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New results for linear-quadratic discrete-time games†

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This paper examines a class of linear-quadratic discrete-time games. Attention is paid to the problem of ascertaining the existence of the open-loop and closed-loop solutions as well as to the forms of these solutions for games with a finite horizon. The new contributions of the paper to existing results for this class of games cover the following points : the introduction of the concepts of nonunique optimal strategies, of the pair of 'minimum energy' optimal strategies, the characterization of the so-called 'no-conjugate-point condition' in these discrete-time games. These results also have various implications, showing that this class of games has peculiar features not shared by its continuous-time counterpart. Finally, an attempt is made to extend the results to the case of games with infinite horizon. Only a few explicit results are obtained in this direction, with the main difficulties indicated.

1. Introduction

Since the publication of the paper by Ho *et al.* (1965), there have appeared in the literature many papers giving results for the class of zero sum, time-invariant linear quadratic games which are deterministic and continuous time. (See, e.g., Rhodes (1967), Schmitendorf and Citron (1969), Bensoussan (1971)). On the other hand, explicit results for the discrete-time counterparts of these games are few and far between ; such discrete-time games have generally not been treated to the same depth as the continuous-time game (see brief dis-discussion in Rhodes (1967) and Bley and Stear (1971)). In this paper we attempt to collect together some results concerning the above class of discrete-time games and to point out some implications of these results. We aim to avoid detailing those results which are obvious parallels of corresponding continuous-time results, seeking rather to emphasize the differences that can be encountered.

As usual, we shall be interested in establishing the existence of (pure) open-loop and (pure) closed-loop saddle-point solutions for this class of discrete-time games and the explicit forms of these solutions. However, we depart from the usual treatment in the sense that we do not insist on the uniqueness of the optimal strategy for each player. As will be seen later, this feature does not increase the difficulty of the solution process for this class of games ; in fact, it results in a slight relaxation on the existence conditions of the saddle-point solutions.

These last results do not have parallels in the continuous-time case. The same situation arises in problems of regulator type, as is well known. A continuous-time linear-quadratic control problem can be either non-singular or singular ; if it is non-singular, the optimal solution, if it exists, is unique and can be determined via a continuous-time Riccati equation, but if it is singular

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(partially or totally) the Riccati equation cannot be defined and different approaches (Moylan and Moore 1971, Moore 1974) must be used to find the optimal controls. On the other hand, for discrete-time linear-quadratic control problems, the singularity question does not arise and the optimal solutions if they exist can be determined directly via a discrete-time Riccati equation (even in the case that the optimal control is not unique).

The paper is structured as follows. Section 2 provides a complete statement of the problem as well as various solution concepts. In § 3 we give results concerning the open-loop game. The first existence condition is not new, being simply a specialized version of a more general existence theorem in game theory (see, e.g., Owen (1969) and Karlin (1959)). However, the second condition (which is only sufficient) for the existence of an open-loop solution as well as the explicit form of the solution may be of some interest. Our main results are those for the closed-loop solution contained in § 4. We shall show that the existence of the closed-loop solution is related to the fact that the well-defined solution of a certain Riccati recursive equation satisfies certain side conditions. These side conditions thus assume the role of the no-conjugate-point condition for this class of discrete-time game (in contrast to the continuous-time case where the same condition takes the form of the required boundedness of the solution of a Riccati differential equation, or, equivalently, the absence of an escape time). In this section we shall discuss a finite-time game as well as an infinite-time game and briefly mention the difficulty of connecting the two via a limiting procedure. Section 5 contains several comments on the results of previous sections and their implications, § 6, the concluding remarks.

2. Statement of the problem

For the deterministic, time-invariant, discrete-time linear dynamic system :

$$\left.\begin{aligned} x(i+1) &= Ax(i) + B_1u_1(i) + B_2u_2(i) \\ x(0) &= x_0 \text{ given} \end{aligned}\right\} \tag{2.1}$$

where A, B_1, B_2 are real matrices of dimension $n \times n$, $n \times m_1$, $n \times m_2$ and $x(i)$, $u_1(i)$, $u_2(i)$ are $n \times 1$, $m_1 \times 1$, $m_2 \times 1$ vectors respectively, consider the game between the two players u_1 and u_2, with u_1 striving to minimize and u_2 to maximize the performance index

$$J[x_0, 0, \alpha, u_1(.), u_2(.)] = \sum_{i=0}^{\alpha-1} [u_1'(i)u_2'(i)x'(i)] \begin{bmatrix} U_1 & U_3 & C_1' \\ U_3' & U_2 & C_2' \\ C_1 & C_2 & Q \end{bmatrix} \begin{bmatrix} u_1(i) \\ u_2(i) \\ x(i) \end{bmatrix} + x'(\alpha)Sx(\alpha) \tag{2.2}$$

Here U_1, U_2, U_3, C_1, C_2, Q and S are real matrices of dimension $m_1 \times m_1$, $m_2 \times m_2$, $m_1 \times m_2$, $n \times m_1$, $n \times m_2$, $n \times n$ and $n \times n$ respectively (with U_1, U_2, Q and S being symmetric without loss of generality) and α is a prescribed integer, $1 < \alpha < \infty$.

In later sections we also use

$$B \triangleq [B_1 \; B_2], \quad C \triangleq [C_1 \; C_2] \quad \text{and} \quad U \triangleq \begin{bmatrix} U_1 & U_3 \\ U_3' & U_2 \end{bmatrix}$$

Depending on the way each player is allowed to carry out the control to achieve his purpose, we have different solution concepts for the above game. In the following we shall be interested only in the cases where both players use open-loop controls and the case where both use closed-loop controls. The first case is called an open-loop game and the second a closed-loop game. Discussions of the case where one player uses an open-loop control while his opponent uses a closed-loop control as well as the case where mixed or randomized controls (in contrast to the pure controls mentioned above) can be employed are beyond the scope of this paper.

We can now give the definitions of the solutions of the open-loop game, corresponding to a prescribed α. For equivalent definitions, see, e.g., Ereshko and Propoi (1970).

Definition 1. (Open-loop game.)

Let $\mathscr{U}_j(\alpha-1)$, $j=1, 2$, denote the control sequences

$$\mathscr{U}_j(\alpha-1) \triangleq [u_j'(\alpha-1)\ u_j'(\alpha-2)\ \ldots\ u_j'(1)\ u_j'(0)]', \quad j=1, 2$$

The α-stage open-loop game is said to have a solution if and only if there exists a pair of control sequences $\mathscr{U}_1^*(\alpha-1)$, $\mathscr{U}_2^*(\alpha-1)$ such that, for any x_0, $\mathscr{U}_1(\alpha-1)$ and $\mathscr{U}_2(\alpha-1)$,

$$\begin{aligned} J[x_0, 0, \alpha, \mathscr{U}_1^*(\alpha-1), \mathscr{U}_2(\alpha-1)] &\leqslant J[x_0, 0, \alpha, \mathscr{U}_1^*(\alpha-1), \mathscr{U}_2^*(\alpha-1)] \\ &\leqslant J[x_0, 0, \alpha, \mathscr{U}_1(\alpha-1), \mathscr{U}_2^*(\alpha-1)] \end{aligned} \tag{2.3}$$

The pair $\mathscr{U}_1^*(\alpha-1)$, $\mathscr{U}_2^*(\alpha-1)$ is then called a pair of open-loop optimal controls [OLOC's] of the game and the optimal value of the performance index, given by $J[x_0, 0, \alpha, \mathscr{U}_1^*(\alpha-1), \mathscr{U}_2^*(\alpha-1)]$, is then called the open-loop value of the game [OLV].

Definition 2. (Closed-loop game.)

Let $T \triangleq [0, \alpha-1]$ and consider the class of mappings $u_j : R^n \times T \to R^{m_j} : x(i), i \to u_j[x(i), i]$ for $j=1, 2$. Then the α-stage closed-loop game is said to have a solution if and only if there exists a pair of mappings $u_1^*[\cdot, \cdot]$, $u_2^*[\cdot, \cdot]$ such that for any x_0 and all $u_1[\cdot, \cdot]$, $u_2[\cdot, \cdot]$

$$\begin{aligned} J[x_0, 0, \alpha, u_1^*[\cdot, \cdot], u_2[\cdot, \cdot]] &\leqslant J[x_0, 0, \alpha, u_1^*[\cdot, \cdot], u_2^*[\cdot, \cdot]] \\ &\leqslant J[x_0, 0, \alpha, u_1[\cdot, \cdot], u_2^*[\cdot, \cdot]] \end{aligned}$$

The pair $u_1^*[\cdot, \cdot]$, $u_2^*[\cdot, \cdot]$ is then called a pair of closed-loop optimal strategies [CLOS] of the game and the optimal value of the performance index, given by $J[x_0, 0, \alpha, u_1^*[\cdot, \cdot], u_2^*[\cdot, \cdot]]$, is then called the closed-loop value [CLV] of the game.

