Degenerate Networks

In the study of network theory the description of networks by means of network matrices (e.g., the impedance matrix, the scattering matrix) has proved exceedingly useful. For linear, passive, time-invariant networks it is known that if an impedance or admittance matrix exists, then so does a scattering matrix (see Youla et al. [1], p. 123); however, the converse is not true, as seen by the ideal transformer. In this letter, by considering linear passive but time-variable elements, we exhibit some networks which are describable by impedance or admittance matrices but not by a scattering matrix. For completeness we discuss various other degenerate networks, tabulating in Table I a set of networks not describable by one or more of the impedance, admittance, or scattering matrix.

TABLE I

SUMMARY OF DEGENERATE NETWORKS

Network	$s(t, \tau)$	$z(t, \tau)$	$y(t, \tau)$
Transformer-coupled inductor, Fig. 1(a)	—	$n(t)\delta'(t-\tau)n(\tau)$	—
Transformer-coupled capacitor, Fig. 1(b)	—	—	$n(t)\delta'(t-\tau)n(\tau)$
Cascade transformer-coupled resistor, Fig. 2(a)	$\cos t\delta(t-\tau)$	—	—
Transformer-coupled resistor, Fig. 2(b)	Exists	$\cos^2 t\delta(t-\tau)$	—
Transformer-coupled resistor, Fig. 2(c)	Exists	—	$\cos^2 t\delta(t-\tau)$
Transformer viewed from one port, Fig. 3(a)	—	—	—
Nullator, Fig. 3(b)	—	—	—
Short circuit	$-\delta(t-\tau)$	0	—
Open circuit	$\delta(t-\tau)$	—	0
Time-invariant transformer	Exists	—	—

In Fig. 1(a) is shown a network which possesses the impedance matrix [2]

$$z(t, \tau) = n(t)\delta'(t-\tau)n(\tau). \tag{1}$$

Here δ' is the derivative of the unit impulse.

The scattering matrix of this network can be calculated [3] to be

$$s(t, \tau) = \delta(t-\tau) - \phi(t)\psi(\tau)u(t-\tau) \tag{2a}$$

where u is the unit step function and the function ϕ and ψ are related by

$$\psi(t) = \frac{\phi(t)}{\int_t^\infty \phi^2(\lambda)d\lambda + c} \tag{2b}$$

(c being a non-negative constant), and $\phi(t)$ and $n(t)$ are related by

$$\frac{1}{n(t)} = \frac{\phi(t)}{\sqrt{2\left\{\int_t^\infty \phi^2(\lambda)d\lambda + c\right\}}}. \tag{2c}$$

From these equations it immediately follows that

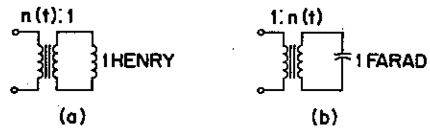

Fig. 1. Networks which may lack scattering matrices.

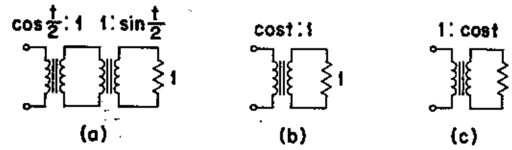

Fig. 2. Networks which lack immittance matrices.

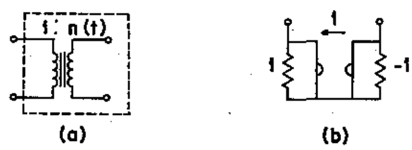

Fig. 3. Networks which lack scattering matrices.

$$\frac{1}{n^2(t)} = \frac{\phi(t)\psi(t)}{2}. \tag{3}$$

Now (1) is well defined if we assume smooth variations in $n(t)$, even if we let $n(t)=0$ for isolated values of t, or over an interval. But (3) shows that at least one of ϕ and ψ behaves in a discontinuous fashion when $n(t)$ goes to zero, and consequently the network cannot possess a (well-defined) scattering matrix if $n(t)$ ever vanishes.

