij} of $F_j(s)$, (24) is

$$\left.\begin{aligned} &a_{ij}=a_{i,\,j-1}+\Delta_j a^*_{j-i,\,j-1}, \quad 1\leqslant i\leqslant j-1 \\ &a_{0j}=1, \quad a_{jj}=\Delta_j \end{aligned}\right\} \tag{25}$$

Having settled the relation between the coefficients a_i of $f(s)$ and the quantities Δ_i, we shall define the analogue of the Schwarz matrix S and establish its similarity to a companion matrix.

Proposition 3

Let A be the companion matrix associated with $f(s)$, as defined in (6); with the Δ_i as defined above, let

$$\Sigma=\begin{bmatrix} -\Delta^*_{n-1}\Delta_n & 1-|\Delta_{n-1}|^2 & 0 & 0 & . & . \\ -\Delta^*_{n-2}\Delta_n & -\Delta^*_{n-2}\Delta_{n-1} & 1-|\Delta_{n-2}|^2 & 0 & . & . \\ -\Delta^*_{n-3}\Delta_n & -\Delta^*_{n-3}\Delta_{n-1} & -\Delta^*_{n-3}\Delta_{n-2} & 1-|\Delta_{n-3}|^2 & . & . \\ . & . & . & . & & . \\ . & . & . & . & . & 1-|\Delta_1|^2 \\ -\Delta_n & -\Delta_{n-1} & -\Delta_{n-2} & . & . & -\Delta_1 \end{bmatrix} \tag{26}$$

and with the a_{ij} as defined above, let

$$T=\begin{bmatrix} 1 & a^*_{1,\,n-1} & a^*_{2,\,n-1} & \dots & a^*_{n-1,\,n-1} \\ 0 & 1 & a^*_{1,\,n-2} & \dots & a^*_{n-2,\,n-2} \\ 0 & 0 & 1 & \dots & a^*_{n-3,\,n-3} \\ 0 & 0 & 0 & \dots & . \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & a^*_{1,\,1} \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{27}$$

Then

$$TAT^{-1}=\Sigma \tag{28}$$

Proof

Using (25), we have

$$\begin{aligned} a_{r,\,n}&=a_{r,\,n-1}+\Delta_n a^*_{n-r,\,n-1} \quad 1\leqslant r\leqslant n-1 \\ &=a_{r,\,n-2}+\Delta_{n-1}a^*_{n-1-r,\,n-2}+\Delta_n a^*_{n-r,\,n-1} \quad 1\leqslant r\leqslant n-2 \\ &=a_{r,\,n-3}+\Delta_{n-2}a^*_{n-2-r,\,n-3}+\Delta_{n-1}a^*_{n-1-r,\,n-2}+\Delta_n a^*_{n-r,\,n-1} \\ &\qquad\qquad 1\leqslant r\leqslant n-3 \end{aligned}$$

and so on, which implies

$$\begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{n-1,n} \\ a_{n,n} \end{bmatrix} = \begin{bmatrix} 1 & a^*_{1,1} & a^*_{2,2} & a^*_{3,3} & \cdots & . & a^*_{n-1,n-1} \\ 0 & 1 & a^*_{1,2} & a^*_{2,3} & \cdots & . & a^*_{n-2,n-1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & . & \vdots \\ 0 & 0 & . & . & \cdots & 1 & a^*_{1,n-1} \\ 0 & 0 & . & . & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ . \\ \Delta_n \end{bmatrix}$$

and in a similar way, one has

$$\begin{bmatrix} a_{1,r} \\ a_{2,r} \\ \vdots \\ a_{r-1,r} \\ a_{r,r} \end{bmatrix} = \begin{bmatrix} 1 & a^*_{1,1} & a^*_{2,2} & . & . & a^*_{r-1,r-1} \\ 0 & 1 & a^*_{1,2} & . & . & a^*_{r-2,r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & 1 & a^*_{1,r-1} \\ 0 & 0 & 0 & . & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{r-1} \\ \Delta_r \end{bmatrix}$$

If one now evaluates TA and ΣT, the relations just stated establish equality of these matrices. Since $\det T = 1$, T^{-1} exists and $TAT^{-1} = \Sigma$.

