Proof of Theorem 6: Notice $x^i(\nu,s,x^i(s;\,t,x))=x^i(\nu;\,t,x)$ is independent of s, hence from (39),

$$\frac{d}{ds}p^i(s,x^i(s;\,t,x))=-p^i(s,x^i(s;\,t,x))f_x^i(s,x^i(s;\,t,x)u^i(s;\,t,x)) - \sum_{\substack{j=1\\ j\neq i}}^{k}\lambda_{ij}p^j(s,x^i(s;\,t,x)). \quad (45)$$

Recall that $x_x^i(s;\,t,x)$ is a solution of (10). Multiplying (45) on the right by $x_x^i(s;\,t,x)$ and (10) on the left by $p^i(s,x^is;\,t,x)$ adding the two equations multiplying the result by $e^{\lambda_{ii}(s-t)}$ and using Lemma 1 gives

$$\frac{d}{ds}[e^{\lambda_{ii}(s-t)}p^i(s,x^i(s;\,t,x))x_x^i(s;\,t,x)] = -\sum_{j\neq i}\lambda_{ij}e^{\lambda_{ii}(s-t)}p^j(s,x^i(s;\,t,x)x_x^i(s;\,t,x)). \quad (46)$$

Integrating from t to T and using that $p^i(T;\,x)=\phi_x(x)$ and $x_x(t;\,t,x)=I$ gives

$$p^i(t,x)=\phi_x(x^i(T;\,t,x))x_x^i(T;\,t,x)e^{\lambda_{ii}(T-t)} + \int_t^T\lambda_{ij}e^{\lambda_{ii}(s-t)}p^j(s,x^i(s;\,t,x)x_x^i(s;\,t,x))\,ds.$$

Thus $p^i(t,x)$ and $\psi_x^i(t,x)$ satisfy the same integral equation (9). It is not difficult to show that this integral equation has a unique solution and hence that (41) holds.

To deduce the sufficiency part of Theorem 5 from Theorem 6 notice that from (41), (40), and Theorem 2 that

$$\psi_t(t,x)+\min_{u\in U}\psi_x^i(t,x)f^i(t,x,u)=\psi_t(t,x)+\psi_x^i(t,x)f^i(t,x,u^i(t,x))=0$$

and hence from Theorem 4 that $u^i(t,x)$ is an optimal control.

Comment: When discontinuous controls are allowed and the optimal control is discontinuous, there is an analog of Theorem 5, however it is much more complicated. Jump terms corresponding to places where $x^i(s;\,t,x)$ crosses the discontinuities of the control must be added in (39). These jump terms are the same as those given by Mirica [10] when he represented adjoint equations for deterministic problems as functions of (t,x). There is another minimum principle whose adjoints are not represented by an integral equation which is also necessary and sufficient for optimality. For details see [13].

Conclusion and Summary: Satisfaction of the dynamic programming partial differential equation was shown to be a necessary and sufficient condition for a continuously differentiable feedback control to be an optimal control. A new minimum principle, expressed in terms of adjoints given by deterministic integral equations, which is necessary and sufficient for optimality was formulated. The adjoints of this minimal principle agree with the partial derivatives of the performance function of the optimal control.

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Relations Between Real and Complex Polynomials for Stability and Aperiodicity Conditions

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***Abstract*—Results are obtained connecting the zero distribution of a real polynomial and a complex polynomial of approximately half the degree of the real polynomial. The results are applied to the relating of real aperiodic polynomials with real and complex Hurwitz polynomials. "Unit circle" results are also outlined.**

I. Introduction

A real monic polynomial of degree $2n$ and a complex monic polynomial of degree n are both described by $2n$ parameters; here, we show how from a real $2n$ degree polynomial which is Hurwitz, i.e., has all its zeros in $\mathrm{Re}[z]<0$, we can construct a complex n degree polynomial (with parameters in 1-1 relationship with those of the real polynomial) which is also Hurwitz. In fact, two different complex polynomials can be found, the result extends to odd degree real polynomials, and a converse can be stated. All this is the content of Theorem 1.

The results of Theorem 1 are probably known to many people, although we are unaware of the existence of a statement as complete as that given here. Further, we are unaware whether all the various proofs (suggested in Section II only in outline) are fully known.

In Section III, Theorem 2 applies the results of Theorem 1 to aperiodic polynomials (those with zeros which are distinct, and negative real). We show how two previously stated criteria for aperiodicity can be immediately related, and we present a minor but new variation on one of these earlier stated criteria.

