*Foundations of System Theory: Multidecomposable Systems**

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ABSTRACT: *The paper discusses multilinear, and more generally multidecomposable, machines. An m-linear machine is shown to be realizable as a network of k-linear machines for $k \leq (m-1)$, linked by certain memoryless m-linear maps. In this way, an m-linear machine can be broken down into linear machines and multilinear memoryless maps.*

I. Introduction

This paper extends the methodology of (2) to cover multilinear systems, in the sense introduced by Kalman (4):

(1) *Definition.* Given vector spaces $I_1, \ldots, I_k$ and Y, for $k \geq 1$, we say a response function†

$$f: I_1^\natural \times \ldots \times I_k^\natural \to Y$$

is *k-linear* if for each $j \in \{1, \ldots, k\}$ and for each choice of $w_i \in I_i$ for $i \neq j$, the map

$$f(w_1, \ldots, w_{j-1}, \cdot, w_{j+1}, \ldots, w_k): I_j^\natural \to Y$$

is linear.

While Kalman (4) focused on showing how to characterize the state-space of a multilinear system as "a variety (= algebraic manifold) in a Euclidean space of rather high dimension", Arbib (1) followed Kalman in studying Nerode–Raney equivalence relations (6, 7) associated with a multilinear response, but used these techniques to show how to realize a multilinear system as a *network* of *linear* systems. He showed that every

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† We use the notation $I^\natural$ for the countable copower of I [see (2) for basic concepts undefined here]. For a vector space I, $I^\natural$ is the vector space of all left-infinite sequences with finite support $(\ldots, i_k, \ldots, i_1, i_0)$ of I-vectors.

bilinear response function ($k = 2$) had a realization of the form given in Fig. 1. His conjecture about the extension of this result for $k > 2$ was wrong, but Fig. 1 provides the starting point for the general theory we present here.

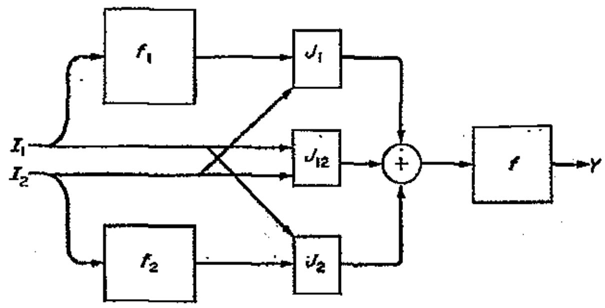


FIG. 1. The realization of a bilinear system as a network of linear systems (1). Each f_i is a linear system; while the J's are memoryless, delayless, bilinear maps.

Marchesini and Picci (5) extended the earlier approaches to analyze time-varying continuous-time bilinear maps

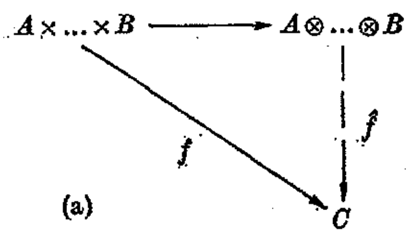
$$\int_{t_0}^{t}\int_{t_0}^{t} k(t,\tau,\sigma)\, u(\tau)\, \hat{u}(\sigma)\, d\tau\, d\sigma,$$

a form which clearly relates the discrete-time version of (1) to Kalman's original motivation, the second-order term of the Wiener–Volterra expansion for a nonlinear input–output function. Like these authors, we shall make use of image factorizations of maps defined on tensor products to obtain multilinear realizations. However, our focus will be on finding the most natural algebraic setting to generalize our theory of decomposable systems (2) to k-linear systems for arbitrary k.

To develop our general theory, we recall from linear algebra that the tensor product of vector spaces has the property that for any vector spaces $A, \ldots, B$ there is a map

$$A \times \ldots \times B \to A \otimes \ldots \otimes B$$

with the property that for any multilinear $f: A \times \ldots \times B \to C$ there exists a unique linear $\hat{f}: A \otimes \ldots \otimes B \to C$ such that

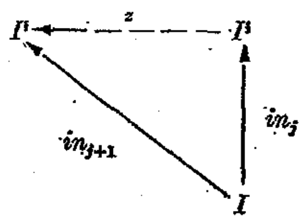


(a)

commutes. Hence, specifying a multilinear $I_1^\natural \times \ldots \times I_k^\natural \to Y$ is equivalent to specifying a linear $I_1^\natural \otimes \ldots \otimes I_k^\natural \to Y$; and we shall henceforth work with the latter form of a multilinear response.