As noted in the introduction of this section, Σ deserves to be regarded as an analogue of S in view of the constructability of a certain Lyapunov function for the discrete-time autonomous linear system $x(k+1) = \Sigma x(k)$. Let us now explore this point. From the theory associated with the table form, the main result follows that $f(s)$ has all zeros inside $|s| < 1$ if and only if $|\Delta_j| < 1$ for all j. We can however give a Lyapunov-type proof of this result. First suppose that $|\Delta_j| < 1$ for all j. Define $n \times n$ matrices

$$P = \operatorname{diag}\{1 - |\Delta_n|^2, (1 - |\Delta_{n-1}|^2)(1 - |\Delta_n|^2), \ldots, (1 - |\Delta_1|^2) \ldots (1 - |\Delta_{n-1}|^2)(1 - |\Delta_n|^2)\}$$

$$Q = \operatorname{diag}\{(1 - |\Delta_n|^2)^2, 0, 0, \ldots, 0\}$$

Evidently, P is positive definite. We also have by an easy calculation

$$P - \Sigma' P \Sigma = Q \tag{29}$$

This states that if $V(x) = x'Px$ is taken as a Lyapunov function for $x(k+1) = \Sigma x(k)$, then $\Delta V = -x'Qx$. It is immediate that all eigenvalues of Σ lie in $|s| \leqslant 1$. Suppose that for all time, $\Delta V \equiv 0$. Then $x_1(k) = 0$ for all k; from the form of Σ, $x_1(k+1) = -\Delta^*_{n-1}\Delta_n x_1(k) + (1 - |\Delta_{n-1}|^2)x_2(k)$, and so one must have $x_2(k) = 0$ for all k also; continuing the process leads to $x_i(k) = 0$ for all k and for all i, i.e. $x(k) = 0$ for all k. Therefore ΔV is not identically zero along a non-zero trajectory, and $x(k+1) = \Sigma x(k)$ is asymptotically stable. This means that the roots of $f(s)$ (which is the characteristic polynomial of A, and therefore $\Sigma = TAT^{-1}$) must lie in $|s| < 1$.

Conversely, suppose $f(s)$ has all roots in $|s| < 1$, so that $x(k+1) = \Sigma x(k)$ is asymptotically stable. From (29), we have

$$P = \sum_{j=0}^{\infty} (\Sigma')^j Q \Sigma^j \geqslant 0$$

and therefore $|\Delta_j| \leqslant 1$ for all j. Suppose P is singular, with $Px(0)=0$ for some non-zero real $x(0)$. Let $x(0)$ initialize $x(k+1)=\Sigma x(k)$. It follows easily from (29) that $Px(k)=0$ for all k and $Qx(k)=0$ for all k. An analysis as in the previous paragraph shows that $x(k)=0$ for all k, contradicting $x(0)\neq 0$. Hence P is non-singular, so that $|\Delta_j|<1$ for all j.

The connection of Σ with a discrete-time stability problem in a way paralleling the connection of S with a continuous-time stability problem suggests that Σ should be designated as a discrete-time Schwarz matrix, S a continuous-time Schwarz matrix.

Now let us indicate how the quantities Δ_i are connected with quantities appearing in the Schur–Cohn test (Cohn 1922, Schur 1918, Fujiwara 1926). The determinantal form of the Schur–Cohn criterion (Cohn 1922, Schur 1918) states that for the nth degree polynomial† $\bar{f}(s)=\bar{a}_0 s^n + \sum_{i=1}^{n} \bar{a}_i s^{n-i}$, all zeros are inside $|s|<1$ if

