Reprinted from JOURNAL OF THE FRANKLIN INSTITUTE, Vol. 281, No. 5, May, 1966
Printed in U. S. A.

Equivalence of Linear Time-Invariant Dynamical Systems

by B. D. O. ANDERSON, R. W. NEWCOMB, R. E. KALMAN
Stanford Electronics Laboratories, Stanford, California
and D. C. YOULA
Department of Electrophysics
Polytechnic Institute of Brooklyn, Farmingdale, New York

ABSTRACT: *Using constant transformations and matrix embeddings it is shown how to find all constant realizations of an arbitrary rational transfer function matrix. Explicit means of calculating the transformations are given and the theory is applied to finding essentially all equivalent networks.*

I. Introduction

The identification of all system realizations of a given transfer function is of considerable theoretical and practical importance. If we conceive of a realization of a transfer function matrix as the specification of a set of state equations yielding the transfer function, then considerable information is available. For example, if by a minimal realization is meant one having the state space of minimum dimension, then there are methods for finding minimal realizations for a linear finite-dimensional time-invariant system (1, p. 175), (2, p. 530), (3, p. 30), (4), (5), (6, p. 14). From any one such realization all other (constant) minimal realizations can be obtained by applying constant transformations to the state (2, p. 525). For nonminimal realizations Youla (3, p. 13) has given a method for finding all realizations via a time-variable coordinate change.

In a similar vein and limiting ourselves to this linear time-invariant situation, we show how all nonminimal constant realizations may be generated, through constant transformations, from any one such realization.

The practical interest in obtaining nonminimal realizations through an equivalence theory is manifold. For example, in some cases we may easily find a minimal realization and for engineering reasons desire to modify this to be able to accomodate fixed, but not minimally realized, equipment. Or, we may be required to use excess state variables in a desire to delete unwanted elements, as gyrators in the cascade synthesis of network theory (7, p. 279), (8, p. 112). Theoretical interest in equivalence theory lies, among other things, in the fact that general properties of systems can be studied by applying appropriate transformations to a fixed but known system.

The structure of this work is as follows: In Section II we review the basic background ideas and introduce appropriate notation. The short Section III presents the primary equivalence result, while Section IV gives an alternate method to those presently available for finding the principal transformations. In Section V is found an application of the theory to network equivalence through the scattering matrix.

II. Preliminary Material

Following Kalman (1) we consider only those real differential systems described by

$$\frac{dx}{dt} = Fx + Gu \tag{1a}$$

$$y = Hx + Ju \tag{1b}$$

where u, x, and y are m, n, and p-vectors respectively (the input, state, and output). We also assume, without further comment, that F, G, H, and J are real constant matrices. Under these conditions the quadruple $R = \{F, G, H, J\}$ is called a (constant) *realization* of the transfer function matrix $W(s)$ if

$$W(s) = J + H[s1_n - F]^{-1}G \tag{2}$$

where 1_n is the $n \times n$ unit matrix. In actual fact $R = \{F, G, H, J\}$ directly yields a physical realization of $W(s)$ as an input-output description for the system since it specifies a set of analog computer connections.

Likewise, for this class of constant differential systems Kalman (1, p. 172) has shown, and we will discuss later, that there exists a real constant nonsingular matrix T_c to bring the realization into *canonical* form $R_c = \{F_c, G_c, H_c, J\}$ with

$$F_c = T_cFT_c^{-1}, \qquad G_c = T_cG, \qquad H_c = HT_c^{-1} \tag{3a}$$

$$= \begin{bmatrix} F^{AA} & F^{AB} & F^{AU} \\ 0 & F^{BB} & F^{BU} \\ 0 & 0 & F^{UU} \end{bmatrix} \qquad = \begin{bmatrix} G^A \\ G^B \\ 0 \end{bmatrix} \qquad = [0, H^B, H^U]. \tag{3b}$$

