The coefficients of the power series can be determined by long division, and one can easily write

$$c_k=\frac{1}{b_0}\left[a_k-\sum_{i=1}^{k} b_i c_{k-i}\right] \tag{4}$$

where

$$a_k=0 \qquad \text{for } k>n-1$$

and

$$b_k=0 \qquad \text{for } i>n.$$

The relations (3) and (4) can be used to determine the moments of the impulse response of the system from its transfer function with significant computational advantage.

Authors' Reply[2]

M. LAL AND R. MITRA

We would like to thank Drs. Bajwa and Khatwani for their comments. The algorithm suggested by them, which also appears in [1], does not seem to offer any computational advantage. The procedure is essentially the same as given by us with the only difference being that whereas in our procedure all the coefficients $A_{k,1}$ in the third and subsequent arrays in (6) are explicitly computed and stored, in the method suggested by them these coefficients are repeatedly computed afresh every time a new moment is evaluated.

References

[1] Y. Shamash, "Model reduction using the Routh stability criterion and Padé approximation technique," *Int. J. Control*, vol. 21, no. 3, pp. 475–484, 1975.

[2]Manuscript received April 22, 1976.
M. Lal is with the Department of Electrical Engineering, Wichita State University, Wichita, KS 67208.
R. Mitra is with the Department of Electronics and Communication Engineering, University of Roorkee, Roorkee, U. P., India.

Comments on "Determination of Matrix Transfer Function in the Form of Matrix Fraction from Input-Output Observations"

S. P. BINGULAC

Although the authors in a recent paper[1] have used the computational example already treated in [1], it happens that in both papers this example is published erroneously. In the above paper[1] on page 395, as well as in [1, p. 399], instead of

$$u(0)=1,\quad u(1)=2,\quad u(2)=3,\quad \cdots$$

it should read

$$u(0)=-1,\quad u(1)=2,\quad u(2)=3,\quad \cdots.$$

Also, in the above paper two additional misprints are encountered.

1) In (18) the element in the first row, second column should read -1 instead of 1.

2) In (20) the element in the second row, second column should read z^2+2z+2 instead of z^2+2z+z.

References

[1] M. A. Budin, "Minimal realization of discrete linear systems from input-output observations," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 395–401, Oct. 1971.

Manuscript received January 12, 1976; revised March 16, 1976.
The author is with the Department of Electrical Engineering, University of Campinas, Campinas, Brazil.
[1]K. Furuta and J.-G. Paquet, *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 392–396, June 1975.

Authors' Reply[2]

KATSUHISA FURUTA AND JEAN-GUY PAQUET

We thank S. P. Bingulac for pointing out errors in our paper. It is not strange that our results are correct in spite of the fact that $u(0)$ should read -1, since we did not use $u(0)$ as Budin [1] did in the example; actually we used $\bar{B}_3(1)$ of [1] to calculate P.

References

[1] M. A. Budin, "Minimal realization of discrete linear systems from input-output observations," *IEEE Trans. Automat. Contr.*, vol. AC-16, pp. 395–401, Oct. 1971.

[2]Manuscript received January 12, 1976.
K. Furuta is with the Department of Control Engineering, Tokyo Institute of Technology, Tokyo, Japan.
J.-G. Pacquet is with the Department of Electrical Engineering, University of Laval, Quebec, P. Q., Canada.

Comments on "Conditions for a Feedback Transfer Matrix to be Proper"

R. W. SCOTT AND B.D.O. ANDERSON

Abstract—**When constant output feedback is applied around a linear system with a rational transfer function matrix which may be improper, the closed-loop transfer function matrix is generically proper.**

Introduction

In a recent paper[1] conditions were examined for a transfer function matrix derived by feedback to be proper. In this note we show that the conditions referred to in that paper hold generically. (We also point out several adjustments necessary to the arguments in that paper.) More specifically, we establish the following.

Manuscript received January 26, 1976. This work was supported by the Australian Research Grants Committee.
The authors are with the Department of Electrical Engineering, University of Newcastle, N. S. W., Australia.
[1]M. Vidyasagar, *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 570–571, Aug. 1975.

Theorem 1: Let $W(s)$ be a $q\times r$ matrix of real rational transfer functions. For a generic constant $r\times q$ matrix F, $W(s)\,[I+FW(s)]^{-1}$ is proper, i.e., $\lim_{s\to\infty} W(s)\,[I+FW(s)]^{-1}$ exists (and is finite).

Remark 1: We remind the reader that to say that the property of the theorem holds for a generic F is the same as saying that it holds for all F, except those whose entries satisfy one or more nontrivial polynomial equalities [1].

