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Correspondence Item

A Note on the Youla–Bongiorno–Lu Condition†

Summary—Youla, Bongiorno and Lu have presented a condition for the stabilizability using stable controllers of an unstable plant. We show this condition may be reformulated in terms of Cauchy indices, allowing its checking in a finite number of rational operations, should the plant not be given in factored form, i.e. should its poles and zeros be unknown. The results applies to discrete and continuous-time, and scalar and multivariable plants.

1. *Introduction*

YOULA, Bongiorno and Lu[1] have considered the following problem. With reference to Fig. 1, suppose that the plant, series and feedback compensators are scalar, linear, time-invariant and finite-dimensional; that any hidden modes of the plant are asymptotically stable, and there are no hidden modes of the series or feedback compensator; and that the closed loop is dynamical. Suppose also that the plant is not asymptotically stable. State when there exist asymptotically stable series and feedback compensators such that the closed-loop system is asymptotically stable.

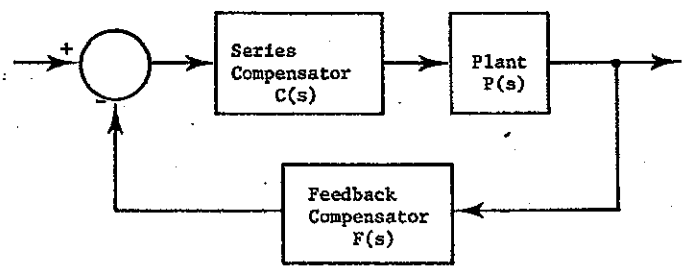

FIG. 1. Closed loop system comprising plant and two compensators.

The answer to this problem is contained in [1]. Let $\sigma_1, \sigma_2, \ldots, \sigma_\ell$ denote the distinct real zeros of $P(s)$ in $Re[s] \geq 0$ (infinity included). Let ν_i be the total number of real poles of $P(s)$, multiplicities being counted, to the right of σ_i. Then compensators with the desired properties exist if and only if all the ν_i are even or all are odd. In case $P(\infty) = 0$, this means that all ν_i must be even.

In this note, we redescribe this criterion in terms of a Cauchy index[2]. This shows that to check satisfaction of the criterion, one does not need to evaluate the real poles and zeros of $P[s]$, because the Cauchy index of any rational function can be computed by a finite number of rational calculations involving the coefficients of the numerator and the denominator polynomial of the function. Though it is our belief that the authors of [1] are well aware of this possibility, the actual language in [1] might lead the reader to suspect that poles and zeros need to be known or evaluated.

2. *The Cauchy index*

Let $r(s)$ be a real rational function of s; the Cauchy index of $r(\cdot)$ between the limits a and b [written $\overset{b}{\underset{a}{I}} r(s)$] for a, b real or $\pm\infty$ is the difference of the number of jumps of $r(s)$ from $-\infty$ to $+\infty$ and from $+\infty$ to $-\infty$ as the argument changes from a to b; no account is taken of jumps at a or b.

Cauchy indices may be computed by computing a Sturm sequence with first two members the denominator and numerator of $r(s)$,[2]; in case (a, b) is $(-\infty, \infty)$, there are three other procedures for determining the value of a Cauchy index—by evaluating the signs of inners determinants akin to Kurwitz determinants with entries determined by the coefficients of the denominator or numerator of $r(s)$[2], or by forming a Hankel matrix with the aid of the Markov parameters associated with $r(\cdot)$[2], or finally by solving some linear equalities and checking the signature of a matrix derived from the solution[3]. Evidently, in all these methods, only a finite number of rational calculations are required; however, one is still at liberty to compute the Cauchy index by factoring the denominator polynomial of $f(\cdot)$ and then studying how $f(\cdot)$ behaves at each of its real poles.

In case one is required to compute the Cauchy index of a real rational $f(\cdot)$ over $(0, \infty)$, one can take advantage of the methods available for the interval $(-\infty, \infty)$ by noting that, provided $f(0)$ is finite,

$$2\overset{\infty}{\underset{0}{I}} f(s) = \overset{+\infty}{\underset{-\infty}{I}} s f(s^2)$$

This follows from a straightforward application of the Cauchy index definition. In case $f(0)$ is infinite, let $f(s) = s^{-k}\hat{f}(s)$ where k is a positive integer and $\hat{f}(0)$ is finite. Then

$$2\overset{\infty}{\underset{0}{I}} f(s) = \overset{+\infty}{\underset{-\infty}{I}} s\hat{f}(s^2)$$

3. *Equivalent statement of the stabilization condition*

Let the denominator polynomial of $P(s)$ be $d(s)$, and be monic. Let the nonnegative real zeros of $P(s)$, including infinity if it is a zero, be denoted by $\sigma_1, \sigma_2, \ldots, \sigma_l$. Then it is shown in [1] that compensators with the desired properties exist if and only if $d(\sigma_i)$ has the same sign for all σ_i. If $s = \infty$ is a zero of $P(s)$, this means that $d(\sigma_i) > 0$ for all σ_i.

4. *A Cauchy index formulation of the stabilization condition*

In order to reinterpret the stabilization condition using Cauchy indices, we make the following two observations.

Lemma 1. Let $f(s)$ be a real polynomial. Then the number of distinct real zeros (counting a multiple zero only once) in the interval (a, b) is $\overset{b}{\underset{a}{I}}[f'(s)/f(s)]$. (Here, $f'(s) = df/ds$).