The system of differential equations for this realization is the same as in Eq. 1 except that subscript c's are inserted and $x_c = T_cx$, $\tilde{x}_c = [\tilde{x}^A, \tilde{x}^B, \tilde{x}^U]$ with a superscript tilde $\sim$ denoting the transpose. The superscript letters physically have the following meanings, which, in fact, leads us to the canonical forms

A: controllable and unobservable
B: controllable and observable
U: uncontrollable

If $\delta = \delta[W(s)]$ is the degree of $W(s)$ (9, p. 580), (2, p. 536) then x^B is a δ-vector and the minimum possible dimension of the state space in any realization of $W(s)$ is given by $n = \delta$ (2, p. 536). In such a case the realization is called *minimal* (or irreducible); all realizations with a state space of greater dimension are naturally called *nonminimal* (or reducible). Through Eqs. 2 and 3 we can readily check that $R_M = \{F^{BB}, G^B, H^B, J\}$ is a minimal canonical realization. Further any other constant minimal realization $\hat{R}_M = \{\hat{F}^{BB}, \hat{G}^B, \hat{H}^B, J\}$ can be found from (4, prop. 3), (2, p. 525)

$$\hat{F}^{BB} = TF^{BB}T^{-1}, \qquad \hat{G}^B = TG^B, \qquad \hat{H}^B = H^BT^{-1} \tag{4}$$

where T is a real constant nonsingular $\delta \times \delta$ matrix.

III. Equivalence Results

With a knowledge of the results of Section II, equivalence results are deceptively simple.

We first comment that if we are given an arbitrary (constant) realization $\hat{R} = \{\hat{F}, \hat{G}, \hat{H}, J\}$ of a transfer function $W(s)$ then a minimal realization $\hat{R}_M$ is again found by reducing to canonical form

$$\hat{F} = \hat{T}_c^{-1}\begin{bmatrix} \hat{F}^{AA} & \hat{F}^{AB} & \hat{F}^{AU} \\ 0 & \hat{F}^{BB} & \hat{F}^{BU} \\ 0 & 0 & \hat{F}^{UU} \end{bmatrix}\hat{T}_c, \qquad \hat{G} = \hat{T}_c^{-1}\begin{bmatrix} \hat{G}^A \\ \hat{G}^B \\ 0 \end{bmatrix}, \qquad \hat{H} = [0, \hat{H}^B, \hat{H}^U]\hat{T}_c \tag{5}$$

and extracting entries for $\hat{R}_M = \{\hat{F}^{BB}, \hat{G}^B, \hat{H}^B, J\}$. Further, since only $\hat{F}^{BB}$, $\hat{G}^B$, and $\hat{H}^B$ enter in $\hat{R}_M$, all other nonzero submatrices in Eq. 5 can be arbitrarily assigned and then $\hat{R}$ is again a realization of $W(s)$. As seen by choosing the $\hat{G}^A$ term zero, the superscript A in an arbitrary assignment may not denote controllable states, yet the B portions continue to have $\hat{R}_M$ for a minimal realization, as seen from Eq. 2. Still, if so desired, a further transformation on the state allows the "arbitrary" realization to again be put into canonical form with $\hat{R}_M$ a minimal realization. In actual fact, absorbing a suitable transformation in $\hat{T}_c$ allows $\hat{R}_M$ to be chosen as any minimal realization, by Eq. 4. From these comments we have the following conclusion:

Main Result: *Every (constant) realization $\hat{R}$ of $W(s)$ results from a given minimal one $\hat{R}_M$ by using Eq. 5 with an arbitrary assignment of the remaining ("nonzero") submatrices (including non-singular $\hat{T}_c$).*

For discussion purposes it is convenient to call $\hat{R}_c = \{\hat{T}_c\hat{F}\hat{T}_c^{-1}, T_c\hat{G}, \hat{H}\hat{T}_c^{-1}, J\}$ an *encirclement* of $\hat{R}_M$.