Remark 2: It is interesting to note that the McMillan degree [2] of $W(s)\,[I+FW(s)]^{-1}$ is generically the same as the McMillan degree of $W(s)$; see [3], [4]. This fact together with Theorem 1 can be used as an efficient tool to develop results of [5] applicable to improper transfer function matrices.

Remark 3: It is clear that at best a generic result can be obtained. For with $W(s)$ improper and $F=0_{r\times q}$, evidently $W(s)\,[I+FW(s)]^{-1}$ is still improper.

Adjustments to the Original Paper

The proof of Lemma 1[1] appears incorrect. The lemma is stated thus:

"*Lemma 1:* Given any polynomial matrix $M(s)$ one can find a constant nonsingular matrix U such that $M(s)U$ is column proper."

Here, if $M(s)$ is an $n\times n$ polynomial matrix in s, and if α_i denotes the highest power of s occurring in the ith column of $M(s)$, then $M(s)$ is column proper if $\det[M(s)]$ is a polynomial of degree $\sum_{i=1}^{n}\alpha_i$.

Define M_H to be the matrix whose ith column consists of the coefficients of $s^{\alpha i}$ in the corresponding entries of the ith column $M(s)$. The crucial step in the proof of Lemma 1[1] says that if one finds a matrix U_1 such that some column (say the kth) of M_HU_1 is identically zero, then the coefficients of $s^{\alpha k}$ in the kth column of $M(s)U_1$ are all zero. This is not so. The entries of the kth column of M_HU_1 are $\sum_{j=1}^{n} m_{ij}u_{jk}$, $(i=1,\cdots,n)$, where $M_H=\{m_{ij}\}$ and $U_1=\{u_{ij}\}$. The entries of the kth column of $M(s)U_1$ are $\sum_{j=1}^{n}[c_{ij}s^{p_{ij}}+\cdots]u_{jk}$, $(i=1,\cdots,n)$, where p_{ij} is the degree of the (i,j) entry of $M(s)$ and c_{ij} is the coefficient of the highest order term of this entry. Clearly, unless $p_{i1}=p_{i2}=\cdots p_{in}=\alpha_k$ in a row of $M(s)$ with an entry of degree α_k, then $\sum_{j=1}^{n} m_{ij}u_{jk}=0 (i=1,\cdots,n)$ does not imply that the coefficients of $s^{\alpha k}$ in the kth column of $M(s)U_1$ are all zero. Moreover, the column index may be greater in $M(s)U_1$ if one of the p_{ij} is greater than α_k.

Lemma 1[1] must be replaced by the following theorem [6, p. 27].

Theorem 2: For any nonsingular polynomial matrix $M(s)$ one can always find a unimodular matrix $U(s)$ such that $M(s)U(s)$ has column proper form.

Taking into account the replacement of Lemma 1[1] with Theorem 2, Fact 4[1] must be replaced by the following theorem.

Theorem 3: Consider a rational $q\times r$ matrix $W(s)$ expressed as the product of the $q\times r$ polynomial matrix $N(s)$ and the inverse of the $r\times r$ polynomial matrix $D(s)$ (see [7, p. 4]), that is, $W(s)=N(s)\,D^{-1}(s)$. For the $r\times q$ constant matrix F select $U(s)$ such that $[D(s)+FN(s)]U(s)$ is column proper. Then $W(s)\,[I+FW(s)]^{-1}$ is proper if and only if, for $i=1,2,\cdots,n$, the highest power of s in the ith column of $N(s)U(s)$ is less than or equal to that in the ith column of $[D(s)+FN(s)]U(s)$.

Demonstration That $W(s)[I+FW(s)]^{-1}$ is Generically Proper

Let the highest power of s in the ith column of a polynomial matrix A be denoted by $\alpha_i\{A\}$. Express $W(s)$ as $N(s)D^{-1}(s)$. Then by Theorem 3, $W(s)\,[I+FW(s)]^{-1}$ is proper if and only if

$$\alpha_i\{N(s)U(s)\}\leqslant\alpha_i\{[D(s)+FN(s)]U(s)\}$$

where $U(s)$ is selected to make $[D(s)+FN(s)]U(s)$ column proper. The existence of $U(s)$ is guaranteed by Theorem 2 and the fact that $D(s)+FN(s)$ is generically nonsingular; this in turn following from the nonsingularity of $D(s)$.