Remark 1. Note that in the formulation of both games given above, we are actually interested in solving a family of open-loop games and a family of closed-loop games, each member of these families corresponding to a given $x_0 \in R^n$. Furthermore, we have not insisted on the uniqueness of the pair of OLOC and the pair of CLOS. The reasons for this have been briefly mentioned in the introduction and will be discussed in detail later.

Let us define the following quantities (assuming that they exist):

$$\min_{\mathscr{U}_1(\alpha-1)} \max_{\mathscr{U}_2(\alpha-1)} J \triangleq w_1^+(x_0)$$

$$\max_{\mathscr{U}_2(\alpha-1)} \min_{\mathscr{U}_1(\alpha-1)} J \triangleq w_1^-(x_0)$$

$$\min_{u_1[\cdot,\cdot]} \max_{u_2[\cdot,\cdot]} J \triangleq w_2^+(x_0)$$

$$\max_{u_2[\cdot,\cdot]} \min_{u_1[\cdot,\cdot]} J \triangleq w_2^-(x_0)$$

$$\min_{u_1[x_0,0]} \max_{u_2[x_0,0]} \cdots \min_{u_1[x(\alpha-2),\alpha-2]} \max_{u_2[x(\alpha-2),\alpha-2]} \min_{u_1[x(\alpha-1),\alpha-1]} \max_{u_2[x(\alpha-1),\alpha-1]} J \triangleq w_3^+(x_0)$$

$$\max_{u_2[x_0,0]} \min_{u_1[x_0,0]} \cdots \max_{u_2[x(\alpha-2),\alpha-2]} \min_{u_1[x(\alpha-2),\alpha-2]} \max_{u_2[x(\alpha-1),\alpha-1]} \min_{u_1[x(\alpha-1),\alpha-1]} J \triangleq w_3^-(x_0)$$

The arguments of J have been omitted for clarity.

Definitions 1 and 2 then imply that the α-stage open-loop game has a solution if and only if $w_1^+(x_0)$ and $w_1^-(x_0)$ both exist and $w_1^+(x_0)=w_1^-(x_0)$; similarly, the α-stage closed-loop game has a solution if and only if $w_2^+(x_0)$ and $w_2^-(x_0)$ both exist and $w_2^+(x_0)=w_2^-(x_0)$. The quantities $w_3^+(x_0)$ and $w_3^-(x_0)$ may be thought of as extreme values for J derived via a dynamic programming approach. For further details, see Ereshko and Propoi (1970) where fundamental relationships are established.

Lemma 1. [Ereshko and Propoi.]

Provided that the quantities $w_j^{\pm}(x_0)$, $j=1, 2, 3$, exist, they are related by

$$w_1^-(x_0) \leqslant w_2^-(x_0) \leqslant w_2^+(x_0) \leqslant w_1^+(x_0)$$
$$w_2^-(x_0) = w_3^-(x_0)$$
$$w_2^+(x_0) = w_3^+(x_0)$$

In the paper by Ereshko and Propoi, this result is established under the assumption that the values taken by u_1 and u_2 are restricted *a priori* to lying in a bounded set. However, the proof of their result can be extended to the situation considered here. Roughly, this is because various game theoretic relationships, such as

$$\max_{z} \min_{y(z)} \psi(y, z) \leqslant \min_{y} \max_{z(y)} \psi(y, z)$$

(where the operation $\min_{y(z)}$ denotes the minimization operation of the player y *knowing the control z of his opponent* and the operation $\max_{y(z)}$ has a similar meaning) which are used to establish Lemma 1 in Ereshko and Propoi (1970) also hold true if $\psi(y, z)$ is a quadratic form, even in the case where y and z are not restricted to lie in bounded sets.

From Lemma 1, it immediately follows that

Corollary 1

(i) If an α-stage open-loop game has a solution, so does the α-stage closed-loop game, while if an α-stage closed-loop game does not have a solution, neither has the α-stage open-loop game. The converse, in general, is not true.

(ii) The α-stage closed-loop game can also be solved by a dynamic programming approach.

3. Open-loop game

To obtain conditions for existence of the α-stage open-loop game associated with (2.1) and (2.2), we shall use the standard approach of reformulating it as an equivalent game in normal form and find conditions relating to this form.

For any α it can be shown by simple manipulations that the α-stage open-loop game is equivalent to a problem of the following type.

Given the quadratic function

$$J[x_0, 0, \alpha, \mathscr{U}_1(\alpha-1), \mathscr{U}_2(\alpha-1)] = [\mathscr{U}_1'(\alpha-1) \vdots \mathscr{U}_2'(\alpha-1) \vdots x_0'] \times \begin{bmatrix} X(\alpha) & Y(\alpha) & M'(\alpha) \\ Y'(\alpha) & Z(\alpha) & N'(\alpha) \\ M(\alpha) & N(\alpha) & P(\alpha) \end{bmatrix} \begin{bmatrix} \mathscr{U}_1(\alpha-1) \\ \mathscr{U}_2(\alpha-1) \\ x_0 \end{bmatrix} \tag{3.1}$$

find (if it exists) any pair of control sequences $\mathscr{U}_1{}^*(\alpha-1)$, $\mathscr{U}_2{}^*(\alpha-1)$ such that condition (2.3) is satisfied.

The matrices $X(\alpha)$, $Y(\alpha)$, $Z(\alpha)$, $M(\alpha)$, $N(\alpha)$ and $P(\alpha)$ are called the open-loop parameters of the α-stage open-loop game. They are matrices with dimensions growing as α increases, and explicit expressions for them are quite cumbersome. They can be described via straightforward recursive calculations, which will be omitted.

The main necessary and sufficient condition for the existence of the α-stage open-loop game solution can now be expressed in terms of these open-loop parameters.

Theorem 1

The α-stage open-loop game has a solution, for any x_0, if and only if

$$X(\alpha) \geqslant 0\,; \quad \mathscr{N}[X(\alpha)] \subset \mathscr{N} \begin{bmatrix} Y'(\alpha) \\ M(\alpha) \end{bmatrix} \tag{3.2 a}$$

$$Z(\alpha) \leqslant 0\,; \quad \mathscr{N}[Z(\alpha)] \subset \mathscr{N} \begin{bmatrix} Y(\alpha) \\ N(\alpha) \end{bmatrix} \tag{3.2 b}$$

The open-loop value, OLV, is then given by

$$\mathrm{OLV} = x_0' \left\{ P(\alpha) - [M(\alpha) N(\alpha)] \begin{bmatrix} X(\alpha) & Y(\alpha) \\ Y'(\alpha) & Z(\alpha) \end{bmatrix}^{\#} \begin{bmatrix} M'(\alpha) \\ N'(\alpha) \end{bmatrix} \right\} x_0 \tag{3.3}$$

The conditions (3.2 *a*, *b*) of Theorem 1 do not ensure uniqueness of the pair of OLOC's for the two players. However, as is well known from the theory of zero-sum two-person games [see Bensoussan (1971), Theorem II 1.2], in such a case, any one OLOC for one player may be combined with any one OLOC for the other player and the OLV is the same. Among those OLOC's, there exists a pair of OLOC's which require a minimum amount of energy on the part of each player [the energy of an OLOC $\mathscr{U}_j(\alpha-1)$ being measured by $\mathscr{U}_j'(\alpha-1)\mathscr{U}_j(\alpha-1)$]; it may be reasonable to expect the players to use this particular pair (called the pair of ' minimum energy ' OLOC's).

Corollary 2

In case more than one pair of OLOC's exist among the various optimal pairs, there exists one pair of minimum energy OLOC's determined (uniquely) by

$$\begin{bmatrix} \mathscr{U}_1^*(\alpha-1) \\ \mathscr{U}_2^*(\alpha-1) \end{bmatrix} = -\begin{bmatrix} X(\alpha) & Y(\alpha) \\ Y'(\alpha) & Z(\alpha) \end{bmatrix}^{\#} \begin{bmatrix} M'(\alpha) \\ N'(\alpha) \end{bmatrix} x_0 \tag{3.4}$$

The proofs of both Theorem 1 and Corollary 2 are given in Appendix A.

Remark 2. The above results indicate that, at least for this class of problems, the non-uniqueness of the OLOC's does not create any extra difficulty in the solution process. Further, by not insisting on the uniqueness of the OLOC's, we actually obtain some relaxation in the existence conditions, for it is obvious that the conditions (3.2) must be strengthened for the uniqueness property to follow. (It is not difficult to see how (3.2) must be strengthened to ensure the uniqueness of the OLOC for one or for both players, but we shall not pursue this point any further.) We shall see in the next section that a similar conclusion holds for the closed-loop game.