If we wish to transfer from one fixed realization R to any other $\hat{R}$ we can proceed by reducing to R_c and $\hat{R}_c$, the respective canonical forms, and then interrelating the resulting minimal realizations R_M and $\hat{R}_M$ by Eq. 4. The steps in obtaining $\hat{R}$ from R are then: 1) reduction of R to R_c; 2) extraction of R_M, 3) derivation of $\hat{R}_M$ from R_M via Eq. 4; 4) encirclement of $\hat{R}_M$ to obtain $\hat{R}_c$; 5) transformation of $\hat{R}_c$ into $\hat{R}$ by $\hat{T}_c$. The procedure is illustrated in Fig. 1.

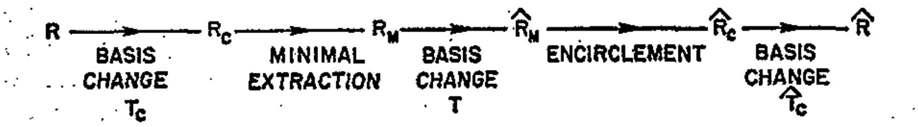

FIG. 1. Derivation of any $\hat{R}$ from a given one R.

IV. Canonical Transformation Determination

Kalman (**1**, p. 172) has shown how to find T_c to transform R into canonical form R_c. However, the method requires integration of products of the fundamental matrix with G and H. Here we describe an alternate method which avoids integration by using columns of

$$Q = [G, FG, \cdots, F^{n-1}G] \tag{6a}$$

$$P = [\tilde{H}, \tilde{F}\tilde{H}, \cdots, \tilde{F}^{n-1}\tilde{H}]. \tag{6b}$$

We first observe (**10**, pp. 500, 504) that the set of controllable states is the space spanned by the columns of Q and that the set of unobservable states is perpendicular to the space spanned by the columns of P.

Following Kalman we obtain T_c as the product of two matrices. Let T_I, the former of these two matrices, have its first q, $q = \text{rank } Q$, columns as any q independent columns of Q with the remaining $n - q$ columns arbitrarily chosen but independent (such that T_I is nonsingular). As is seen by considering q independent state vectors with a single unit in one of the first q positions and all other entries zero, the first transformation yields

$$T_I^{-1}FT_I = \begin{bmatrix} F^{KK} & F^{KU} \\ 0 & F^{UU} \end{bmatrix}, \qquad T_I^{-1}G = \begin{bmatrix} G^K \\ 0 \end{bmatrix}, \qquad HT_I = [H^K, H^U]. \tag{7a}$$

Here the superscript K refers to controllable portions with U uncontrollable, as before. It is worthwhile noting that, by premultiplying T_I^{-1} by an upper triangular orthogonalizing matrix, it is possible to use an orthogonal T_I to obtain Eq. 7a.

We next consider the subsystem realized by $R^K = \{F^{KK}, G^K, H^K, J\}$ for which we form P^K according to Eq. 6b. If $p_K = \text{rank } P^K$, we select p_K linearly independent columns of P^K and orthogonalize these, via the Gram Schmidt procedure (say), and then extend them to a complete orthogonal set of vectors for the q dimensional state space for R^K. If we use these q vectors, ordered reverse to their order of generation, as columns for a matrix T_K, then the relation of P^K to the unobservable states for R^K shows that

$$T_K F^{KK} T_K^{-1} = \begin{bmatrix} F^{AA} & F^{AB} \\ 0 & F^{BB} \end{bmatrix}, \qquad T_K G^K = \begin{bmatrix} G^A \\ G^B \end{bmatrix}, \qquad H^K T_K^{-1} = [0, H^B]. \quad (7b)$$

If k is the order of F^{KK} and 1_{n-k} the identity matrix of order $n - k$ then

$$T_a = [T_K \dotplus 1_{n-k}] T_I^{-1} \quad (7c)$$

where $\dotplus$ denotes the direct sum.

