Degenerate Networks

In the study of network theory the description of networks by means of network matrices (e.g., the impedance matrix, the scattering matrix) has proved exceedingly useful. For linear, passive, time-invariant networks it is known that if an impedance or admittance matrix exists, then so does a scattering matrix (see Youla et al. [1], p. 123); however, the converse is not true, as seen by the ideal transformer. In this letter, by considering linear passive but time-variable elements, we exhibit some networks which are describable by impedances or admittance matrices but not by a scattering matrix. For completeness we discuss various other degenerate networks, tabulating in Table I a set of networks not describable by one or more of the impedance, admittance, or scattering matrix.

**TABLE I**

<table>
<thead>
<tr>
<th>Network</th>
<th>( z(t, \tau) )</th>
<th>( \delta(t, \tau) )</th>
<th>( \psi(t, \tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transformer-coupled inductor, Fig. 1(a)</td>
<td>-</td>
<td>( u(0) \delta'(\tau) )</td>
<td>-</td>
</tr>
<tr>
<td>Transformer-coupled capacitor, Fig. 1(b)</td>
<td>-</td>
<td>-</td>
<td>( u(0) \delta(\tau - \tau) )</td>
</tr>
<tr>
<td>Cascade transformer-coupled resistor, Fig. 2(a)</td>
<td>( \cos \delta t(\tau - \tau) )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Transformer-coupled resistor, Fig. 2(b)</td>
<td>( \delta(t - \tau) )</td>
<td>( \cos \delta t(\tau - \tau) )</td>
<td>-</td>
</tr>
<tr>
<td>Transformer viewed from one port, Fig. 3(a)</td>
<td>-</td>
<td>-</td>
<td>( \cos \delta t(\tau - \tau) )</td>
</tr>
<tr>
<td>Nullator, Fig. 3(b)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Short circuit</td>
<td>( \delta(t - \tau) )</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>Open circuit</td>
<td>( \delta(t - \tau) )</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Time-invariant transformer</td>
<td>Exists</td>
<td>Exists</td>
<td>-</td>
</tr>
</tbody>
</table>

In Fig. 1(a) is shown a network which possesses the impedance matrix [2]

\[
z(t, \tau) = u(0) \delta'(\tau - \tau) \delta(t).
\]

Here \( \delta' \) is the derivative of the unit impulse.

The scattering matrix of this network can be calculated [3] to be

\[
z(t, \tau) = \delta(\tau - \tau) - \delta(\delta(t)\delta(\tau) u(\tau - \tau))
\]

where \( \delta \) is the unit step function and the function \( \delta \) and \( \psi \) are related by

\[
\delta(t) = \int_{-\infty}^{t} \phi(\lambda) d\lambda + \epsilon
\]

(\( \epsilon \) being a non-negative constant), and \( \psi(t) \) and \( n(t) \) are related by

\[
\frac{1}{n(t)} = \frac{\phi(t)}{\sqrt{2} \left[ \int_{-\infty}^{t} \phi(\lambda) d\lambda + \epsilon \right]^2}.
\]

From these equations it immediately follows that

\[
\frac{1}{n(t)} = \frac{\phi(t)}{\sqrt{2} \left[ \int_{-\infty}^{t} \phi(\lambda) d\lambda + \epsilon \right]^2}.
\]

Now (1) is well defined if we assume smooth variations in \( n(t) \), even if we let \( n(t) = 0 \) for isolated values of \( t \), or over an interval. But (3) shows that at least one of \( \delta \) and \( \psi \) behaves in a discontinuous fashion when \( n(t) \) or zero, and consequently the network cannot possess a (well-defined) scattering matrix if \( n(t) \) ever vanishes.

The admittance of the network of Fig. 1(a) is given formally as

\[
\psi(t, \tau) = \frac{1}{n(t)} u(t - \tau) - \frac{1}{n(t)} \delta(t).
\]

This will not be defined—even as a distributional kernel ([4], sec. 2)—precisely when the scattering matrix is not defined. Notice that a dual result holds for the network of Fig. 1(b).

The use of time-variation allows us to construct other examples of pathological networks which have no time-invariant parallel. For a one-port linear time-invariant network of lumped passive elements, impedance and admittance matrices (which are in fact functions) always exist, save in the case of the open and short circuit. However, for completeness, we mention two other pathological networks which have been discussed elsewhere [4], [5]. Fig. 3(a) shows a trans-

![Fig. 1. Networks which may lack scattering matrices.](image)

![Fig. 2. Networks which lack admittance matrices.](image)

![Fig. 3. Networks which lack scattering matrices.](image)

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former with only one port accessible; when \( n(t) \) goes to zero at some finite time, this one-port network possesses no scattering matrix \([4]\).

The network of Fig. 3(b) is the nullator, first introduced by Tellegen \([6]\); it has \( v = i = 0 \), where \( v \) and \( i \) are the port voltage and current. As shown by Newcomb \([5]\) and Carlin and Youla \([7]\), p. 908, this network has no scattering matrix.

Table I summarizes the preceding material. A dash indicates lack of existence of a given matrix, where any necessary assumptions for pathological behavior are assumed made [e.g., \( n(t) = 0 \) for some \( t \)]. We include for completeness the short circuit, the open circuit, and the standard time-invariant transformer.

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References


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