As noted earlier (Remark 1) we are interested in solving a family of α-stage open-loop games, each one corresponding to a value of $x_0 \in R^n$. This results in the presence of the conditions $\mathscr{N}[X(\alpha)] \subset \mathscr{N}[M(\alpha)]$ and $\mathscr{N}[Z(\alpha)] \subset \mathscr{N}[N(\alpha)]$ in (3.2) where $M(\alpha)$ and $N(\alpha)$ are the parameters associated with x_0 which affect the solvability of the games. Now we shall show that, with two additional assumptions, we can obtain an existence condition (only however sufficient) depending only on the parameters $X(\cdot)$, $Y(\cdot)$ and $Z(\cdot)$.

Theorem 2

Assume $[A, B_1]$ and $[A, B_2]$ are completely reachable and let $l \triangleq \max [l_1, l_2]$, where l_1, l_2 denote the controllability indices of the pair $[A, B_1]$ and $[A, B_2]$ respectively. The α-stage open-loop game has a solution for any x_0 if

$$X(\alpha+l) \geqslant 0\ ; \quad \mathscr{N}[X(\alpha+l)] \subset \mathscr{N}[Y'(\alpha+1)] \tag{3.5 a}$$

$$Z(\alpha+l) \leqslant 0\ ; \quad \mathscr{N}[Z(\alpha+l)] \subset \mathscr{N}[Y(\alpha+l)] \tag{3.5 b}$$

Essentially, (3.5) are the necessary and sufficient conditions for the solvability of a particular $(\alpha+l)$-stage open-loop game, namely, the one with $x_0 = 0$, whereas the structural assumptions on (A, B_1) and (A, B_2) guarantee that either one of the players can bring any initial state x_0 to zero in l steps, *provided his* *opponent is inactive.* Theorem 2 thus shows that the combinations of (3.5) and the structural assumptions on (A, B_1) and (A, B_2) are also sufficient for the α-stage open-loop game, *for any* x_0, to have a solution. The proof of this apparently quite non-trivial result is also contained in Appendix A.

Remark 3. For a more narrow class of games (pursuit–evasion type) where U_3, C_1, C_2 are zero matrices, Q, S are non-negative definite, U_1 is positive definite and U_2 is negative definite, results from continuous-time (Ho *et al.* 1965, Rhodes 1967, Schmitendorf and Citron 1969) seem to suggest that a 'relative controllability' condition (a notion to be made precise later) also suffices to ensure the existence of the game solution. That this is not true for the discrete-time case will be discussed in detail later.

Remark 4. We note that the results of this section require amplification in case $\alpha \to \infty$ because the parameters $X(\cdot)$, $Y(\cdot)$ ... involved are infinite dimensional and may possibly possess unbounded entries. What the various conditions mean is not clear, and how one might check them we cannot envisage. Even when α is finite, although explicit results for the open-loop game solution are obtained, the task of verifying the various existence conditions (as well as that of computing the minimum energy OLOC's) is quite cumbersome. We shall not discuss the latter problem any further, because using the results of Lemma 1 and Corollary 1, we know when the open-loop game has a solution, so does the closed-loop game and for our class of problems the CLOS can be readily found via a recursive Riccati equation, as will be shown in the next section.

We end this section with some comments on the approach and the results. The approach of transforming a dynamic (open-loop) game into a static one in normal form to find existence conditions has been used quite extensively in previous works (see, e.g., Rhodes (1967), Porter (1967) and especially Bensoussan (1971)). These references mostly deal with continuous-time games, but by viewing the ideas in a Hilbert space setting, the results come to have connections with the particular class of games considered here. In particular, with $\mathscr{U}_1(\cdot)$, $\mathscr{U}_2(\cdot)$, x_0 as elements of appropriate Hilbert spaces and the parameters $X(\cdot)$, $Y(\cdot)$ ... as operators or mappings between these spaces, Theorem 1 here is somewhat similar to Bensoussan's (1971, Corollary I.1). However, the above references do not give a complete account of the case where non-uniqueness of the OLOC's (and also of the CLOS) occur as we do here in this (and the next) section.

4. Closed-loop game

For any finite α, the main result for the α-stage closed-loop game as defined in § 2 is

Theorem 3

The necessary and sufficient conditions for the α-stage closed-loop game to have a solution, for any x_0, are

$$[U_1 + B_1'\phi(i)B_1] \geqslant 0 \; ; \quad \mathscr{N}[U_1 + B_1'\phi(i)B_1] \subset \mathscr{N} \begin{bmatrix} U_3' + B_2'\phi(i)B_1 \\ C_1 + A'\phi(i)B_1 \end{bmatrix} \qquad (4.1\,a)$$

$$[U_2+B_2'\phi(i)B_2]\leqslant 0\,;\quad \mathcal{N}[U_2+B_2'\phi(i)B_2]\subset\mathcal{N}\begin{bmatrix}U_3+B_1'\phi(i)B_2\\ C_2+A'\phi(i)B_2\end{bmatrix} \qquad (4.1\,b)$$
$$\forall i\in T=[0,\,\alpha-1]$$

where $\phi(i)$ is the solution of the recursive Riccati equation

$$\left.\begin{aligned}&\phi(i+1)=A'\phi(i)A-(A'\phi(i)B+C)[U+B'\phi(i)B]^{\#}(B'\phi(i)A+C')+Q\\ &\phi(0)=S\end{aligned}\right\} \qquad (4.2)$$

The CLV is then given by $x_0'\phi(\alpha)x_0$. Further, there may exist more than one pair of CLOS; however, there exists a unique minimum energy pair of CLOS determined by

$$u^*(i)=\begin{bmatrix}u_1^*(i)\\ u_2^*(i)\end{bmatrix}=-[U+B'\phi(\alpha-i)B]^{\#}(B'\phi(\alpha-i)A+C')x(i) \qquad (4.3)$$

The proof of this theorem is given in Appendix B.

Remark 5. Note that, for any finite α, $\phi(i)$ for all $i\in T$ is well defined. The question of whether $x_0'\phi(\alpha)x_0$ constitutes the CLV of the α-stage closed-loop game (and of course, the question of existence of the α-stage closed-loop game) depend solely on the side conditions (4.1 *a*) and (4.1 *b*). If one or all of these conditions fail for some $i\in T$, we conclude that the α-stage closed-loop game has no solution (and by Corollary 1(i), the α-stage open-loop also does not have a solution). The relation between the properties of the solution of a Riccati equation and the solvability of a linear-quadratic variational problem is well known in control problems (for both continuous and discrete-time versions) and in game problems (for the continuous-time version, see e.g. Ho *et al.* (1965), Rhodes (1967) and Schmitendorf and Citron (1969)). It is usually referred to as the no-conjugate point condition. Theorem 3 thus gives the no-conjugate point condition for our class of discrete-time games. It is of interest to compare it with its continuous-time counterpart, which requires boundedness of the solution of a differential Riccati equation and which implicitly requires the uniqueness of the pair of CLOS. Again, we wish to note the fact that Remark 2 for open-loop games applies here as well, i.e. our result covers the case where non-uniqueness of the pair of CLOS occurs.

Remark 6. Perhaps in contrast to variational problems of control type, many difficulties for our class of games arise when the time interval is allowed to become infinite. Even if $\phi(i)$ satisfies the side conditions (4.1) for all i and is known to converge to a limit Φ satisfying the steady-state version of (4.2), i.e.

$$\Phi=A'\Phi A-(A'\Phi B+C)[U+B'\Phi B]^{\#}(B'\Phi A+C')+Q \qquad (4.5)$$

one still has to show that the sequence of ' minimum energy ' CLOS (4.3) also converges to a limiting strategy defining a pair of square-summable controls before one can conclude that the CLV of the infinite-time game exists and is given by $x_0'\Phi x_0$. The presence of the pseudo-inverse in (4.3) and the lack of knowledge of the asymptotic behaviour of $\phi(i)$ are the two sources of difficulties for this task. Similarly, it is not trivial to prove the converse statement: suppose that the infinite-time closed-loop game is known to have a solution and the CLV is $x_0'\Phi x_0$, then $\phi(i)$ will satisfy (4.1) for all i with $\lim_{i\to\infty} \phi(i) = \Phi$.

There also remains the difficult problem of ascertaining when an infinite-time closed-loop game has a solution. Although for finite α, it may be possible to use the conditions of Theorem 1 to ensure the existence of the α-stage open-loop game and by Corollary 1 (i), also of the α-stage closed-loop game, this becomes impossible for $\alpha = \infty$.

Following is a result which is applicable to an infinite-time closed-loop game when an extra constraint, of stability, is imposed.