V. Passive Network Equivalence

As an application of the theory let us find all finite equivalent circuits for an arbitrary finite passive m-port network (assumed linear, time-invariant, and solvable (11, p. 9). Such an m-port possesses a rational bounded-real $m \times m$ scattering matrix $S(p)$, $p = \sigma + j\omega$ (12, p. 116), (13, Sec. 5), from which a realization can be found from analysis of the circuit structure, using only passive elements, obtained by standard synthesis techniques (14, p. 163).

To begin such an analysis it is most convenient to replace each inductor by a gyrator loaded in a unit capacitor and then to normalize all other capacitors to unity by perhaps the use of transformers. If there are n capacitors, these can be removed and considered as a load on a gyrator-resistor-transformer network N_Σ, as shown in Fig. 2 (where $\frac{1-p}{1+p} 1_n$ is the scattering matrix of the capacitors). If we set

$$s = \frac{1+p}{1-p} \quad (8)$$

and treats τ as the time associated with s, and if further $v_c^i(\tau)$ and $v_c^r(\tau)$ are the capacitor incident and reflected n-vector voltages, then

$$v_c^i = \frac{d}{d\tau} v_c^r \quad (9a)$$

since $1/s$ is the reflection coefficient of a capacitor. Similarly, if $v^i(\tau)$ and $v^r(\tau)$ are the input port incident and reflected m-vector voltages and we partition Σ, the constant scattering matrix of N_Σ, as the ports of N_Σ, then

$$\begin{bmatrix} v^r \\ v_c^i \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} v^i \\ v_c^r \end{bmatrix}. \quad (9b)$$

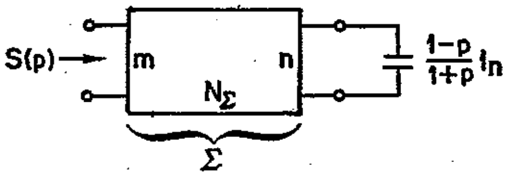

FIG. 2. Cascade loading for network equivalence.

Combining Eqs. 9 we immediately find

$$\frac{dv_c^r}{d\tau} = \Sigma_{22} v_c^r + \Sigma_{21} v^i \tag{10a}$$

$$v^r = \Sigma_{12} v_c^r + \Sigma_{11} v^i \tag{10b}$$

which is Eq. 1 in disguise; thus

$$W(s) = \Sigma_{11} + \Sigma_{12}[s1_n - \Sigma_{22}]^{-1}\Sigma_{21} = S(p). \tag{11}$$

The left equality in Eq. 11 follows from Eq. 2 while the right follows from direct analysis of Fig. 2 (**11**, p. 11); this shows that $v^i(t)$ and $v^r(t)$ are the actual input and output variables in true time t.

We conclude that a realization in the sense of Section II is $R = \{\Sigma_{22}, \Sigma_{21}, \Sigma_{12}, \Sigma_{11}\}$, and that a specification of R yields the constant coupling network matrix Σ.

Applying the ideas of Section III, we can find all R or, what is the same, all Σ to yield $S(p)$, once a given realization is known (say, from synthesis). In actual fact, besides yielding all passive circuits, this yields all active structures for which Σ exists. We can sift out the passive ones by the bounded-real requirement on Σ, that is by retaining only those for which $1_{n+m} - \tilde{\Sigma}\Sigma$ is positive semidefinite. We remark that $\Sigma_{11} = S(1)$ which is well-defined, as is $S(\infty)$, when $S(p)$ is bounded-real.

To complete the network equivalence study we observe that the theory does not consider the topological structure of the circuits synthesizing Σ. However, the Howitt theory (**15**) immediately applies to derive all circuits for Σ from a given one. To go along with this is the fact that the transformations of Section IV yield $\Sigma_e = [1_n \dotplus T_e]\Sigma[1_m \dotplus T_e^{-1}]$, which need not be bounded-real when Σ is.