In (2) we saw that the proper setting for a general theory of (discrete-time) decomposable systems—which embraced linear systems and group machines—was a category $\mathscr{K}$ which had countable powers $(in_j: I \to I^\natural \mid j \geq 0)$ and copowers $(\pi_k: Y_\natural \to Y \mid k \geq 0)$*. A *system* was then specified by a *dynamics* $F: Q \to Q$, an *input* map $G: I \to Q$ and an *output* map $H: Q \to Y$. We defined the free dynamics $z: I^\natural \to I^\natural$ by

(2) *Definition.*

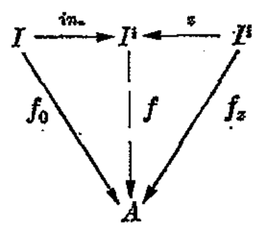


(b)

It is immediate that:

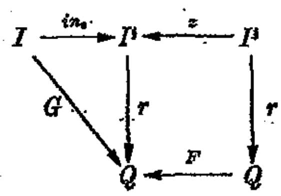
(3) The diagram $I \xrightarrow{in_0} I^\natural \xleftarrow{z} I^\natural$ is a coproduct. □

In other words, given any maps $f_0: I \to A$ and $f_z: I^\natural \to A$, there is a unique f such that



(c)

(4) *Definition.* We define the *reachability map* of (F, G) to be the unique dynamorphic extension $r: (I^\natural, z) \to (Q, F)$ of G:



(d)

We then define the *response map* of (F, G, H) as $f = H \cdot r: I^\natural \to Y$.

* The reader may consult (2) for basic concepts not defined here. For intuition, we note that in the case of vector spaces: $I^\natural$ is the space of left-infinite sequences of finite support $(\ldots, i_j, \ldots, i_1, i_0)$ of vectors i_j from I, with $in_j: I \to I^\natural$ sending u to the sequence whose only non-zero element is $i_j \to u$.

$Y_\natural$ is the space of right infinite sequences, not necessarily of finite support; $(y_0, y_1, \ldots, y_k, \ldots)$ of vectors y_k from Y, with $\pi_k: Y_\natural \to Y$ sending each sequence to its kth element y_k.

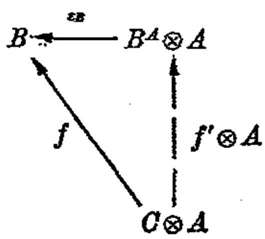
In the rest of this section, we shall present a general categorical setting in which we may talk about $I_1^\dagger \otimes \dots \otimes I_k^\dagger$. In Section II we shall express the situation shown in Fig. 1 in algebraic language to get our general definition of a *k-line system*, and then define the reachability and response maps for such a system, and verify that each k-line system has a response map of the form $I_1^\dagger \otimes \dots \otimes I_k^\dagger \to Y$. In Section III we shall establish that each $I_1^\dagger \otimes \dots \otimes I_k^\dagger \to Y$ is the response map of a k-line system.

A general setting which embraces vector spaces with tensor products—and sets with cartesian products—as a special case is a closed category [see, e.g., (3) for a more careful definition than we need here].

(5) *Definition.* A *closed category*, $(\mathscr{K}, \otimes)$ is a category $\mathscr{K}$ together with a functor $\otimes: \mathscr{K} \times \mathscr{K} \to \mathscr{K}$ which has the property that, for each object A of $\mathscr{K}$, the functor $-\otimes A: \mathscr{K} \to \mathscr{K}$ has a right adjoint. (Moreover, $\otimes$ has certain pleasant properties such as $A \otimes B \cong B \otimes A$, and $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ which we shall use without explicit formalization.)

For ease of reference we spell out the condition that each $-\otimes A$ has a right adjoint:

(6) Fixing A, to each B in $\mathscr{K}$ there corresponds an object B^A and a morphism $\varepsilon_B: B^A \otimes A \to B$ (called *evaluation*) such that to each $f: C \otimes A \to B$ there corresponds



(e)

a unique $f': C \to B^A$ such that $\varepsilon_B \cdot (f' \otimes A) = f$.