Theorem 4

Consider an infinite-time closed-loop game as defined previously, except that $S = 0$ and an extra constraint is imposed, namely $\lim_{i\to\infty} x(i) = 0$. If there exists a solution $\hat{\Phi}$ of eqn. (4.5) satisfying

$$|\lambda_i\{A - B[U + B'\hat{\Phi}B]^{\#}(B'\hat{\Phi}A + C')\}| < 1 \tag{4.6}$$

$$[U_1 + B_1'\hat{\Phi}B_1] \geqslant 0\,; \quad \mathcal{N}[U_1 + B_1'\hat{\Phi}B_1] \subset \mathcal{N}\begin{bmatrix} U_3' + B_2'\hat{\Phi}B_1 \\ \\ C_1 + A'\hat{\Phi}B_1 \end{bmatrix} \tag{4.7 a}$$

$$[U_2 + B_2'\hat{\Phi}B_2] \geqslant 0\,; \quad \mathcal{N}[U_2 + B_2'\hat{\Phi}B_2] \subset \mathcal{N}\begin{bmatrix} U_3 + B_1'\hat{\Phi}B_2 \\ \\ C_2 + A'\hat{\Phi}B_2 \end{bmatrix} \tag{4.7 b}$$

then the above infinite-time closed-loop game (with constraint) will have a solution and the CLV is given by $x_0'\hat{\Phi}x_0$.

Note

Any solution Φ of (4.5) which satisfies (4.6) is called a stabilizing solution. When it exists, it can be shown to be unique. In contrast to the control case where existence conditions for the stabilizing solution can be simply stated, the only available existence conditions for the stabilizing solution for the game case considered here are quite complicated to state and, furthermore, only applicable under some restrictions on the parameters. Therefore, they will not be stated here. It suffices to note that these conditions are the discrete-time equivalent of the conditions given by Coppel (1974) for the continuous-time problem.

The proof of this result is also given in Appendix B.

Remark 7. In the control case it is common to identify the infinite-time control problem with constraint $\lim_{i\to\infty} x(i) = 0$ with the infinite-time control problem with $S = +\infty(I)$ in the performance index and it is known that the solution (i.e. the optimal cost) (if it exists) is given by $x_0'\Phi x_0$, where Φ is the stabilizing solution of the steady-state discrete-time Riccati equation associated with the problem and Φ satisfies some side conditions. In that case, it also follows that the solution of the recursive Riccati equation initialized by

$\phi(0)=S=+\infty(I)$ will converge to Φ with $\phi(i)$ satisfying the same side conditions for all i, implying that each $\phi(i)$ can be connected with the optimal cost, $x_0'\phi(i)x_0$, for any i-stage control problem, with $S=+\infty(I)$. In the game case the difficulties mentioned in Remark 6 again prevent us from verifying a similar conjecture.

For the same reasons the following question of computational interest is also unresolved. Given the parameters A, B, C, U, Q as defined previously, suppose it is known that the stabilizing solution $\hat{\Phi}$ of eqn. (4.5) exists with $[U+B'\hat{\Phi}B]$ indefinite and with (4.7) holding. (If $[U+B'\hat{\Phi}B]\geqslant 0$ or $\leqslant 0$ we return to the case of control problems (Ereshko and Propoi 1970)). Characterize (if it exists) the set of all real symmetric matrices S such that $\phi(i)$, the solution of eqn. (4.3), satisfies the side conditions (4.1) for all i, and furthermore converges to $\hat{\Phi}$ as $i\to\infty$. In the game context this question can be stated as : find all real symmetric matrices S such that, for any α, the α-stage closed-loop game has a solution and this solution converges to that of an infinite-time game.

In the next section we indicate a partial answer to the last question.

5. Some implications of the main results

It is a standard result, see Ho *et al.* (1965) and Rhodes (1967), that the linear-quadratic differential game

$$\left.\begin{aligned} \dot{x}&=Fx+G_1u_1+G_2u_2 \\ x(0)&=x_0 \end{aligned}\right\} \tag{5.1}$$

$$J[x_0,0,T,u_1(\cdot),u_2(\cdot)]=\int_0^T [u_1'u_2'x']\begin{bmatrix}\mathscr{R}_1 & 0 & 0\\ 0 & \mathscr{R}_2 & 0\\ 0 & 0 & Q_c\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ x\end{bmatrix}+x(T)Sx(T) \tag{5.2}$$

where $\mathscr{R}_1>0$, $\mathscr{R}_2<0$, $S\geqslant 0$ and $Q_c\geqslant 0$ with $[\mathscr{F}, Q_c^{1/2}]$ completely observable, has a closed-loop solution for any x_0, and any $\infty>T>0$, provided the following condition (called ' relative controllability ') is satisfied for all $\tau_1>0$

$$\int_0^{\tau_1}\{\exp(F\tau)[\mathscr{G}_1\mathscr{G}_2]\begin{bmatrix}\mathscr{R}_1^{-1} & 0\\ 0 & \mathscr{R}_2^{-1}\end{bmatrix}\begin{bmatrix}\mathscr{G}_1'\\ \mathscr{G}_2'\end{bmatrix}\exp(F'\tau)\}\,d\tau>0 \tag{5.3}$$

The CLV, for any T, is given by $x_0'\mathscr{P}(T)x_0$, where $\mathscr{P}(T)$ is determined by the differential Riccati equation.

$$\left.\begin{aligned} \dot{\mathscr{P}}&=\mathscr{P}\mathscr{F}+\mathscr{F}'\mathscr{P}-[\mathscr{G}_1\mathscr{G}_2]\begin{bmatrix}\mathscr{R}_1^{-1} & 0\\ 0 & \mathscr{R}_2^{-1}\end{bmatrix}\begin{bmatrix}\mathscr{G}_1'\\ \mathscr{G}_2'\end{bmatrix}\mathscr{P}+Q_c \\ \mathscr{P}(0)&=S \end{aligned}\right\} \tag{5.4}$$

and furthermore, as $T\to\infty$, $\mathscr{P}(T)$ converges to $\hat{\mathscr{P}}$, the steady-state stabilizing solution of (5.4) and $x_0'\hat{\mathscr{P}}x_0$ is the CLV of the infinite-time game.

This result seems to suggest that, for the discrete-time linear-quadratic game of pursuit–evasion (i.e. the game as defined in § 2 with $U_3=0$, $C_1=0$,

$C_2 = 0$, $U_1 > 0$, $U_2 < 0$, $S \geqslant 0$ and $Q \geqslant 0$ with $[A, Q^{1/2}]$ completely observable) one can expect that the analogue of condition (5.3) in discrete time, viz.

$$[B_1 B_2] \begin{bmatrix} U_1^{-1} & 0 \\ 0 & U_2^{-1} \end{bmatrix} \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} \geqslant 0 \tag{5.5 a}$$

$$\sum_{k=0}^{l} \left\{ A^k [B_1 B_2] \begin{bmatrix} U_1^{-1} & 0 \\ 0 & U_2^{-1} \end{bmatrix} \begin{bmatrix} B_1' \\ B_2' \end{bmatrix} A'^k \right\} > 0 \quad \text{(for suitably large } l) \tag{5.5 b}$$

would ensure the existence of solution of the α-stage closed-loop game. We shall now demonstrate that (5.5 a, b) are, in general, *not* sufficient to draw this conclusion.

Consider the recursive equation

$$\left. \begin{aligned} \phi(i+1) &= A'\{\phi(i)[I + BU^{-1}B'\phi(i)]^{-1}\}A + Q \\ \phi(0) &= S(\geqslant 0) \end{aligned} \right\} \tag{5.6}$$

From the conditions (5.5 a, b) we can replace $BU^{-1}B$ by DD', where D is a full column rank matrix such that $[A, D]$ is completely reachable. Equation (5.6) becomes

$$\begin{aligned} \phi(i+1) &= A'\{\phi(i)[I + DD'\phi(i)]^{-1}\}A + Q \\ \phi(0) &= S(\geqslant 0) \end{aligned}$$

or, after some manipulations,

$$\left. \begin{aligned} \phi(i+1) &= A'\phi(i)A - A'\phi(i)D[I + D'\phi(i)D]^{-1}D'\phi(i)A + Q \\ \phi(0) &= S(\geqslant 0) \end{aligned} \right\} \tag{5.7}$$

Equation (5.7) can be recognized as the recursive Riccati equation associated with a regulator problem. The following result is standard, see, e.g., Hitz and Anderson (1972). The conditions $[A, D]$ completely reachable and $[A, Q^{1/2}]$ completely observable imply that $\phi(i)$ exists and is non-negative definite for all i, and converges to a positive definite matrix $\hat{\Phi}$ such that

$$|\lambda_i\{A - D[I + D'\hat{\Phi}D]^{-1}D'\hat{\Phi}A\}| < 1 \tag{5.8}$$

As it can be shown, again by some simple manipulations, that (5.6) is equivalent to

$$\left. \begin{aligned} \phi(i+1) &= A'\phi(i)A - A'\phi(i)B[U + B'\phi(i)B]^{-1}B\phi(i)A + Q \\ \phi(0) &= S(\geqslant 0) \end{aligned} \right\} \tag{5.9}$$

and that $\hat{\Phi}$ is such that

$$|\lambda_i\{A - B[U + B'\hat{\Phi}B]^{-1}B'\hat{\Phi}A\}| < 1 \tag{5.10}$$

we have thus shown that (5.9) has a well-defined solution $\phi(i)$ for all i which converges to a steady-state solution of (5.9) which is precisely the stabilizing solution as defined earlier. However, since for some $S(>0)$, the side conditions (4.1) may fail for some i (e.g. when $S = +\beta I$ with β a large enough positive constant, $U_2 + B_2'SB_2 \nleq 0$) and since there may exist cases where $\hat{\Phi}$ fails to satisfy the conditions (4.7), although the solution $\phi(i)$ of (5.9) exists, we cannot immediately associate this with the existence of a solution to the closed-loop game. Thus, in general, the discrete 'relative controllability' conditions (5.5) are not sufficient to ensure the existence of the CLV for the discrete-time pursuit–evasion games.