Fig. 1(b) shows the dual network, which possesses an admittance matrix but no scattering matrix if $n(t)$ ever vanishes.

The admittance of the network of Fig. 1(a) is given formally as

$$y(t, \tau) = \frac{1}{n(t)} u(t-\tau) \frac{1}{n(\tau)}. \tag{4}$$

This will not be defined—even as a distributional kernel ([4], sec. 2)—precisely when the scattering matrix is not defined. Notice that a dual result holds for the network of Fig. 1(b).

The use of time-variation allows us to construct other examples of pathological networks which have no time-invariant parallel. For a one-port linear time-invariant network of lumped passive elements, impedance and admittance matrices (which are in fact functions) always exist, save in the case of the open and short circuit. However,

1) $s(t, \tau) = \cos t\delta(t-\tau)$, realized in Fig. 2(a), has neither an impedance nor an admittance matrix
2) $z(t, \tau) = \cos^2 t\delta(t-\tau)$, realized in Fig. 2(b), has no corresponding admittance matrix, but $s = (\cos^2 t+1)^{-1}(\cos^2 t-1)\delta(t-\tau)$
3) $y(t, \tau) = \cos^2 t\delta(t-\tau)$, realized in Fig. 2(c), has no corresponding impedance matrix, but $s = (\cos^2 t+1)^{-1}(1-\cos^2 t)\delta(t-\tau)$.

For completeness, we mention two other pathological networks which have been discussed elsewhere [4], [5]. Fig. 3(a) shows a trans-

Manuscript received February 11, 1966. This work was sponsored by the Air Force Office of Scientific Research under Contract AF-AFOSR 337-63.

former with only one port accessible; when $n(t)$ goes to zero at some finite time, this one-port network possesses no scattering matrix [4]. The network of Fig. 3(b) is the nullator, first introduced by Tellegen [6]; it has $v=i=0$, where v and i are the port voltage and current. As shown by Newcomb [5] and Carlin and Youla [7], p. 908, this network has no scattering matrix.

Table I summarizes the preceding material. A dash indicates lack of existence of a given matrix, where any necessary assumptions for pathological behavior are assumed made [e.g., $n(t)=0$ for some t]. We include for completeness the short circuit, the open circuit, and the standard time-invariant transformer.

B. D. Anderson
R. W. Newcomb
Stanford Electronics Lab.
Stanford, Calif.

References

[1] D. C. Youla, L. J. Castriota, and H. J. Carlin, "Bounded real scattering matrices and the foundations of linear passive network theory," *IRE Trans. on Circuit Theory*, vol. CT-4, pp. 102–124, March 1959.
[2] D. A. Spaulding, "Foster-type time-varying lossless synthesis," *Electronics Lett.*, vol. 1, pp. 248–249, November 1965.
[3] B. D. O. Anderson, "Cascade synthesis of time-varying nondissipative networks," submitted for publication.
[4] B. D. O. Anderson and R. W. Newcomb, "Functional analysis of linear passive networks," *Internat'l J. Engrg. Sci.*, to be published.
[5] R. W. Newcomb, "The foundations of network theory," *IEEE Trans. on Electrical and Mechanical Engineering (Australia)*, vol. EM-6, pp. 7–12, May 1964.
[6] B. D. H. Tellegen, "La recherche pour une série complète d'éléments de circuits idéaux non linéaires," *Rendiconti Seminario Matematico e Fisico (Milano)*, vol. 25, pp. 134–144, 1953–1954.
[7] H. J. Carlin and D. C. Youla, "Network synthesis with negative resistors," *Proc. IRE*, vol. 49, pp. 907–920, May 1961.

Reprinted from the PROCEEDINGS OF THE IEEE
VOL. 54, NO. 4, APRIL, 1966
pp. 694–695
Copyright 1966, and reprinted by permission of the copyright owners
PRINTED IN THE U.S.A.