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On Eigenvalues of Complex Matrices in a Sector

B. D. O. ANDERSON, N. K. BOSE, AND E. I. JURY

***Abstract*—It is shown that the test to determine whether all eigenvalues of a complex matrix of order n lie in a certain sector can be replaced by an equivalent test to find whether all eigenvalues of a real matrix of order $4n$ lie in the left half plane.**

In a recent article [1], it was mentioned that when

$$AB=BA \tag{1}$$

the eigenvalues of a $n\times n$ complex matrix $A+jB$ (where A and B are real matrices), are in the shaded sector shown in [1, Fig. 1] if and only if the eigenvalues of the $2n\times 2n$ real matrix

$$X_{2n}=\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \tag{2}$$

are in the same sector. The substantiation was dependent on the commutavity condition in (1). It was pointed out to the authors by Jones [2] that results mentioned above still hold when matrices A and B do not commute under multiplication. The proof for this, as indicated by Jones [2], is based on simple column and row operations which enable one to conclude that the characteristic equations of

$$X_{2n}=\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

and

$$Y_{2n}=\begin{bmatrix} A+jB & 0 \\ 0 & A-jB \end{bmatrix}$$

are identical in all cases. Thus, the test for eigenvalues of a complex matrix in the prescribed sector is equivalent to test for eigenvalues of a real matrix, of double order, in the same sector. Complex algebra, is thus replaced by real algebra [3]. Also, the matrix counterpart of the result given for polynomials in [1, Theorem 2], readily follows.

Theorem: The eigenvalues of the $n\times n$ complex matrix $A+jB$ lie in the shaded sector shown in [1, Fig. 1] if and only if the eigenvalues of the $4n\times 4n$ real matrix

$$X_{4n}=\begin{bmatrix} X_{2n}\cos\delta & -X_{2n}\sin\delta \\ X_{2n}\sin\delta & X_{2n}\cos\delta \end{bmatrix} \tag{3}$$

lie in the left half plane.

Several comments are in order. First, the $n\leftrightarrow 4n$ relationship is valid for matrices, in the sense that the apparently difficult problem of determining whether all eigenvalues of a $n\times n$ complex matrix lie in the prescribed sector can be reduced to the more straightforward problem of determining whether all the eigenvalues of a $4n\times 4n$ real matrix lie in the left half plane. Second, it has been noticed that the characteristic equations of X_{2n} and Y_{2n} are identical. Of course, the eigenvalues of two matrices could be located in the prescribed sector without their characteristic equations being identical. This suggests that a sparser matrix than X_{2n} might exist, on which the implementation of the test could be computationally simpler.

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B. D. O. Anderson is with the Department of Electrical Engineering, University of Newcastle, N. S. W., Australia.
N. K. Bose is with the Department of Electrical Engineering, University of Pittsburgh, Pittsburgh, Pa. 15261.
E. I. Jury is with the Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, Calif. 94720.

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An Improved Test for the Zeros of a Polynomial in a Sector

G. H. HOSTETTER

***Abstract*—A recently proposed method of determining whether or not the eigenvalues of a matrix lie within a sector of the complex plane is modified, giving both insight and computational advantages.**

I. INTRODUCTION

Recently, Anderson, Bose, and Jury [1] have called attention to and extended an interesting and useful result by Davison and Ramesh [2] concerning the eigenvalues of a matrix. The essence of the result is that an $n\times n$ matrix A has eigenvalues s outside of the "damping factor" sector

$$-90^\circ-\delta<\arg\cdot s<90^\circ+\delta \tag{1}$$

if and only if

$$A^*=\begin{bmatrix} A\cos\delta & -A\sin\delta \\ A\sin\delta & A\cos\delta \end{bmatrix}$$

has eigenvalues in the right half of the complex plane.

The application of this result to the sector bounding of system eigenvalues involves forming the matrix A^* from the $n\times n$ system state coupling matrix A, determining the characteristic equation of the $(2n)\times(2n)$ matrix A^*, then Routh–Hurwitz testing that polynomial. This procedure appears to be simpler than that in the mathematical literature [3].

Further refinement may be made by simplifying or eliminating the step involving determination of the characteristic equation of A^*, which is likely to be of high order in a practical problem. The widely used Faddeeva–Leverrier method of characteristic equation determination [4] would require $2n-1$ multiplications of $(2n)\times(2n)$ matrices, for instance.

While special canonical forms for A will yield simplifications, there is really no need to cast the problem in terms of matrices. The approach to be described concentrates, instead, upon the characteristic polynomials.

II. CHARACTERISTIC POLYNOMIAL RELATIONS

For the nth-order polynomial

$$P(s)=s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0,$$

the related $(2n)$th-order polynomial

$$F(s)=P(s)|_{s\to se^{j\delta}}P(s)|_{s\to se^{-j\delta}}$$

has two roots corresponding to each root of $P(s)$. For each root of $P(s)$, one root of $F(s)$ is the $P(s)$ root rotated on the complex plane by the angle δ. The other root of $F(s)$ is the $P(s)$ root rotated by the angle $-\delta$.

If the coefficients of $P(s)$ are real numbers so that the roots of $P(s)$

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The author is with the Department of Electrical Engineering, California State University, Long Beach, Calif. 90840.