Also, since the closed-loop game does not always have a solution, by Corollary 1, the open-loop game cannot be expected to have a solution in general. This clarifies Remark 3 of § 3.

Following is a simple example with dimensions $n = m_1 = m_2 = 1$. Let $A = 1$, $B = [2 \vdots b]$, where b is any real scalar $\neq 0$;

$$U = \begin{bmatrix} 1 & 0 \\ 0 & -b^2 \end{bmatrix}, \quad C = [0 \vdots 0],$$

$Q = \frac{3}{4}$ and $S = s$, any real non-negative scalar. It can be easily checked that the conditions (5.5) are satisfied and the solution $\phi(i)$ of (5.7), for any $S \geqslant 0$, converges to the stabilizing solution $\hat{\Phi}$. In particular, for $1 > s \geqslant 0$,

$$\hat{\Phi} = 1 \geqslant \phi(i+1) \geqslant \phi(i) \geqslant 0$$

for all i and the side conditions (4.1) are satisfied for all i, *except at the limit*, since

$$[U + B'\Phi B] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -b^2 \end{pmatrix} + \begin{pmatrix} 2 \\ b \end{pmatrix} 1 (2 \; b) \right] = \begin{bmatrix} 5 & 2b \\ 2b & 0 \end{bmatrix}$$

On the other hand, if $s \geqslant 1$, the side conditions (4.1 *a*, *b*) can be checked to fail for all i.

Finally, we comment that there are techniques for transforming linear-quadratic continuous-time problems to linear-quadratic discrete-time problems, e.g. the bilinear transformation technique (Hitz and Anderson 1972). The above observation shows that when a differential game is transformed into a discrete-time game, the discrete-time game may not have a solution even when the continuous-time game does. This is in contrast to the situation observed for the regulator problem.

6. Conclusions

In this paper, we have examined some aspects of a class of zero-sum time-invariant, linear-quadratic games which are deterministic and discrete-time. The results obtained show that this class of games has many features not shared by its continuous-time counterpart. There remain a number of unresolved questions for further investigation.

Appendix A

Proof of Theorem 1

The proof can be obtained via the following assertions.

Assertion 1

The quadratic form

$$V[x,y]=[x'y']\begin{bmatrix} P & R \\ R' & Q \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

for a fixed y [for a fixed x] has a minimum with respect to x [has a maximum with respect to y] if and only if $P\geqslant 0$ and $\mathcal{N}[P]\subset\mathcal{N}[R]$ [$Q\leqslant 0$ and $\mathcal{N}[Q]\subset\mathcal{N}[R]$], the minimum being $y'[Q-RP^{\#}R']y$ (the maximum being $x'[P-R'Q^{\#}R]x$).

Proof

Consider the case of the minimization problem. The proof is standard: for sufficiency, use the standard procedure to complete the square; for necessity, show that if either $P\ngeqslant 0$ or $\mathcal{N}[P]\subset\mathcal{N}[R']$, then for any fixed y, there exists x^* such that $V[x^*,y]$ can be made arbitrarily large in the negative direction, obtaining a contradiction.

The proof for the maximization problem is analogous.

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Consider now the quadratic form

$$V[x,y,z]=[x'y'z']\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}' & P_{22} & P_{23} \\ P_{13}' & P_{23}' & P_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{A 1}$$

By identifying appropriate block matrices and vectors, we obtain from Assertion 1:

Assertion 2

The necessary and sufficient conditions for $\min_x V[x,y,z]$, over all x, to exist for all y, z are

$$P_{11}>0$$

$$\mathcal{N}[P_{11}]\subset\mathcal{N}\begin{bmatrix} P_{12}' \\ P_{13}' \end{bmatrix} \tag{A 2a}$$

and the minimum is

$$[y'z']\left\{\begin{bmatrix} P_{22} & P_{23} \\ P_{23}' & P_{33} \end{bmatrix}-\begin{bmatrix} P_{12}' \\ P_{13}' \end{bmatrix}P_{11}^{\#}[P_{12}\,P_{13}]\right\}\begin{bmatrix} y \\ z \end{bmatrix} \tag{A 3a}$$

Analogously, the necessary and sufficient conditions for $\max_y V[x,y,z]$ to exist for all x, z are

$$P_{22}\leqslant 0$$

$$\mathcal{N}[P_{22}]\subset\mathcal{N}\begin{bmatrix} P_{12} \\ P_{23}' \end{bmatrix} \tag{A 2b}$$

and the maximum is

$$[x'z']\left\{\begin{bmatrix} P_{11} & P_{13} \\ P_{13}' & P_{33}\end{bmatrix} - \begin{bmatrix} P_{12} \\ P_{23}'\end{bmatrix} P_{22}^{\#}[P_{12}' \; P_{23}]\right\}\begin{bmatrix} x \\ z\end{bmatrix} \qquad (A\ 3b)$$

Now, by applying Assertion 1 to (A 3*b*), we obtain

Assertion 3

The necessary and sufficient conditions for $\min_x \max_y V$ to exist for all z are that (A 2*b*) should hold and that

$$[P_{11} - P_{12}P_{22}^{\#}P_{12}'] \geqslant 0$$

$$\mathcal{N}[P_{11} - P_{12}\,P_{22}^{\#}P_{12}'] \subset \mathcal{N}[P_{13}' - P_{23}'\,P_{22}^{\#}P_{12}] \qquad (A\ 4)$$

and

$$\min_x \max_y V = z'[P_{33} - P_{23}'P_{22}^{\#}P_{23} - (P_{13}' - P_{23}'P_{22}^{\#}P_{12})(P_{11} - P_{12}P_{22}^{\#}P_{12}')^{\#}(P_{13} - P_{12}'P_{22}^{\#}P_{23})]z \qquad (A\ 5)$$

Furthermore, if (A 2*b*) are satisfied, then (A 2*a*) implies and is implied by (A 4). Thus, the conditions (A 2*b*) and (A 4) together are equivalent to (A 2*b*) and (A 2*a*).

Remark. Assume that (A 2*a*) and (A 2*b*) hold ; then

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22}\end{bmatrix} = \begin{bmatrix} I & 0 \\ P_{12}'P_{11}^{\#} & I\end{bmatrix}\begin{bmatrix} P_{11} & 0 \\ 0 & (P_{22} - P_{12}'P_{11}^{\#}P_{12})\end{bmatrix}\begin{bmatrix} I & P_{11}^{\#}P_{12} \\ 0 & I\end{bmatrix}$$

whence

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22}\end{bmatrix}^{\#} = \begin{bmatrix} I & -P_{11}^{\#}P_{12} \\ 0 & I\end{bmatrix}\begin{bmatrix} P_{11}^{\#} & 0 \\ 0 & (P_{22} - P_{12}'P_{11}^{\#}P_{12})^{\#}\end{bmatrix}\begin{bmatrix} I & 0 \\ -P_{12}'P_{11}^{\#} & I\end{bmatrix}$$

By changing the role of P_{11} and P_{22} in the decomposition above, one also obtains

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22}\end{bmatrix}^{\#} = \begin{bmatrix} I & 0 \\ -P_{22}^{\#}P_{12}' & I\end{bmatrix}\begin{bmatrix} (P_{11} - P_{12}P_{22}^{\#}P_{12}')^{\#} & 0 \\ 0 & P_{22}^{\#}\end{bmatrix}\begin{bmatrix} I & -P_{12}P^{\#}{}_{11} \\ 0 & I\end{bmatrix}$$

From (A 5), we see that

$$\min_y \max_x V = z'\left\{P_{33} - [P_{13}'P_{23}']\begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22}\end{bmatrix}^{\#}\begin{bmatrix} P_{13} \\ P_{23}\end{bmatrix}\right\}z \qquad (A\ 6)$$

By variations of arguments in Assertion 3 and its following remark, one obtains

Assertion 4

(A 2*a*) and (A 2*b*) are the necessary and sufficient conditions for the existence of $\max_y \min_x V$ for all z. Furthermore

$$\max_y \min_x V = z' \left\{ P_{33} - [P_{13}' P_{23}'] \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}^{\#} \begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix} \right\} z \qquad \text{(A 7)}$$

In other words, we have also shown that, under the conditions (A 2*a*), (A 2*b*),

$$\min_x \max_y V = \max_y \min_x V$$

Return now to the α-stage open-loop game problem. By identifying $\mathscr{U}_1(\alpha-1)$, $\mathscr{U}_2(\alpha-1)$, x_0, $X(\alpha)$, $Y(\alpha)$, $Z(\alpha)$, $M'(\alpha)$, $N'(\alpha)$, $P(\alpha)$ with x, y, z, P_{11}, P_{12}, P_{22}, P_{13}, P_{23} and P_{33} respectively, we obtain from Assertion 4 the results of this theorem.