Since F is a constant matrix the elements in the ith column of $FN(s)U(s)$ are linear combinations of the elements in the ith column of $N(s)U(s)$. Therefore,

$$\alpha_i\{FN(s)U(s)\}=\alpha_i\{N(s)U(s)\}\tag{1}$$

generically. If the highest coefficients of the highest powers of s in the ith column of $N(s)U(s)$ cancel out under the linear combination caused by premultiplication by F, then

$$\alpha_i\{FN(s)U(s)\}<\alpha_i\{N(s)U(s)\}.\tag{2}$$

However, this implies that a polynomial in the entries of F is zero. That is, (1) holds generically.

Consider now the following three cases for the ith column.

Case 1:

$$\alpha_i\{D(s)U(s)\}>\alpha_i\{N(s)U(s)\}.$$

Equivalently,

$$\alpha_i\{D(s)U(s)\}>\alpha_i\{FN(s)U(s)\}.$$

Therefore,

$$\alpha_i\{D(s)U(s)+FN(s)U(s)\}=\alpha_i\{D(s)U(s)\}>\alpha_i\{N(s)U(s)\}$$

and the condition of Theorem 3 is satisfied for the ith column.

Case 2:

$$\alpha_i\{D(s)U(s)\}<\alpha_i\{N(s)U(s)\}.$$

Equivalently,

$$\alpha_i\{D(s)U(s)\}<\alpha_i\{FN(s)U(s)\}$$

generically. Therefore,

$$\alpha_i\{D(s)U(s)+FN(s)U(s)\}=\alpha_i\{FN(s)U(s)\}=\alpha_i\{N(s)U(s)\}$$

generically and the condition of Theorem 3 is satisfied except on the solutions of a finite number of polynomial equations.

Case 3:

$$\alpha_i\{D(s)U(s)\}=\alpha_i\{N(s)U(s)\}.$$

Equivalently,

$$\alpha_i\{D(s)U(s)\}=\alpha_i\{FN(s)U(s)\}$$

generically. Therefore,

$$\alpha_i\{D(s)U(s)+FN(s)U(s)\}=\alpha_i\{FN(s)U(s)\}=\alpha_i\{N(s)U(s)\}$$

except when the coefficients of the highest power of s in the ith columns of $D(s)U(s)$ and $FN(s)U(s)$ cancel out. In this situation $\alpha_i\{N(s)U(s)\}>\alpha_i\{D(s)U(s)+FN(s)U(s)\}$. Then, for Case 3, the condition of Theorem 3 is satisfied for column i, except if the coefficient of the highest power of s, in the ith column of $D(s)U(s)+FN(s)U(s)$, is zero. This implies that a finite number of polynomials in the entries of F are zero. So, for Case 3, the conditions of Theorem 3 are satisfied generically.

For an arbitrary column i, Cases 1–3 show that the conditions of Theorem 3 are satisfied generically, and so, applying Theorem 3, $W(s)[I+FW(s)]^{-1}$ is generically proper.

Example: Consider Example 2:[1]

$$W(s)=\begin{bmatrix}\dfrac{s+1}{s+2} & \dfrac{2s+1}{s+3}\\[2mm] \dfrac{-s+1}{s+1} & \dfrac{-2s+1}{s+1}\end{bmatrix}.\tag{3}$$

Then $W(s)=N(s)D(s)^{-1}$ where

$$N(s)=\begin{bmatrix}s^2+2s+1 & 2s^2+9s+4\\ -s^2-s+2 & -2s^2-5s+3\end{bmatrix}$$

$$D(s)=\begin{bmatrix} s^2+3s+2 & 0 \\ 0 & s^2+7s+12 \end{bmatrix}.$$

Let

$$F=\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}.$$

$$D(s)+FN(s)=\begin{bmatrix} (f_{11}-f_{12}+1)s^2+(2f_{11}-f_{12}+3)s+(f_{11}+2f_{12}+2) & (2f_{11}-2f_{12})s^2+(9f_{11}-5f_{12})s+(4f_{11}+3f_{12}) \\ (f_{21}-f_{22})s^2+(2f_{21}-f_{22})s+(f_{21}+2f_{22}) & (2f_{21}-2f_{22}+1)s^2+(9f_{21}-5f_{22}+7)s+(4f_{21}+3f_{22}+12) \end{bmatrix}$$

which is column proper unless

$$f_{11}-f_{12}+2f_{21}-2f_{22}+1=0. \quad (4)$$

(Note that (4) is satisfied by $F=I$ as was the case in the original paper.) Since F generically does not satisfy (4), in Theorem 3 we may take $U=I$. Then, the conditions of Theorem 3 are satisfied unless either

$$\begin{aligned} f_{11}-f_{12}+1&=0 \\ f_{21}-f_{22}&=0 \end{aligned} \quad (5)$$

or

$$2f_{11}-2f_{12}=0$$

or

$$2f_{21}-2f_{22}+1=0 \quad (6)$$

hold. So the conditions of Theorem 3 hold generically and so for the $W(s)$ defined in (3), $W(s)[I+FW(s)]^{-1}$ is generically proper.