For a proof, see[2]. Note that $a = -\infty$, $b = \infty$ are not precluded.

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Lemma 2. Let $f(s)$ and $g(s)$ be real polynomials. Then $I_a^b(f'(s)g(s)/f(s))$ = [number of distince real zeros in (a, b) of $f(\cdot)$ at which $g(\cdot)$ is positive] – [number of distinct real zeros in (a, b) of $f(\cdot)$ at which $g(\cdot)$ is negative].

Proof. Suppose that $f(s) = a_0(s-\alpha_1)^{n_1}\ldots(s-\alpha_m)^{n_m}$ with $\alpha_1, \ldots, \alpha_p$ real, $\alpha_{p+1}, \ldots, \alpha_m$ not real. Then

$$\frac{f'(s)}{f(s)} = \sum_{i=1}^{p}\frac{n_i}{s-\alpha_i} + R(s)$$

where $R(s)$ is rational with no real poles. The contribution to $I_a^b[f'(s)/f(s)]\ g(s)$ from a zero at $s = \alpha_i$ evidently $+1$ if $g(\alpha_i) > 0$, -1 if $g(\alpha_i) < 0$, and zero otherwise. This establishes the lemma.

Combining Lemmas 1 and 2 we have the following:

Corollary. Let $f(\cdot)$ and $g(\cdot)$ be two real polynomials. The polynomial $g(\cdot)$ is positive at all real zeros of $f(\cdot)$ in (a, b) and only if

$$I_a^b\frac{f'(s)}{f(s)} = I_a^b\frac{f'(s)g(s)}{f(s)}$$

and has the same sign at all real zeros of $f(\cdot)$ in (a, b) if and only if

$$I_a^b\frac{f'(s)}{f(s)} = \pm I_a^b\frac{f'(s)g(s)}{f(s)}$$

Turning to the stabilization problem again, suppose that $P(s) = n(s)/\mathrm{d}(s)$. If $s = 0, \infty$ are not zeros of $P(s)$, it is evident that $\mathrm{d}(\sigma_i)$ has the same sign for all zeros σ_i of $n(\cdot)$ in $[0, \infty)$ if and only if

$$I_0^\infty\frac{n'(s)}{n(s)} = \pm I_0^\infty\frac{n'(s)}{n(s)}\mathrm{d}(s) \qquad (1)$$

If $s = \infty$ is a zero of $P(s)$, we require that

$$I_0^\infty\frac{n'(s)}{n(s)} = I_0^\infty\frac{n'(s)}{n(s)}\mathrm{d}(s) \qquad (2)$$

If $s = 0$ is a zero of $P(s)$, we require additionally that $\mathrm{d}(0)$ is positive or negative according as the plus or minus sign applies in (1); of course, if $s = \infty$ is also a zero, $\mathrm{d}(0)$ positive is the only possibility. In summary, we have:

Reformulation of Youla–Bongiorno–Lu condition. The plant with transfer function $P(s) = n(s)/\mathrm{d}(s)$ is stabilizable via stable compensation under the conditions listed in the introduction if and only if (1) holds provided $s = 0, \infty$ are not zeros of $P(s)$, (2) holds if $s = \infty$ is a zero and $\mathrm{d}(0) > 0$ or $\mathrm{d}(0) < 0$ in case $s = 0$ is a zero, according as the plus or minus sign holds in (1).

Reference has already been made to the evaluation of Cauchy indices.

5. *Discrete-time system*

The bilinear transformation $s = (1+z)(1-z)^{-1}$ maps the s-plane interval $[0, \infty)$ into the z-plane interval $[-1, 1]$ with $s = \infty$ corresponding to $z = 1$. This observation may be used to develop a discrete-time variant of the stabilization condition. Cauchy indices over $[-1, 1]$ appear.

6. *Multivariable plants*

Let $P(s)$ be an $n \times m$ matrix of real rational transfer functions. Write $P(s) = (n_{ij}(s)/\mathrm{d}(s))$ where $\mathrm{d}(s)$ is the nomic least common denominator of all entries of $P(s)$. Following[1], let the nonnegative real zeros of $P(s)$, including ∞ if it is a zero, be denoted by $\sigma_1, \sigma_2, \ldots, \sigma_l$, where in this context, a zero of $P(s)$ is understood to be a zero of every entry of $P(s)$. Then, as shown in[1], stabilizing and stable compensators exist if and only if $\mathrm{d}(\sigma_i)$ has the same sign for all σ_i.

The question arises as to whether this condition can be checked in a finite number of rational calculations. Indeed it can; for the greatest common divisor $\phi(s)$ of all $n_{ij}(s)$ can be determined in a finite number of operations and the zeros of $\phi(s)$ in $[0, \infty)$ are precisely the finite σ_i. The scalar result accordingly applies save that in (1) and (2), $n(s)$ is replaced by $\phi(s)$.

7. *Conclusions*

In case $P(s)$ is not given in a factored form, it is possible to check the stabilizability condition of Youla, Bongiorno and Lu by a finite number of rational calculations; the concept of the Cauchy index provides a helpful tool for defining what calculations need to be executed. We can therefore add the problem posed in[1] to the class which can be tackled by decision algebra methods, used in[3] to tackle a different output feedback stabilization problem.

References

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