This same transformation carries over to the impedance matrix, a fact which may assist in realizability studies. Nevertheless, if T_e is chosen orthogonal, as in Section IV where transformations among minimal realizations are omitted, then $\Sigma_e = [1_m \dotplus T_e]\Sigma[1_m \dotplus \tilde{T}_e]$ allows Σ_e to be derived from Σ through the use of orthogonal transformers.

In summary, since any circuit structure yielding $S(p)$ takes the form of Fig. 2, the cited method yields all passive connections, and almost all active ones, which produce a rational bounded-real $S(p)$.

VI. Discussion

Given a rational transfer function $W(s)$ it is usually quite easy to find one realization; here, by using only constant transformations we have shown how to find all realizations if $W(s)$ has no pole at infinity. The basic idea is to pick out a minimal realization by transforming to canonical form and then encircle by adjoining pertinent submatrices. Although the theory was framed for differential systems it can of course be directly applied to discrete systems (**2**, p. 524).

Since we have specified a realization to mean an assignment of $R = \{F, G, H, J\}$ it is worth pointing out that this assignment is equivalent to an analog computer set-up, and often to an equipment interconnection. As a consequence there is some interest in finding all realizations since we may have only certain equipment on hand, as flip-flops in the discrete case. Alternately we may wish to pass from one nonminimal realization to another realization to eliminate unstable states.

The restriction to $W(s)$ which are finite at infinity can be bypassed by a simple transformation analogous to the conversion of an impedance matrix to the scattering matrix in network theory (**16**). For square $W(s)$ let $\mathfrak{R}$ be a real positive definite (symmetric) matrix and define new input and output variables u_r and y_r by

$$2u_r = y + \mathfrak{R}u \tag{12a}$$

$$2y_r = y - \mathfrak{R}u \tag{12b}$$

then the transfer function matrix $W_r(s)$ for these new variables is given by

$$W_r(s) = [W(s) - \mathfrak{R}][W(s) + \mathfrak{R}]^{-1}. \tag{13}$$

In actual fact $W_r(s)$ will exist and be "bounded" at infinity for almost all such $\mathfrak{R}$; we choose any suitable one and carry out the theory on $W_r(s)$. If $W(s)$ is not square it can be made so by augmenting with rows or columns of zeros, which can be ignored at the end of the analysis.

In the network case the notion of Eq. 13 allows Section V to apply to any active network. Also in the network case we can apply the ideas of Section V directly to immittances (**3**, p. 30) except that the theory is somewhat restrictive; we can also interpret $s = d/dt$ directly for equations of the form of Eq. 10 using $S(s)$, since $S(\infty)$ is "bounded," but the physical interpretation of the realization R is not as clear. Further, synthesis from Eq. 11 can be investigated with profitable results (**17**).

Concerning the transformations of Section IV we can apply the ideas used on R^K to $R^U = \{F^{UU}, 0, H^U, J\}$, but this is unnecessary for the theory. Along the same lines we can break x^U into observable and unobservable parts following Kalman (**1**, p. 174), but the final transformation seems to require use of the transition matrix. Using ideas similar to those of Section IV Youla (**18**) has described an alternate method of canonical form construction.

Finally we mention that this theory adds another very promising view-point to the already existing theories of equivalence (**15**), (**19**), (**20**, p. 623), (**21**), and especially (**14**).

Acknowledgments

For the preparation of the manuscript the authors acknowledge the support of the National Science Foundation through Grant NSF GK-237 as well as the excellent assistance of Barbara Serrano. The first author would also like to acknowledge the support of an Agency of the Australian Government, the Services Canteens Trust Fund, and the United States Educational Foundation in Australia, for a Fulbright Grant. The assistance of Professor C. A. Desoer in correcting the manuscript is gratefully acknowledged.

References

(1) R. E. Kalman, "Mathematical Description of Linear Dynamical Systems," *Jour. SIAM Control*, Vol. 1, No. 2, pp. 152–192, 1963.