For vector spaces and tensor products, B^A is the vector space of all linear maps $A \to B$ and $\varepsilon_B(f, a) = f(a)$. The case of sets and cartesian products is similar, using all functions $A \to B$. In both cases, $f'(c) := f(c, \cdot)$.

We now decree:

(7) *The setting for the algebraic theory of multidecomposable systems is a closed category* $(\mathscr{K}, \otimes)$ *with countable powers and copowers.* Henceforth, we assume such a setting. Thus we can introduce the following useful notation:

We say a subset α of $K = \{1, \dots, k\}$ is *proper* if $\alpha \neq K$. For any $\alpha \subset K$ we set

$$I^\alpha = B_1 \otimes \dots \otimes B_k \quad \text{where} \quad B_i = \begin{cases} I_i^\dagger & \text{if } i \in \alpha \\ I_i & \text{if } i \notin \alpha \end{cases}$$

so that $I^\phi = I_1 \otimes \dots \otimes I_k$.

We set $I^{[\alpha]} = \otimes (I_j^\dagger \mid j \in \alpha)$ and $I^{\bar{\alpha}} = \otimes (I_j \mid j \notin \alpha)$ so that $I^\alpha \cong I^{[\alpha]} \otimes I^{\bar{\alpha}}$, and $I^K = I^{[K]}$. Then for each k-tuple of non-negative integers $p = (p_1, \dots, p_k) \in \mathbf{N}^k$ we set

$$in_p: I^\phi \to I^K = in_{p_1} \otimes \dots \otimes in_{p_k}.$$

$$in_\alpha: I^\alpha \to I^K = \beta_1 \otimes \ldots \otimes \beta_k \quad \text{with } \beta_i = \begin{cases} z_i & \text{if } i \in \alpha \\ in_0 & \text{if } i \notin \alpha \end{cases}$$

while

$$z: I^K \to I^K = z_1 \otimes \ldots \otimes z_k.$$

II. Reachability and Response for k-line Systems

We fixed a closed category $(\mathscr{K}, \otimes)$ and define:

(1) *Definition.* A *k-decomposable response function* is any $\mathscr{K}$-morphism

$$f: I_1^? \otimes \ldots \otimes I_k^? \to Y.$$

We next formalize the situation shown in Fig. 1.

(2) *Definition.* A *k-line system M* with input objects $I_1, \ldots, I_k$ and output object Y is defined by induction on k as follows:
For $k = 1$: M is a decomposable system $M = (Q, F, I_1, G, Y, H)$

$$G: I_1 \to Q, \quad F: Q \to Q, \quad H: Q \to Y.$$

For $k > 1$: M is specified by:

(i) a decomposable dynamics $F: Q \to Q$ and output $H: Q \to Y$;

(ii) for each proper non empty subset α of $\{1, \ldots, k\}$ a $|\alpha|$-line system M_α with input objects $(I_i \mid i \in \alpha)$ and output object Y_α;

(iii) for each proper non empty α, a morphism $J_\alpha: Y_\alpha \otimes I^{\bar{\alpha}} \to Q$, where Y_α is as in (ii), and Q is as in (i); and

(iv) a morphism $J_\phi: I^\phi \to Q$.

When $(\mathscr{K}, \otimes)$ is ⟨vector spaces and tensor products⟩, $k = 1$ just says that a 1-line system is a linear system. Let us now check that $k = 2$ satisfies all the ingredients in the network of Fig. 1:

(i) Q, F and H are the state-space, dynamics and output map, respectively, of the "ultimate" linear system f.

(ii) $\{1, 2\}$ has only two proper non empty subsets α, namely $\{1\}$ and $\{2\}$, so that in each case the $|\alpha|$-line system M_α is a linear system. We have $M_{\{1\}} = f_1$ and $M_{\{2\}} = f_2$.