$$(-1)^k D_k > 0, \quad k=1, 2, \ldots, n$$

where

$$D_k = \det \left[\begin{array}{cccc|cccc} \bar{a}_n & 0 & \ldots & 0 & \bar{a}_0 & \bar{a}_1 & \ldots & \bar{a}_{k-1} \\ \bar{a}_{n-1} & \bar{a}_n & \ldots & 0 & 0 & \bar{a}_0 & \ldots & \bar{a}_{k-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{a}_{n-k+1} & \bar{a}_{n-k+2} & \ldots & \bar{a}_n & 0 & 0 & \ldots & \bar{a}_0 \\ \hline \bar{a}^*_0 & 0 & \ldots & 0 & \bar{a}^*_n & \bar{a}^*_{n-1} & \ldots & \bar{a}^*_{n-k+1} \\ \bar{a}^*_1 & \bar{a}^*_0 & \ldots & 0 & 0 & \bar{a}^*_n & \ldots & \bar{a}^*_{n-k+2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{a}^*_{k-1} & \bar{a}^*_{k-2} & \ldots & \bar{a}^*_0 & 0 & 0 & \ldots & \bar{a}^*_n \end{array}\right]$$

The hermitian matrix form of the criterion (Fujiwara 1926) says that if one defines a hermitian matrix with $i-j$ entry

$$\sum_{k=1}^{\min(i,\,j)} (\bar{a}_{i-k}\bar{a}^*_{j-k} - \bar{a}^*_{n-i+k}\bar{a}_{n-j+k})$$

then this matrix is positive definite if and only if $f(s)$ has all zeros inside $|s|<1$. Let C_k be the kth leading principal minor. As shown by Wilf (1959)

$$C_k = (-1)^k D_k \tag{30}$$

Evidently, it is enough to relate the C_k to the Δ_j, which are defined as the a_{jj} entries of the table (21), constructed for the polynomial (20) with leading coefficient unity. Now the C_k are also related to the entries of a table, as shown by Jury (1965, 1971). The table is however different from that of (21) both because the polynomial used as the basis of construction may have

† Note that we have relaxed the constraint that $a_0=1$ to simply $a_0\neq 0$.

a non-unity leading coefficient, and because the rules for constructing successive rows of the table are slightly varied. The modified table is as follows :

$$\left.\begin{array}{cccccc} \bar{a}_{0n}=\bar{a}_0 & \bar{a}_{1n}=\bar{a}_1 & \bar{a}_{2n}=\bar{a}_2 & \cdots & . & \bar{a}_{nn}=\bar{a}_n \\ \bar{a}^*_{nn} & \bar{a}^*_{n-1,\,n} & \bar{a}^*_{n-2,\,n} & \cdots & . & \bar{a}^*_{0n} \\ \bar{a}_{0,\,n-1} & \bar{a}_{1,\,n-1} & \bar{a}_{2,\,n-1} & \cdots & \bar{a}_{n-1,\,n-1} & . \\ \bar{a}^*_{n-1,\,n-1} & \bar{a}^*_{n-2,\,n-1} & \bar{a}^*_{n-3,\,n-1} & \cdots & \bar{a}^*_{0,\,n-1} & . \\ \bar{a}_{0,\,n-2} & \bar{a}_{1,\,n-1} & . & \cdots & . & . \\ \bar{a}^*_{n-2,\,n-2} & \bar{a}^*_{n-3,\,n-2} & . & \cdots & . & . \end{array}\right\} \tag{31}$$

with

$$\left.\begin{array}{l} \bar{a}_{i,\,n-1}=\bar{a}^*_{0,\,n}\bar{a}_{in}-\bar{a}_{nn}\bar{a}^*_{n-i,\,n} \\ \bar{a}_{i,\,n-2}=\bar{a}^*_{0,\,n-1}\bar{a}_{i,\,n-1}-\bar{a}_{n-1,\,n-1}\bar{a}^*_{n-i-1,\,n-1} \\ \bar{a}_{i,\,j-1}=\dfrac{\bar{a}^*_{0,\,j}\bar{a}_{i,\,j}-\bar{a}_{jj}\bar{a}^*_{j-i,\,j}}{\bar{a}_{0,\,j+1}}, \quad j<n-1 \end{array}\right\} \tag{32}$$