(2) R. E. Kalman, "Irreducible Realizations and the Degree of a Rational Matrix," *Jour. SIAM*, Vol. 13, No. 2, pp. 520–544, June 1965.

(3) D. C. Youla, "The Synthesis of Linear Dynamical Systems from Prescribed Weighting Patterns," Polytechnic Inst. Brooklyn, Rep. PIBMRI-1271-65, June 1, 1965.

(4) B. L. Ho and R. E. Kalman, "Effective Construction of Linear State-Variable Models from Input/Output Data," 1965 *Allerton Conf. Proc.*

(5) B. D. O. Anderson and R. W. Newcomb, "A Canonical Simulation of a Transfer Function Matrix," *IEEE Trans. on Automatic Control*, Vol. AC-11, No. 2, April 1966.

(6) C. A. Desoer, "The Minimal Realization of Impulse Response Matrices," ERL Tech. Memo. M-128, Univ. of Calif., Berkeley, Sept. 13, 1965.

(7) V. Belevtich, "Factorization of Scattering Matrices with Applications to Passive-Network Synthesis," *Philips Res. Rep.*, Vol. 18, No. 4, pp. 275–317, Aug. 1963.

(8) D. C. Youla, "Cascade Synthesis of Passive n-Ports," Polytechnic Inst. Brooklyn, Rep. PIBMRI 1213-64, Aug. 1964.

(9) B. McMillan, "Introduction to Formal Realizability Theory–II," *Bell System Tech. Jour.*, Vol. 31, No. 3, pp. 541–600, May 1952.

(10) L. A. Zadeh and C. A. Desoer, "Linear System Theory," New York, McGraw-Hill Book Co., 1963.

(11) R. W. Newcomb, "The Foundations of Network Theory," The Institution of Engineers Australia, *Elec. and Mech. Eng. Trans.*, Vol. EM-6, No. 1, pp. 7–12, May 1964.

(12) D. C. Youla, L. J. Castriota, and H. J. Carlin, "Bounded Real Scattering Matrices and the Foundations of Linear Passive Network Theory," *IRE Trans. on Circuit Theory*, Vol. CT-6, No. 1, pp. 102–124, Mar. 1959.

(13) B. D. O. Anderson and R. W. Newcomb, "On the Cascade Connection for Time-Invariant *n*-Ports," Proceedings IEE, to be published.

(14) Y. Oono and K. Yasuura, "Synthesis of Finite Passive 2*n*-Terminal Networks with Prescribed Scattering Matrices," *Memoirs of the Faculty of Engineering*, Kyushu Univ., Fukuoka, Japan, Vol. 14, No. 2, pp. 125–177, May 1954.

(15) N. Howitt, "Group Theory and the Electric Circuit," *Phys. Rev.*, Vol. 37, pp. 1583–1595, June 15, 1931.

(16) R. W. Newcomb, "Reference Terminations for the Scattering Matrix," *Electronics Letters*, Vol. 1, No. 4, pp. 82–83, June 1965.

(17) D. C. Youla and P. Tissi, "*n*-Port Synthesis via Reactance Extraction-Part I," 1966 *IEEE Internat. Convention Record*, to be published.

(18) D. C. Youla to R. W. Newcomb, private correspondence, Nov. 1965.

(19) R. S. Burington, "On the Equivalence of Quadrics in *m*- Affine *n*- Space and Its Relation to the Equivalence of 2*m*- Pole Networks," *Trans. Amer. Math. Soc.*, Vol. 38, No. 1. pp. 163–176, July 1935.

(20) W. Cauer, "Synthesis of Linear Communication Networks," New York, McGraw-Hill Book Co., 1958.

(21) J. D. Schoeffler, "Continuously Equivalent Networks and Their Applications," *IEEE Trans. on Commun. and Electr.*, Vol. 83, No. 75, pp. 763–767, Nov. 1964.