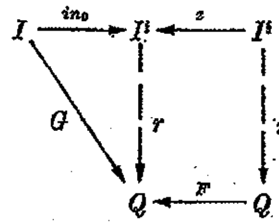
(iii) We then define $J_{\{1\}}: Y_{\{1\}} \otimes I_2 \to Q$ to be the linearization of the bilinear map $G \cdot J_1: Y_{\{1\}} \times I_2 \to Q$ of Fig. 1. Similarly for $J_{\{2\}}$.

(iv) Finally, $J_\phi: I_1 \otimes I_2 \to Q$ is the linearization of $G \cdot J_{12}: I_1 \times I_2 \to Q$.

That Definition 2 also holds good for $k > 2$ will be evidenced by Definition 11 below, which shows that we may associate a k-decomposable response function with each k-line system. In Section III, we show that each k-decomposable response function is realized by a k-line system.

Given a decomposable dynamics (Q,F) with input map $G: I \to Q$, we defined its reachability map, I 4, by the diagram:

(3)



(f)

That r exists and is unique is proved by using the fact that

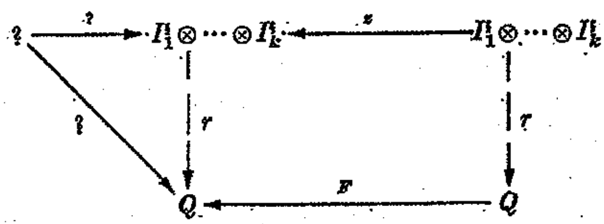
$$in_j: I \to I^@ \mid j \geqslant 0$$

is a coproduct, and the fact that z is defined by $z \cdot in_j = in_{j+1}$ to deduce from (3) that

$$r \cdot in_j = F^j G: I \to Q \quad \text{for } j \geqslant 0$$

thus defining $r: I^@ \to Q$ uniquely.

We now assume $(\mathscr{K}, \otimes)$ is a closed category. The form $f: I_1^@ \otimes \dots \otimes I_k^@ \to Y$ for a k-decomposable response function suggests that (3) be generalized to



(g)

How are the ?'s to be filled in? Because $A \otimes -$ has a right adjoint, it preserves coproducts (see [3, Corollary 8.14] for an exposition). Thus if $A_i \xrightarrow{a_i} A$ and $B_j \xrightarrow{b_j} B$ are coproducts, so is $A_i \otimes B_j \xrightarrow{a_i \otimes B_j} A \otimes B_j$ for each j, as is $A \otimes B_j \xrightarrow{A \otimes b_j} A \otimes B$. Pasting these together, we deduce:

(4) $\qquad (a_i \otimes b_j: A_i \otimes B_j \to A \otimes B)$ is also a coproduct.

Thus, for p running over $\mathbf{N}^K$, $I^p \xrightarrow{in_p} I^{[K]}$ is a coproduct. Moreover, (4) shows that for α running over proper subsets of $K = \{1, \dots, k\}$,

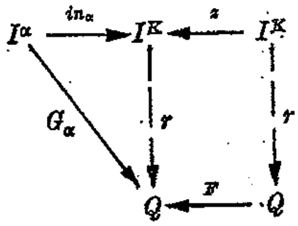
$$I^\alpha \xrightarrow{in_\alpha} I^K \xleftarrow{z} I^K$$

is a coproduct, generalizing the $k = 1$ situation in which

$$I \xrightarrow{in_0} I^@ \xleftarrow{z} I^@$$

is a coproduct. This suggests the plausibility of:

(5) *Principle of Simple k-recursion.* Given a dynamics $F: Q \to Q$ and a morphism $G_\alpha: I^\alpha \to Q$ for each proper α, there exists a unique r satisfying



(h)

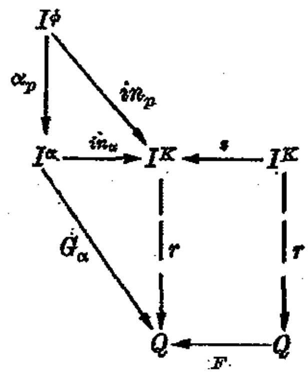
Proof: In proving (1) we determined r by finding each $r \cdot in_j$.
In this case, we determine $r \cdot in_p: I_1 \otimes \ldots \otimes I_k \to Q$ for each $p = (p_1, \ldots, p_k) \in \mathbb{N}^k$
For each proper α, let $A_\alpha = \{p \in \mathbb{N}^k \mid p_j > 0$ if and only if $j \in \alpha\}$. Then:

(6) $$(I^{\phi} \xrightarrow{\alpha_p} I^\alpha \mid p \in A_\alpha) \text{ is a coproduct}$$

where $\alpha_p = \alpha_{p,1} \otimes \ldots \otimes \alpha_{p,k}$ with

$$\alpha_{p,i} = \begin{cases} I_i \xrightarrow{in_{p_i-1}} I_i^{\natural} & \text{if } i \in \alpha, \\ id_{I_i} & \text{if } i \notin \alpha. \end{cases}$$

This is immediate from (4), noting that $\otimes$ is commutative.
Now it is clear that the upper triangle commutes in



(i)

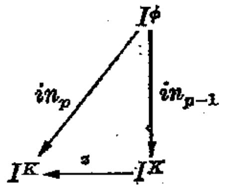
and that our task is to find an r which makes the rest commute. We do this by induction on $m_p = \min(p_1, \ldots, p_k)$.

Basis: If $m_p = 0$, let $\alpha = (i \mid p_i = 0)$ and set:

(7) $$r \cdot in_p = G_\alpha \cdot \alpha_p$$

which is equivalent to $r \cdot in_\alpha = G_\alpha$ because of (6) and $in_\alpha \cdot \alpha_p = in_p$.

Induction: If $m_p > 0$, assume that $r \cdot in_{p-1}$ has been defined for $p-1 = (p_1-1, \ldots, p_k-1)$; then use the observation that



(j)

to define:

$$(8) \qquad r \cdot in_p = F \cdot r \cdot in_{p-1}.$$

Clearly, the unique r defined by (7) and (8) satisfies (5). □

Before proceeding further with the general theory we need some motivation. Consider the linear case—$(\mathscr{K}, \otimes)$ = vector spaces and tensor products—and compute the response of the subsystem of Fig. 1 shown in Fig. 2.

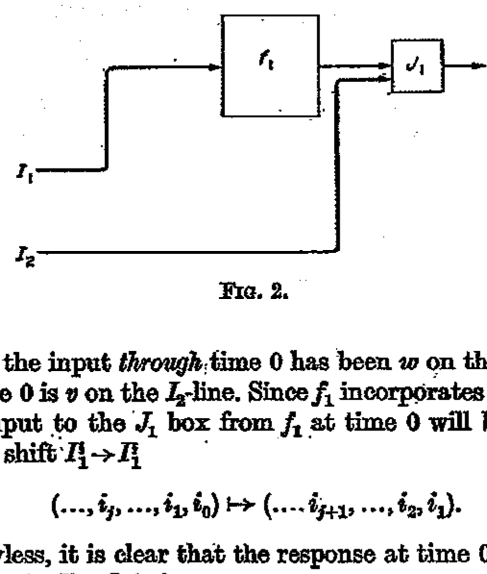


FIG. 2.

Suppose that the input *through* time 0 has been w on the I_1-line, and that the input *at* time 0 is v on the I_2-line. Since f_1 incorporates a unit delay in its response, the input to the J_1 box from f_1 at time 0 will be $f_1 z_1^{-1}(w)$ where z_1^{-1} is the right shift $I_1^{1} \to I_1^{1}$

$$(\ldots, i_j, \ldots, i_1, i_0) \mapsto (\ldots, i_{j+1}, \ldots, i_2, i_1).$$

Since J_1 is delayless, it is clear that the response at time 0 of the subsystem of Fig. 2 to (w, v) in $I_1^{1} \times I_2$ is just

$$J_1(f_1 z_1^{-1}(w), v).$$

Now the principle of simple k-recursion invites us to look at each $r \cdot in_\alpha \colon I^{[\alpha]} \otimes I^{\bar{\alpha}} \to Q$, which, in our Fig. 1 example with $\alpha = \{1\}$, is the state of the f-box obtained by inserting an input of the form

$$in_{\{1\}}(w, v) = (z_1 w, (\ldots, 0, \ldots, 0, v)).$$

It is clear that the only non-zero input to f is the last input

$$J_1(f_1 z_1^{-1}(z_1 w), v) = J_1(f_1 w, v))$$

received via J_1. Since, via (2), we have now decreed that f has input map id_Q, we deduce that