The relation between the C_i and $\bar{a}_{ij}$ is

$$C_i=\bar{a}_{0,\,n-i} \tag{33}$$

Now the difference between the tables (21) and (31) lies essentially in the fact that the leading entry of each odd-numbered row in (21) is scaled to unity, whereas this is not done in (31). If one arranges for the two tables to correspond to essentially the same polynomial, by taking $a_i=\bar{a}_i(\bar{a}_0)^{-1}$, then this sort of scaling can be checked to propagate, yielding $a_{ij}=\bar{a}_{ij}(\bar{a}_{0j})^{-1}$, $j=n, n-1, \ldots, 0$. Then we have

$$C_1=\bar{a}_{0,\,n-1}=|\bar{a}_{0n}|^2-|\bar{a}_{nn}|^2=|\bar{a}_{0n}|^2[1-|a_{nn}|^2]=|\bar{a}_0|^2[1-|\Delta_n|^2]$$

$$C_2=\bar{a}_{0,\,n-2}=|\bar{a}_{0,\,n-1}|-|\bar{a}_{n-1,\,n-1}|^2=|\bar{a}_{0,\,n-1}|^2[1-|a_{n-1,\,n-1}|^2]$$
$$=[1-|\Delta_{n-1}|^2][1-|\Delta_n|^2]^2|\bar{a}_0|^4$$

$$C=\bar{a}_{0,\,n-3}=\frac{|\bar{a}_{0,\,n-2}|^2-|\bar{a}_{n-2,\,n-2}|^2}{\bar{a}_{0,\,n-1}}=\frac{|\bar{a}_{0,\,n-2}|^2}{\bar{a}_{0,\,n-1}}[1-|\bar{a}_{n-2,\,n-2}|^2]$$
$$=[1-|\Delta_{n-2}|^2][1-|\Delta_{n-1}|^2]^2[1-|\Delta_n|^2]^3|\bar{a}_0|^6$$

and in general

$$C_i=(1-|\Delta_{n+1-i}|^2)(1-|\Delta_{n+2-i}|^2)^2\ldots(1-|\Delta_{n-1}|^2)^{i-1}(1-|\Delta_n|^2)^i|\bar{a}_0|^{2i} \tag{34}$$

This formula relates entries of the discrete-time Schwarz matrix associated with the polynomial $f(s)=s^n+\sum_{i=1}^{n}a_is^{n-i}$ and determinant values appearing in the Schur-Cohn criterion associated with the polynomial $\bar{f}(s)=\bar{a}_0s^n+\sum_{i=1}^{n}\bar{a}_is^{n-i}$, assuming that $\bar{f}(s)=\bar{a}_0f(s)$, $\bar{a}_0\neq 0$.

In case the polynomial $f(s)$ does not have all zeros inside $|s| < 1$, it is still possible to obtain information about the root distribution relative to the unit circle, so long as none of the quantities $1 - |\Delta_i|^2$ is zero. As noted by Fujiwara (1926), so long as the Schur–Cohn hermitian matrix is non-singular, i.e. $C_n \neq 0$, the number of positive eigenvalues n_+ of the matrix is the number of roots of $f(s)$ within the unit circle, and the number of negative eigenvalues n_- is the number of roots of $f(s)$ in $|s| > 1$. If all the C_i are non-zero, n_+ and n_- are given by Gantmacher (1959)

$$\left.\begin{aligned} n_+ &= \text{permanences of sign in the sequence } (1, C_1, C_2, \ldots, C_n) \\ n_- &= \text{variations of sign in the sequence } (1, C_1, C_2, \ldots, C_n) \end{aligned}\right\} \tag{35}$$