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Proof of Corollary 2

Using first-order necessary conditions, one sees that a pair of OLOC's must satisfy

$$\begin{bmatrix} X(\alpha) & Y(\alpha) \\ Y'(\alpha) & Z(\alpha) \end{bmatrix} \begin{bmatrix} \mathscr{U}_1^*(\alpha-1) \\ \mathscr{U}_2^*(\alpha-1) \end{bmatrix} + \begin{bmatrix} M(\alpha) \\ N(\alpha) \end{bmatrix} x_0 = 0 \qquad \text{(A 9)}$$

The conditions (3.2) together yield

$$\mathscr{N} \begin{bmatrix} X(\alpha) & Y(\alpha) \\ Y'(\alpha) & Z(\alpha) \end{bmatrix} \subset \mathscr{N}[M'(\alpha) \;\; N'(\alpha)]$$

implying that the linear eqn. (A 9) is consistent and possesses solutions of the form

$$\begin{bmatrix} \mathscr{U}_1^*(\alpha-1) \\ \mathscr{U}_2^*(\alpha-1) \end{bmatrix} = -\begin{bmatrix} X(\alpha) & Y(\alpha) \\ Y'(\alpha) & Z(\alpha) \end{bmatrix}^{\#} \begin{bmatrix} M(\alpha) \\ N(\alpha) \end{bmatrix} x_0 + t \qquad \text{(A 10)}$$

for arbitrary

$$t \in \mathscr{N} \begin{bmatrix} X(\alpha) & Y(\alpha) \\ Y'(\alpha) & Z(\alpha) \end{bmatrix}$$

Since $\mathscr{N}[X(\alpha)] \subset \mathscr{N}[Y'(\alpha)]$ and $\mathscr{N}[Z(\alpha)] \subset \mathscr{N}[Y(\alpha)]$, t can be decomposed as

$$t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

for arbitrary $t_1 \in \mathscr{N}[X(\alpha)]$ and $t_2 \in \mathscr{N}[Z(\alpha)]$.

It can be checked that, because $\mathscr{N}[X(\alpha) \subset \mathscr{N}[Y(\alpha)]$ the choice of the particular vector $t_2 \in \mathscr{N}[X(\alpha)]$ by the second player in $\mathscr{U}_2^*(\alpha-1)$ does not affect the final expression of $\mathcal{U}_1^*(\alpha-1)$ and vice versa; $\mathcal{U}_1^*(\alpha-1)$ and $\mathcal{U}_2^*(\alpha-1)$ are independently given by expressions of the forms

$$\mathcal{U}_1^*(\alpha-1)=X^\#(\alpha)[\cdot]_1+t_1 \tag{A 11}$$

$$\mathcal{U}_2^*(\alpha-1)=Z^\#(\alpha)[\cdot]_2+t_2 \tag{A 12}$$

where $[\cdot]_i$ $(i=1,\ 2)$ denote appropriate expressions which are omitted for simplicity.

Conversely, the expressions (A 11) and (A 12), when used together, can be shown to achieve the OLV given by (3.3). Thus, (A 11) and (A 12) give the form of all pairs of OLOC's for the α-state open-loop game.

Among the OLOC's for player 1, the OLOC corresponding to $t_1=0$ is easily checked to have minimum norm and hence constitutes the OLOC which requires the least amount of energy. Similarly, the OLOC for player 2 corresponding to $t_2\equiv 0$ also requires the least amount of energy.

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Proof of Theorem 2

With the additional assumptions we have

Assertion 5

The conditions (3.5 a) ensure the existence of a minimum of $J[x_0, 0, \alpha, \mathcal{U}_1(\alpha-1), \mathcal{U}_2(\alpha-2)]$ with respect to $\mathcal{U}_1(\alpha-1)$, for arbitrary x_0, $\mathcal{U}_2(\alpha-1)$.

Proof

We shall use a new notation for control sequences in this proof. Let, for all $n>m$,

$$\mathcal{U}_{j[m,\ n]}\triangleq[u_j'(m)\vdots u_j'(m+1)\vdots\dots\vdots u_j'(n)]', \quad j=1,\ 2$$

By the assumption of complete reachability of the pair $[A, B_1]$ and the definition of the integer l, there exists a control sequence $\mathcal{U}^*_{1[-l,\ -1]}$ such that $\mathcal{U}^*_{[-l,\ -1]}\triangleq[\mathcal{U}^{*\prime}_{1[-l,\ -1]}\vdots\mathcal{U}^*_{2[-l,\ -1]}]'$ with $\mathcal{U}^*_{2[-l,\ -1]}\equiv 0$, will bring the state $x(-l)=0$ to $x(0)=x_0$, for any prescribed x_0.

Clearly $\mathcal{U}^*_{1[-l-1]}$ depends on the particular x_0. Also, there may be more than one such control sequence $\mathcal{U}^*_{1[-l-1]}$ in association with any x_0. However, one can always choose one among those candidates, ensuring that the corresponding to $x_0=0$, $\mathcal{U}^*_{1[-l-1]}=0$ also. In the following discussion, the notation $\mathcal{U}^*_{1[-l-1]}$ is used to indicate the particular control sequence chosen. Thus, corresponding to any x_0 (and the corresponding control sequences $\mathcal{U}^*_{1[-l-1]}$ and $\mathcal{U}^*_{2[-l-1]}$ as chosen above) we have $J[0, -l, 0, \mathcal{U}^*_{1[-l-1]}, \mathcal{U}^*_{2[-l-1]}]=f_1(x_0)$, a function depending purely on x_0, and in particular, if $x_0=0$, $f_1(\mathrm{p})=0$.

Let $\mathcal{U}_{1[0,\ \alpha-1]}$ and $\mathcal{U}_{2[0,\ \alpha-1]}$ be respectively a free and a fixed (but arbitrary) control sequence. Then

$$J[x_0, 0, \alpha, \mathcal{U}_{1[0,\ \alpha-1]}, \mathcal{U}_{2[0,\ \alpha-1]}]=J[0, -l, \alpha, \mathcal{U}^\dagger_{1[-l,\ \alpha-1]}, \mathcal{U}^\dagger_{2[-l,\ \alpha-1]}] \\ -J[0, -l, 0, \mathcal{U}^*_{1[-l\ -1]}, \mathcal{U}^*_{2[-l,\ 1]}]$$

where

$$\mathcal{U}^{\dagger}_{1[-l,\ \alpha-1]} \triangleq \begin{cases} \mathcal{U}^{*}_{1[-l,\ -1]} & \text{as defined} \\ \mathcal{U}_{1[0,\ \alpha-1]} & \text{free} \end{cases}$$

$$\mathcal{U}^{\dagger}_{2[-l,\ \alpha-1]} \triangleq \begin{cases} \mathcal{U}^{*}_{2[-l,\ -1]} \equiv 0 & \\ \mathcal{U}_{2[0\ \ \alpha-1]} & \text{free} \end{cases}$$

And so

$$\begin{aligned} &\inf_{\mathcal{U}_{1[0,\ \alpha-1]}} J[x_0, 0, \mathcal{U}_{1[0,\ \alpha-1]}, \mathcal{U}_{2[0,\ \alpha-1]}] \\ &\quad = \inf_{\mathcal{U}_{1[0,\ \alpha-1]}} J[0, -l, \alpha, \mathcal{U}^{\dagger}_{1[-l,\ \alpha-1]}, \mathcal{U}^{\dagger}_{2[-l,\ \alpha-1]}] - f_1(x_0) \\ &\qquad \geqslant \inf_{\mathcal{U}_{1[-l,\ \alpha-1]}} J[0, -l, \alpha, \mathcal{U}_{1[-l,\ \alpha-1]}, \mathcal{U}^{\dagger}_{2[-l,\ \alpha-1]}] - f_1(x_0) \\ &\quad = \inf_{\mathcal{U}_{1[0,\ \alpha+l-1]}} J[0, 0, \alpha+l, \mathcal{U}_{1[0,\ \alpha+l-1]}, \mathcal{U}^{\dagger}_{2[0,\ \alpha+l-1]}] - f_1(x_0) \end{aligned} \tag{A 13}$$

The inequality is obtained because of the difference between a constrained and an unconstrained problem. The last equality follows from the time-invariant nature of the problem (note that

$$\mathcal{U}^{\dagger}_{2[0,\ \alpha+l-1]} \equiv \mathcal{U}^{\dagger}_{2[-l,\ \alpha-1]}$$

except for a translation of time indices).