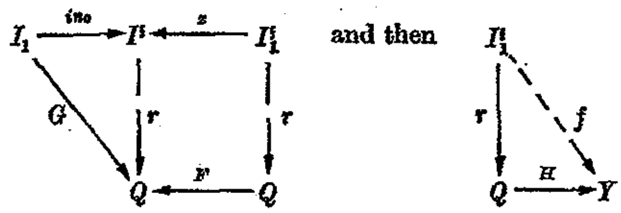
(9) $$r \cdot in_{\{1\}}(w, v) = J_1(f_1 w, v).$$

In (12) below, we shall use this equation in the general form

(10) $$G_\alpha(\text{our new name for } r \cdot in_\alpha) = J_\alpha \cdot (f_\alpha \otimes id).$$

With the principle of simple k-recursion and this motivation we have the following definition:

(11) *Definition.* The *reachability map* $r: I_1^\natural \otimes \ldots \otimes I_k^\natural \to Q$ and *response map* $f: I_1^\natural \to Y$ of the k-line system M of (2) are defined inductively as follows:
For $k = 1$: We have the usual definition, i.e.

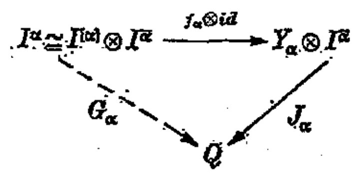


(k)

For $k > 1$: Let G_ϕ be the $J_\phi: I^\phi \to Q$ of (2) (iv).

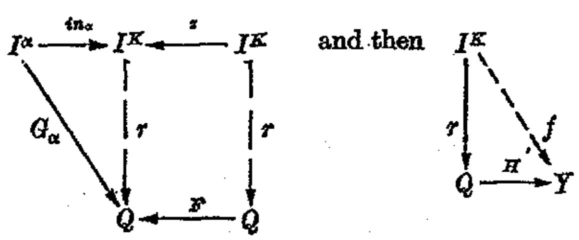
Let f_α, for α proper and nonempty, be the response map $I^{[\alpha]} \to Y_\alpha$ of the M_α of (2) (ii) (already defined, by the induction hypothesis) and then use the J_α of (2) (iii) to define G_α by:

(12)



(l)

Then we use (5), and the (F, H) of (2) (i) to define r and f by



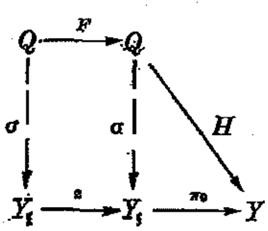
(m)

With this, we have shown that the k-line systems of (2) do indeed have $\mathscr{K}$-morphisms of the form $f: I_1 \otimes \ldots \otimes I_k \to Y$ as their response maps. We must now show that every k-decomposable response function is realized by some k-line system.

III. Observability and Realization Theory

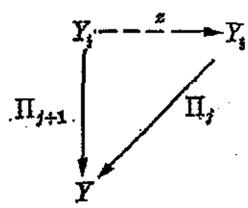
The observability map of a k-line system is just that (2) of the "ultimate" decomposable system:

(1) *Definition.* Let a k-line system M have $F: Q \to Q$ and $H: Q \to Y$ as in definition II (2). Then the *observability map* is the dynamorphic coextension $\sigma: (Q, F) \to (Y_\natural, z)$ of H:



(n)

where



(o)

so that $\pi_j \cdot \sigma = H \cdot F^j$ for all $j \geqslant 0$.

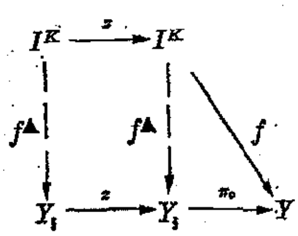
(2) *Definition.* Let a k-line system M have reachability map

$$r: I^K = I_1^\natural \otimes \ldots \otimes I_k^\natural \to Q$$

and observability map $\sigma: Q \to Y_\natural$. Then the *total response* map of M is