Now if $\delta_i = (1 - |\Delta_{n+1-i}|^2)$, we have from (34) that

$$C_1 = \delta_1 |\bar{a}_0|^2$$

and

$$\frac{C_i}{C_{i-1}} = \delta_i \delta_{i-1} \ldots \delta_1 |\bar{a}_0|^2$$

Therefore

$$\left.\begin{aligned} & n_+ = \text{number of positive terms in the sequence} \\ & \qquad\qquad (\delta_1, \delta_1\delta_2, \delta_1\delta_2\delta_3, \ldots, \delta_1\delta_2\delta_3 \ldots \delta_n) \\ \text{and} \quad & \\ & n_- = \text{number of negative terms in the sequence} \\ & \qquad\qquad (\delta_1, \delta_1\delta_2, \delta_1\delta_2\delta_3, \ldots, \delta_1\delta_2\delta_3 \ldots \delta_n) \end{aligned}\right\} \tag{36}$$

As with the Liénard–Chipart simplification discussed in § 3 for the continuous case, we can also obtain a like simplification for the discrete case. If we assume $f(s)$ to be a real polynomial, i.e. all the a_i are real, then the stability condition $|\Delta_j| < 1$ for all j can be significantly simplified. Indeed, we obtain conditions analogous to the Liénard–Chipart criterion for the continuous case.

In the following discussion for the real case, we will only indicate the final results and leave the details for the interested reader to pursue in the pertinent references. Using the Schur–Cohn determinants C_i, we state the following :

1. For stability we require that $C_i > 0$ for $i = 1, 2, \ldots, n$. When the polynomial is real, this condition is equivalent to the following simplified criterion (Jury 1971) :

$$\bar{f}(1) > 0, \quad (-1)^n \bar{f}(-1) > 0, \quad \text{with } \bar{a}_0 > 0 \tag{37}$$

and

$$D^-_{n-1},\ D^+_{n-1} \text{ are positive innerwise} \tag{38}$$

where the matrices $D^{\pm}{}_{n-1} = X_{n-1} \pm Y_{n-1}$, with

$$X_{n-1} = \begin{bmatrix} \bar{a}_0 & \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_{n-2} \\ 0 & \bar{a}_0 & \bar{a}_1 & \dots & \bar{a}_{n-3} \\ 0 & 0 & \bar{a}_0 & \dots & \cdot \\ 0 & \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \cdot & \dots & \bar{a}_0 \end{bmatrix}, \quad Y_{n-1} = \begin{bmatrix} 0 & 0 & \dots & & \bar{a}_n \\ 0 & 0 & & \bar{a}_n & \bar{a}_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \bar{a}_n & \cdot & \dots & \bar{a}_3 & \bar{a}_2 \end{bmatrix}$$

Note that the condition $C_i > 0$, $i = 1, 2, \dots, n$ can also be replaced by one requiring positive definiteness of two half-size symmetric matrices, with principal minors denoted as C_i^+ and C_i^-, $i = 1, 2, \dots, n/2$ for n-even or $n + 1/2$ for n-odd. These conditions are obtained by Anderson and Jury (1963).

We note further that both the sign of the inner determinants of $D^-{}_{n-1}$, $D^+{}_{n-1}$ and the sign of the principal minors of C_i^+, C_i^- can be obtained from the modified table of (31). These are respectively the sign of the difference between the first and last entries of either the *odd* or *even* rows of the modified table in (31). This fact can be readily ascertained from Wall (1945, see p. 104). Also from the table form of Jury (1964), one can obtain the determinants of the inners of $D^-{}_{n-1}$, $D^+{}_{n-1}$ and the principal minors C_i^+, C_i^-.

2. The stability condition just noted can be further simplified as shown by Anderson and Jury (1963, 1974) to yield the reduced Schur–Cohn criterion: with $\bar{a}_0 > 0$,

(*a*) The matrix $D^-{}_{n-1}$ is positive innerwise. (40)

(*b*) For n-odd, say $n \triangleq 2m - 1$, either

$$B_{2i} > 0, \quad B_{2m-1} > 0, \quad \text{or} \quad B_{2i+1} > 0, \quad B_0 > 0, \qquad i = 0, 1, \dots, m-1 \tag{41}$$

where

$$B_i \triangleq \sum_{r=0}^{2m-1} \left[\sum_j (-1)^{r+i-j+1} \bar{a}_r \binom{r}{j} \binom{2m-1-r}{i-j} \right] \tag{42}$$

For n-even, say $n = 2m$, either

$$B_{2i} > 0, \quad i = 0, 1, \dots, m \tag{43}$$

or

$$B_{2i+1} > 0, \quad B_0 > 0, \quad B_{2m} > 0, \quad i = 0, 1, \dots, m-1 \tag{44}$$

where

$$B_i = \sum_{r=0}^{2m} \left[\sum_j (-1)^{r+i-j} \bar{a}_r \binom{r}{j} \binom{2m-r}{i-j} \right] \tag{45}$$

The above condition is the analogue of the Liénard–Chipart criterion for the continuous case.