Conclusion

We have shown that for all $q\times r$ rational transfer function matrices, almost all constant feedback matrices cause the resulting closed-loop transfer function matrix to be proper.

References

[1] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, vol. 2. London: Cambridge Univ. Press, 1968.
[2] B. McMillan, "Introduction to formal realizability theory," *Bell Syst. Tech. J.*, vol. 31, pp. 217–279, Mar. 1952; also, *Bell Syst. Tech. J.*, vol. 31, pp. 541–600, May 1952.
[3] R. J. Duffin and D. Hazony, "The degree of a rational matrix function," *J. Soc. Ind. Appl. Math.*, vol. 11, pp. 645–658, Sept. 1963.
[4] R. E. Kalman, "Irreducible realizations and the degree of a matrix of the rational functions," *J. Soc. Ind. Appl. Math.*, vol. 13, pp. 520–544, June 1965.
[5] B. D. O. Anderson and E. I. Jury, "Generalized Bezoutian and Sylvester matrices in multivariable linear control," this issue, pp. 551–556.
[6] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer, 1974.
[7] C. A. Desoer and J. D. Schulman, "Zeros and poles of matrix transfer functions and their dynamical interpretation," *IEEE Trans. Circuits Syst.*, vol CAS-21, pp. 3–8, Jan. 1974.

Author's Reply[2]

M. VIDYASAGAR

I wish to thank Dr. Scott and Dr. Anderson for their interest in my paper. Their comments are correct.

I would also like to point out that in Fact 4[1] $N(s)+KD(s)$ should be changed to $D(s)+KN(s)$ throughout.

[2]Manuscript received March 4, 1976.
The author is with the Department of Electrical Engineering, Concordia University, Montreal, P.Q., Canada.

Comments on "Design of Piecewise Constant Gains for Optimal Control via Walsh Functions"

N. GOPALSAMI AND B. L. DEEKSHATULU

The above paper[1] considers the use of Walsh functions in the design of piecewise constant gains for the linear optimal control problem with quadratic performance criteria. The use of Walsh functions is illustrated in the context of solving a vector differential equation. A Walsh series is assumed for each rate variable and the unknown Walsh coefficients are determined by solving the resulting set of simultaneous equations. In this correspondence it is shown that the same solution can be easily obtained by proceeding directly with the piecewise constant approximation for the rate variables.

Consider (15) of Chen and Hsiao's paper:[1]

$$C=ACP+G+BH. \quad (1)$$

Taking the transpose of (1),

$$C'=P'C'A'+G'+H'B', \quad (2)$$

i.e.,

$$[c_1c_2\cdots c_n]=P'[c_1c_2\cdots c_n]A'+G'+[h_1h_2\cdots h_n]B' \quad (3)$$

where c_i represents the Walsh series coefficients of the ith rate variable

$$c_i=[c_{i0},c_{i1},\cdots,c_{i(m-1)}]',$$

and h_i corresponds to the Walsh series coefficients of the ith input u_i,

$$h_i=[h_{i0},h_{i1},\cdots,h_{i(m-1)}]',$$

i.e.,

$$\dot{x}_i\cong c_{i0}\phi_0+c_{i1}\phi_1+\cdots+c_{i(m-1)}\phi_{m-1}$$
$$u_i\cong h_{i0}\phi_0+h_{i1}\phi_1+\cdots+h_{i(m-1)}\phi_{m-1}.$$

It is well known that when a function is approximated by a Walsh series of finite order m ($m=$some integral power of 2), the approximated function is of piecewise constant form and the magnitude of the step in each subinterval is equal to the average value of the function in that subinterval. If f_{ij} is the average value of $\dot{x}_i$ and g_{ij} is the average value of u_i in the jth subinterval, $j=0,1,\cdots,(m-1)$, then the following relations are true.

$$c_i=\frac{1}{m}Wf_i \quad (4)$$

$$h_i=\frac{1}{m}Wg_i \quad (5)$$

Manuscript received February 2, 1976.
The authors are with the Department of Electrical Engineering, Indian Institute of Science, Bangalore, India.
[1]C. F. Chen and C. H. Hsiao, *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 596–602, Oct. 1975.