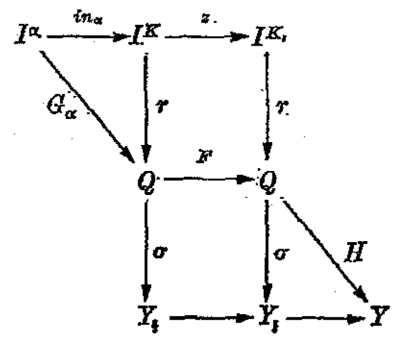
$$f^\blacktriangle = \sigma \cdot r: I^K \to Y_\natural.$$

It is consistent to define $f^\blacktriangle$ as the unique dynamorphic coextension of the response $f: I^K \to Y$:



(p)

We have the commutative diagram



(q)

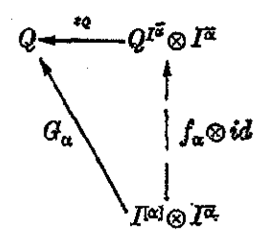
Our task is now to reverse the process, and go from $f^{\blacktriangle}$ to the G_{α}, r, F and H of a *canonical realization*, where, given some fixed image factorization system* $(\mathscr{E}, \mathscr{M})$, we have:

(3) *Definition.* We say a k-line system M is *canonical* if M is both *reachable* in the sense that r is in $\mathscr{E}$, and *observable* in the sense that σ is in $\mathscr{M}$; and if, moreover for $k>1$, each system M_{α} involved in the definition of M is also canonical.

This so clearly follows the story for $k=1$ that the production of canonical realizations is a straightforward inductive generalization of (2):

(4) *Canonical realization.* Given a dynamorphism $f^{\blacktriangle}: (I^{K}, z) \to (Y_{\natural}, z)$, let $f^{\blacktriangle} = I^{K} \xrightarrow{r} Q \xrightarrow{\sigma} Y_{\natural}$ be an image factorization, and let $F: Q \to Q$ be the unique dynamics (2) such that $r: (I^{K}, z) \to (Q, F)$ and $\sigma: (Q, F) \to (Y_{\natural}, z)$ are dynamorphisms. Set $H = \pi_0 \cdot \sigma$.
For $k=1$: Set $G = r \cdot in_0: I_1 \to Q$, and we are done.
For $k>1$: Set $J_{\phi} = r \cdot in_{\phi}: I_1 \otimes \ldots \otimes I_k \to Q$.

For proper nonempty α, set $G_{\alpha} = r \cdot in_{\alpha}: I^{\alpha} \to Q$. Then use $I^{\alpha} \simeq I^{[\alpha]} \otimes I^{\bar{\alpha}}$ and Definition I (6)



(r)

to define $f_{\alpha}: I^{[\alpha]} \to Q^{I^{\bar{\alpha}}}$. Set $Y_{\alpha} = Q^{I^{\bar{\alpha}}}$, and take $J_{\alpha} = \varepsilon_Q: Q^{I^{\bar{\alpha}}} \otimes I^{\bar{\alpha}} \to Q$. Then $J_{\alpha} \cdot (f_{\alpha} \otimes id) = G_{\alpha}$ satisfying Definition II (12). Then, since $|\alpha| < k$, we may

* Again, see (2) for the formal definition. In the case of vector spaces, $\mathscr{E}$ = onto maps, $\mathscr{M}$ = one-to-one maps, and we have that any map $f: A \to B$ factors as $m \cdot e$ where $e: A \to f(A)$ is in $\mathscr{E}$, $m: f(A) \to B$ is in $\mathscr{M}$, and $f(A)$ is unique up to isomorphism.

invoke the induction hypothesis to construct a canonical realization M_α for f_α, and we are done.

There is an *open problem* here, namely "Under what restrictions can we expect the M_α's in a canonical realization to be unique up to isomorphism". It is easy to see that each M_α is unique up to isomorphism if we restrict its output space to be $Q_{I^{\bar{\alpha}}}$. However, we have no useful results about the trade-off between choices of Y_α and the size of the state-space of an $f_\alpha: I^{\alpha} \to Y_\alpha$ when f_α is subject only to the restriction that

$$J_\alpha \cdot (f_\alpha \otimes id) = G_\alpha$$

for some possible choice of $J_\alpha: Y_\alpha \otimes I^{\bar{\alpha}} \to Q$.

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