3. To obtain a 'reduced' criterion in terms of the quantities Δ_j appearing in the Schwarz form statement of the stability conditions, we shall convert the statement (40) to one involving the Δ_j. As one might expect, one obtains a subset of the conditions $-1<\Delta_j<1$; under the assumption $\bar{a}_0>0$, this subset, together with the condition noted in (41) to (45), constitutes the reduced Schwarz form criterion.

Let $\alpha_k^{\pm}$ denote the $k\times k$ inner determinants constructed from D^+_{n-1}. If n is even, k takes odd values only, while if n is odd, k takes even values only. Now it is shown by Jury (1964) that

$$\left.\begin{aligned}
\alpha_1^{\pm} &= \bar{a}_{0n} \pm \bar{a}_{nn} = \bar{a}_{0n}\left(1 \pm \frac{\bar{a}_{nn}}{\bar{a}_{0n}}\right| = \bar{a}_0(1 \pm \Delta_n)\\
\alpha_2^{\pm} &= \bar{a}_{0,\,n-1} \pm \bar{a}_{n-1,\,n-1} = C_1(1 \pm \Delta_{n-1}) = \alpha_1^+\alpha_1^-(1 \pm \Delta_{n-1})\\
\alpha_3^{\pm} &= \frac{\bar{a}_{0,\,n-2} \pm \bar{a}_{n-2,\,n-2}}{\bar{a}_{0n} \mp \bar{a}_{nn}} = \frac{\alpha_2^+\alpha_2^-}{\alpha_1^{\mp}}(1 \pm \Delta_{n-2})\\
\alpha_4^{\pm} &= \frac{\alpha_3^+\alpha_3^-}{\alpha_2^{\mp}}(1 \pm \Delta_{n-3})\\
&\vdots
\end{aligned}\right\} \tag{46}$$

Suppose for the moment that n is even. We shall translate the positivity of the α_i^-, $i=1, 3, \ldots, n-1$ into conditions on the Δ_j. Note first from (47) that if

$$0<\alpha_3^- = \frac{\alpha_2^+\alpha_2^-}{\alpha_1^+}(1-\Delta_{n-2}) = \frac{(\alpha_1^+\alpha_1^-)^2(1-\Delta_{n-1}^2)(1-\Delta_{n-2})}{\alpha_1^+}$$

then one must have α_1^+, α_1^-, α_2^+, α_2^- non-zero. One can continue this argument in fact to show that if $\alpha_k^->0$ for odd k, then $\alpha_j^{\pm}$ is non-zero for all $j<n-1$. Now with $\bar{a}_0>0$, $\alpha_1^->0$ if and only if

$$1-\Delta_n>0 \tag{47}$$

Next, for $n>k>2$, we have

$$\alpha_k^- = \frac{\alpha^+_{k-1}\alpha^-_{k-1}}{\alpha^+_{k-2}}(1-\Delta_{n-k+1})$$

Replace $\alpha^{\pm}_{k-1}$ and α^+_{k-2} by the expressions available for them in (46). This yields, with α^-_0 and α^-_{-1} both taken as 1,

$$\alpha^-_k = \frac{(\alpha^+_{k-2}\alpha^-_{k-2})^2}{(\alpha^+_{k-3}\alpha^-_{k-3})^2}\alpha^-_{k-4}(1-\Delta_{n-k+1})(1-\Delta^2_{n-k+2})(1+\Delta_{n-k+3})^{-1} \tag{48}$$