By Assertion 2, the condition (3.5 a) ensures the existence of

$$\inf_{\mathcal{U}_{1[0,\ \alpha+l-1]}} J[0, 0, \alpha, l, \mathcal{U}_{1[0,\ \alpha+l-1]}, \mathcal{U}^{\dagger}_{2[0,\ \alpha+l-1]}]$$

Then, (A 13) and the quadratic character of $J[x_0, 0, \alpha, \mathcal{U}_{1[0,\ \alpha-1]}, \mathcal{U}_{2[0,\ \alpha-1]}]$ imply that $\min_{\mathcal{U}_{1[0,\ \alpha-1]}} J[x_0, 0, \alpha, \mathcal{U}_{1[0,\ \alpha-1]}, \mathcal{U}_{2[0,\ \alpha-1]}]$ exists (as opposed to merely $\inf J[\quad]$).

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Assertion 6

The conditions (3.5) ensure that

$$\sup_{\mathcal{U}_{2[0,\ \alpha-1]}} \min_{\mathcal{U}_{1[0,\ \alpha-1]}} J[x_0, 0, \alpha, \mathcal{U}_{1[0,\ \alpha-1]}, \mathcal{U}_{2[0,\ \alpha-1]}] > -f_1(x_0)$$

where $f_1(x_0)$ is a function depending purely on x_0.

Proof

By Assertion 2, under the condition (3.5 a)

$$\min_{\mathcal{U}_{1[-r,\ -1]}} J(0, 0, \alpha+r, \mathcal{U}_{1[0,\ \alpha+r-1]}, \mathcal{U}^{\dagger}_{2[0,\ \alpha+r-1]}]$$

exists and is of the form $[\mathcal{U}'^{\dagger}_{2[0,\ \alpha+r-1]}]M[\mathcal{U}^{\dagger}_{2[0,\ \alpha+r-1]}]$ where

$$M \triangleq \{Z(\alpha+r) - Y'(\alpha+r)X^{\#}(\alpha+r)Y(\alpha+r)\}$$

and $\mathcal{U}^{\dagger}_{2[0,\ \alpha+r-1]}$ is defined as $\mathcal{U}^{\dagger}_{2[-r,\ \alpha-1]}$ except for a translation of time indices (i.e. $\mathcal{U}^{\dagger}_{2[0,\ \alpha+r-1]} \triangleq [\tilde{\mathcal{U}}'_{2[0,\ r-1]}\mathcal{U}'_{2[r,\ \alpha+r-1]}]'$ with $\tilde{\mathcal{U}}_{2[0,\ r-1]} \equiv \tilde{\mathcal{U}}_{2[-r,\ -1]} \equiv 0$ and with $\mathscr{U}_{2[r,\ \alpha+r-1]} \equiv \mathscr{U}_{2[0,\ \alpha-1]}$ fixed but arbitrary). From (A 13), we then have corresponding to any arbitrary $\mathscr{U}_{2[0,\ \alpha-1]} \equiv \mathscr{U}_{2[r,\ \alpha+r-1]}$:

$$\min_{\mathscr{U}_{1[0,\ \alpha-1]}} J[x_0, 0, \alpha, \mathscr{U}_{1[0,\ \alpha-1]}, \mathscr{U}_{2[0,\ \alpha-1]}] > [0, \mathscr{U}'_{2[r,\ \alpha+r-1]}]M \begin{bmatrix} 0 \\ \mathscr{U}_{2[r,\ \alpha+r-1]} \end{bmatrix}$$

$$-f_1(x_0) = [0, \mathscr{U}'_{2[0,\ \alpha-1]}]M \begin{bmatrix} 0 \\ \mathscr{U}_{2[0,\ \alpha-1]} \end{bmatrix} - f_1(x_0)$$

The conditions (3.5) together also imply that M is non-positive definite. Hence

$$\sup_{\mathscr{U}_{2[0,\ \alpha-1]}} \min_{\mathscr{U}_{1[0,\ \alpha-1]}} J[x_0, 0, \alpha, \mathscr{U}_{1[0,\ \alpha-1]}, \mathscr{U}_{2[0,\ \alpha-1]}]$$

$$\geqslant \sup_{\mathscr{U}_{2[0,\ \alpha-1]}} [\mathscr{U}'_{2[0,\ \alpha-1]} \vdots 0]M \begin{bmatrix} \mathscr{U}_{2[0,\ \alpha-1]} \\ 0 \end{bmatrix} - f_1(x_0) \geqslant -f_1(x_0)$$

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A symmetric argument leads to

Assertion 5′

The condition (3.5 *b*) ensures the existence of a maximum of

$$J[x_0, 0, \alpha, \mathscr{U}_{1[0,\ \alpha-1]}, \mathscr{U}_{2[0,\ \alpha-1]}]$$

with respect to $\mathscr{U}_{2[0,\ \alpha-1]}$ for arbitrary x_0, $\mathscr{U}_{1[0,\ \alpha-1]}$.

Assertion 6′

The conditions (3.5) also ensure that

$$\inf_{\mathscr{U}_{1[0,\ \alpha-1]}} \max_{\mathscr{U}_{2[0,\ \alpha-1]}} J[x_0, 0, \alpha, \mathscr{U}_{1[0,\ \alpha-1]}, \mathscr{U}_{2[0,\ \alpha-1]}] \leqslant -f_2(x_0)$$

where $f_2(x_0)$ is a function depending purely on x_0. Combining Assertions 5, 6, 5′ and 6′, we obtain

Assertion 7

Under the assumptions of Theorem 2, the conditions (3.5) ensure the following:

$$-f_2(x_0) \geqslant \min \max J[\ ] \geqslant \max \min J[\ ] \geqslant -f_1(x_0)$$

Proof

The proof follows from the results of Assertions 5, 6, 5′ and 6′, the standard result

$$\inf \sup J[\ ] \leqslant \sup \inf J[\ ]$$

and the fact that, by the quadratic nature of $J[\ ]$ whenever an infimum [a supremum] exists, it can be replaced by a minimum [a maximum].

$\nabla\nabla\nabla$

By arguments similar to those used in the proof of Theorem 1, one can proceed to show that in case both min max $J[\ ]$ and max min $J[\ ]$ exist for our class of problems, they must also be equal to each other and correspond to the OLV of the α-stage open-loop game.

This observation completes the proof of Theorem 2.

$\nabla\nabla\nabla$

Appendix B

Proof of Theorem 3

This result can be proved in two ways. First, one can use Corollary 1 (ii) and develop inductive proofs based on the dynamic programming approach for the problems

$$\max_{u_1[x_0,0]} \min_{u_2[x_0,0]} \dots \min_{u_1[x(\alpha-1),\alpha-1]} \max_{u_2[x(\alpha-1),\alpha-1]} J[\ ]$$

and

$$\max_{u_2[x_0,0]} \min_{u_1[x_0,0]} \dots \max_{u_2[x(\alpha-1),\alpha-1]} \min_{u_1[x(\alpha-1),\alpha-1]} J[\ ]$$

For this class of problems the solutions of these two problems, when they exist, will be the same and equal to the CLV of the game. Furthermore, the conditions for existence, obtained in the course of the proofs, can be shown to be precisely (4.1 *a*, *b*). We shall not give a detailed discussion because it is rather lengthy. The second method, also based on a familiar technique for linear-quadratic control problems, is as follows.