Hence $\alpha^-_3>0$, $\alpha^-_5>0$, ... if and only if

$$\left.\begin{aligned}
(1+\Delta_n)(1-\Delta^2_{n-1})(1-\Delta_{n-2})>0\\
(1+\Delta_{n-2})(1-\Delta^2_{n-3})(1-\Delta_{n-4})>0\\
\vdots\qquad\\
(1+\Delta_4)(1-\Delta_3^2)(1-\Delta_2)>0
\end{aligned}\right\} \tag{49}$$

Equations (47) and (49) *are together necessary and sufficient for D^-_{n-1} to be positive innerwise with n-even.* A minor variation applies for n-odd. A second reduced criterion can be obtained by replacing the requirement that D^-_{n-1} be positive innerwise in (40) by one requiring D_n^+ to be positive innerwise. The details are omitted.

5. Conclusions

The paper has two directions of thrust. The lesser one involves showing that the usual Schwarz matrix of a complex polynomial can be related to that of a real polynomial, with this fact being tied in with a Liénard–Chipart-type simplification of the stability properties of a real polynomial, as described by the Schwarz matrix.

The second main thrust develops an analogue of the Schwarz matrix applicable to root-distribution problems relative to the unit circle. Parallels of the results concerning the usual Schwarz matrix are obtained.

References

Anderson, B. D. O., and Jury, E. I., 1963, *I.E.E.E. Trans. autom. Control*, **18,** 157 ; 1974, *Int. J. Control*, **19,** 877.

Anderson, B. D. O., Jury, E. I., and Chaparro, F. L. C., 1974, *I.E.E.E. Conference on Decision and Control*, p. 245 ; 1975, *I.E.E.E. Trans. autom. Control*, **20,** 244.

Barnett, S., 1971, *Matrices in Control Theory* (London : Van Nostrand Reinhold Company).

Barnett, S., and Storey, C., 1967, *I.E.E.E. Trans. autom. Control*, **12,** 117.

Brown, B. M., 1965, *Quart. J. Math.*, **16,** 241.

Chen, C. F., 1974, *Int. J. Control*, **19,** 757.

Chen, C. F., and Chu, H., 1966, *I.E.E.E. Trans. autom. Control*, **11,** 303 ; 1967, *Ibid.*, **12,** 458.

Cohn, A., 1922, *Math. Zeitschrift*, **14,** 110.

Fujiwara, M., 1926, *Math. Zeitschrift*, **24,** 160.

Gantmacher, F. R., 1959, *The Theory of Matrices* (New York : Chelsea).

Jury, E. I., 1964, *Theory and Application of the z-Transform Method* (New York : John Wiley & Sons Inc.) ; 1965, *Proc. I.E.E.E.*, **53,** 184 ; 1971, *I.E.E.E. Trans. autom. Control*, **16,** 233 ; 1974, *Inners and Stability of Dynamic Systems* (New York : John Wiley).

Jury, E. I., and Blanchard, J., 1961, *Proc. Inst. Radio Engrs*, **44,** 1947.

Kalman, R. E., and Bertram, J. E., 1960, *J. bas. Engng*, D, **82,** 371.

Liénard, A., and Chipart, M. H., 1914, *J. Math. pure appl.*, **10,** 291.

Loo, S. G., 1968, *I.E.E.E. Trans. autom. Control*, **13,** 309.

Mansour, M., 1965, *Scientia elect.*, **11,** 87.

Parks, P. C., 1962, *Proc. Camb. phil. Soc. math. phys. Sci.*, **58,** 694.

Power, H. M., 1969, *I.E.E.E. Trans. autom. Control*, **14,** 205.

Puri, N. N., and Weygandt, C. N., 1963, *J. Franklin Inst.*, **276,** 365.

Schur, J., 1918, *J. reine angew. Math.*, **148,** 122.

Schwarz, H. R., 1956, *Z. angew. Math. Phys.*, **7,** 473.

Wall, H., 1945, *Am. Math. Monthly*, **52,** 308.

Wilf, H. S., 1959, *J. ACM*, **6,** 363.