Using the recursive Riccati eqn. (4.2), the dynamic eqn. (2.1) and the properties of pseudo-inverses, the performance index for the α-stage closed-loop game can be expressed as

$$\begin{aligned} J[x_0, 0, \alpha, u_1[x(i), i], u_2[x(i), i]] &= x'(\alpha)Sx(\alpha) + [x_0'\phi(\alpha)x_0 - x'(\alpha)\phi(0)x(\alpha)] \\ &\quad + \sum_{i=0}^{\alpha-1} [u(i)-u^*(i)]'\{U+B'\phi(\alpha-i)B\}[u(i)-u^*(i)] \\ &\quad + \sum_{i=0}^{\alpha-1} 2x'(i)[A'\phi(\alpha-i)B+C][I-(U+B'\phi(\alpha-i)B)^{\#}(U+B'\phi(\alpha-i)B)]u(i) \end{aligned} \tag{B 1}$$

where $u^*(i)$ is given by

$$u^*(i) = \begin{bmatrix} u_1^*(i) \\ u_2^*(i) \end{bmatrix} \triangleq -[U+B'\phi(\alpha-i)B]^{\#}[B'\phi(\alpha-i)A+C']x(i)+t(i) \tag{B 2}$$

with $t(i)$ any vector $\in \mathcal{N}[U+B'\phi(\alpha-1)B]$.

Note that the first and third term in (B 1) cancels since $\phi(0)=S$. For notational simplicity, let

$$[U+B'\phi(\alpha-i)B] = \begin{bmatrix} U_1+B_1'\phi(\alpha-i)B_1 & U_3+B_1'\phi(\alpha-i)B_2 \\ U_3'+B_2'\phi(\alpha-i)B_1 & U_2+B_2'\phi(\alpha-i)B_2 \end{bmatrix} = \begin{bmatrix} M_1(i) & M_3(i) \\ M_3'(i) & M_2(i) \end{bmatrix} \tag{B 3}$$

and

$$N(i) \triangleq [N_1(i) \vdots N_2(i)]$$
$$= [A'\phi(\alpha-i)B+C][I-(U+B'\phi(\alpha-i)B)^{\#}(U+B'\phi(\alpha-i)B)] \quad \text{(B 4)}$$

(i) *Necessity part.* The hypothesis that the α-stage closed-loop game has a solution, for all x_0, necessarily implies that the conditions

$$M_1(i) \geqslant 0 \quad \text{(B 5a)}$$

$$\mathscr{N}[M_1(i)] \subset \mathscr{N} \begin{bmatrix} M_3'(i) \\ N_1(i) \end{bmatrix} \quad \text{(B 5b)}$$

$$M_2(i) \leqslant 0 \quad \text{(B 5c)}$$

$$\mathscr{N}[M_2(i)] \subset \mathscr{N} \begin{bmatrix} M_3(i) \\ N_2(i) \end{bmatrix} \quad \text{(B 5d)}$$

must be satisfied for all $i \in [0, \alpha-1]$, because if one of these conditions fail for some j, one can construct a control for player 1 or for player 2 at that instant to make the performance index arbitrarily positive or arbitrarily negative, thus contradicting the hypothesis.

Now, using $\mathscr{N}[M_1] \subset \mathscr{N}[M_3']$ and $\mathscr{N}[M_2] \subset \mathscr{N}[M_3]$, from (B 5b, d) we obtain (dropping temporarily the index i)

$$\begin{bmatrix} M_1 & M_3 \\ M_3' & M_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ M_3'M_1^{\#} & I \end{bmatrix} \begin{bmatrix} M_1 & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} I & M_1^{\#}M_3 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} M_1 & M_3 \\ M_3' & M_2 \end{bmatrix}^{\#} = \begin{bmatrix} I & -M_1^{\#}M_3 \\ 0 & I \end{bmatrix} \begin{bmatrix} M_1^{\#} & 0 \\ 0 & L^{\#} \end{bmatrix} \begin{bmatrix} I & 0 \\ -M_3'M_1^{\#} & I \end{bmatrix}$$

where $L = M_2 - M_3'M_1^{\#}M_3$. Since $\mathscr{N}[M_2] \subset \mathscr{N}[M_3]$, we have $\mathscr{N}[M_2] \subset \mathscr{N}[L]$. Also, since $M_1 \geqslant 0$ (B 5a) and $M_2 \leqslant 0$ (B 5b), $L \leqslant M_2 \leqslant 0$ and $\mathscr{N}[L] \subset \mathscr{N}[M_2]$. Therefore

$$\mathscr{N}[L] \equiv \mathscr{N}[M_2] \subset \mathscr{N} \begin{bmatrix} M_3 \\ N_2 \end{bmatrix} \quad \text{(B 6)}$$

As a result, we have

$$\begin{bmatrix} M_1 & M_3 \\ M_3' & M_2 \end{bmatrix}^{\#} \begin{bmatrix} M_1 & M_3 \\ M_3' & M_2 \end{bmatrix} = \begin{bmatrix} M_1^{\#}M_1 & M_1^{\#}M_3(I-L^{\#}L) \\ 0 & L^{\#}L \end{bmatrix} = \begin{bmatrix} M^{\#}M & 0 \\ 0 & L^{\#}L \end{bmatrix}$$

and

$$N_1(i) = [A'\phi(\alpha-i)B_1 + C_1](I - M_1^{\#}(i)M_1(i)) \quad \text{(B 7a)}$$

$$N_2(i) = [A'\phi(\alpha-i)B_2 + C_2](I - L^{\#}(i)L(i)) \quad \text{(B 7b)}$$

Let $z \in \mathcal{N}[M_1(i)]$, then $z \in \mathcal{N}[N_1(i)]$ by (B 5*b*) and (B 7*a*) then implies $z \in \mathcal{N}[A'\phi(\alpha-i)B_1+C_1]$ also, i.e. $\mathcal{N}[M_1(i)] \subset \mathcal{N}[A'\phi(\alpha-i)B_1+C_1]$. In a similar manner, (B 6) and (B 7*b*) can be shown to imply

$$\mathcal{N}[M_2(i)] \equiv \mathcal{N}[L(i)] \subset \mathcal{N}[A'\phi(\alpha-i)B_2+C_2].$$

We have thus shown that the conditions (B 5*a*, *b*, *c*, *d*) together also imply the conditions (4.1 *a*, *b*) and this completes the proof of the necessity part.

(ii) *Sufficiency part.* Suppose now that the conditions (4.1 *a*, *b*) are satisfied for all i. These conditions can be easily checked to imply

$$\mathcal{N}[U+B'\phi(\alpha-i)B] \subset \mathcal{N}[A'\phi(\alpha-i)B+C]$$

which in turn, implies that the final term in (B 1) is identically zero. The performance index now becomes

$$J[\ ]=x_0'\phi(\alpha)x_0+\sum_{i=0}^{\alpha-1}[u(i)-u^*(i)]'(U+B'\phi(\alpha-i)B)[u(i)-u^*(i)] \qquad \text{(B 8)}$$

It is then immediate that the conditions (4.1 *a*, *b*) also imply that any pair $u(i)=u^*(i)$ given in (B 2) constitutes a pair of CLOS of the game. The CLV is then $x_0'\phi(\alpha)x_0$.

The minimum energy result is easily obtained.

This completes the proof of Theorem 3.

Proof of Theorem 4

Using the fact that $\hat{\Phi}$ is the stabilizing solution of (4.5), and the fact that any *admissible* control strategies for the two players together must satisfy the state constraint $\lim_{i\to\infty} x(i)=0$, it can be shown by algebraic manipulations that the performance index of the infinite-time game problem with state constraint as defined in the statement of Theorem 4 can be expressed as :

$$J[x_0, 0, \infty, u_1(\cdot), u_2(\cdot)]=x_0'\hat{\Phi}x_0+\sum_{i=0}^{\infty}[u(i)-u^*(i)][U+B'\hat{\Phi}B][u(i)-u^*(i)]$$

$$+\sum_{i=0}^{\infty}2x'(i)[A'\hat{\Phi}B+C][I-(U+B'\hat{\Phi}B)^{\#}(U+B'\hat{\Phi}B)]u(i) \qquad \text{(B 9)}$$

where $\{u(i)\}$ denotes any *admissible* pair of closed-loop strategies of the two players and $\{u^*(i)\}$ are particular *admissible* pairs given by the common formula

$$\begin{bmatrix} u_1^*(i) \\ u_2^*(i) \end{bmatrix} = u^*(i) = -[U+B'\hat{\Phi}B]^{\#}(B'\hat{\Phi}A+C')x(i)+t(i) \qquad \text{(B 10)}$$

where $t(i)$ is any vector in $\mathcal{N}[U+B'\hat{\Phi}B]$, *such that the resulting closed-loop system is asymptotically stable* (e.g. $t(i)\equiv 0$).

The proof then proceeds as for the sufficiency part Theorem 3 : using the fact that $\hat{\Phi}$ satisfies the side conditions (4.7 *a*, *b*) it is shown that the CLV is given by $x_0\hat{\Phi}x_0$ and the pair of minimum energy CLOS in this case is determined by (B 12) with $t(i)\equiv 0$.

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