In Section IV, we discuss a "unit circle" version of the results.

II. Association of Real and Complex Polynomials

The main result relating the zero distribution of a real polynomial and a complex polynomial of approximately half the degree is as follows.

Theorem 1: For real a_i, $i=0,1,\cdots,m$, consider the three polynomials

$$F_0(z)=\sum_{i=0}^{m}a_iz^i \quad (1)$$

$$F_1(z)=\sum_{i=0}^{[m/2]}[a_{2i}(j)^i+a_{2i+1}(j)^{i+1}]z^i \quad (2)$$

$$F_2(z)=\sum_{i=0}^{[m/2+1]}[a_{2i}(-j)^i-a_{2i-1}(-j)^{i+1}]z^i. \quad (3)$$

Here $[r]$ denotes the greatest integer less than or equal to the real number r, and in (2) and (3) one sets $a_k=0$ if k is outside the range $[0,m]$ as required. Suppose that $a_0>0$, $a_m>0$, and $a_{2i}>0$ or $a_{2i+1}>0$ for $i=1,2,\cdots,[m/2]$. Then if any of the polynomials $F_k(z)$, $k=0,1,2$ has its zeros in $\mathrm{Re}[z]<0$, the other two polynomials have this property.

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Remarks

1) If $F_0(z)$ is Hurwitz, it is trivial to show that $a_m>0$ implies $a_i>0$ for all i. However, if $F_1(z)$ or $F_2(z)$ in (2) or (3) are Hurwitz, it does not necessarily follow that the sign conditions on the a_i of the theorem statement are fulfilled.

2) The theorem shows how to associate with any real Hurwitz polynomial two complex Hurwitz polynomials. Not every complex Hurwitz polynomial however has the form of $F_1(z)$ or $F_2(z)$ with the sign constraints on the a_i. Thus, the mapping {*real Hurwitz polynomials of* degree m}→{complex Hurwitz polynomials of degree $[m/2]$ or $[m/2+1]$}, $F_0(z)\mapsto F_1(z)$ or $F_2(z)$, is not onto.

3) There is an obvious mapping {complex Hurwitz polynomials of degree n}→{real Hurwitz polynomials of degree $2n$}, viz. $f(z)\to f(z)f^*(z)$, where f^* is obtained from f by coefficient conjugation. This mapping (which again is not onto) is not related to that in the theorem.

4) If we write $F_0(z)$ in terms of its even and odd parts as

$$F_0(z)=G(z^2)+zH(z^2) \tag{4}$$

then

$$F_1(z)=G(jz)+jH(jz) \quad F_2(z)=G(-jz)+zH(-jz). \tag{5}$$

5) Some conclusions can be obtained from the fact that if $\sum_{i=0}^m b_i z^i$ is Hurwitz, so is $\sum_{i=0}^m b_{m-i}z^i$. (The zeros of the two polynomials are reciprocals.) Thus, suppose $F_0(z)$ is known to be Hurwitz, and suppose the claims of Theorem 1 involving only $F_0(z)$ and $F_1(z)$ for m even have been established. With $m=2n$, the following polynomials are then Hurwitz:

$$F_3(z)=\sum_{i=0}^{2n} a_{2n-i}z^i, \qquad \text{(by coefficient reversal)}$$

$$F_4(z)=\sum_{i=0}^{n}\left[a_{2n-2i}(j)^i+a_{2n-2i-1}(j)^{i+1}\right]z^i, \qquad \text{(by Theorem 1)}$$

$$F_5(z)=\sum_{i=0}^{n}\left[a_{2i}(j)^{n-i}+a_{2i-1}(j)^{n-i+1}\right]z^i, \qquad \text{(by coefficient reversal).}$$

Observe that $F_5(z)$ is none other than $j^nF_2(z)$. This means that if the claims regarding $F_0(z)$ and $F_1(z)$ are established, part at least of those involving $F_2(z)$ follow easily. For m odd however, one cannot deduce claims concerning $F_2(z)$ from those involving $F_0(z)$ and $F_1(z)$ as the polynomial corresponding to $F_5(z)$ becomes a multiple of $F_1(z)$.

6) One proof of this result easily follows using Hurwitz determinants. The Hurwitz character of $F_2(z)$ can be described by positivity conditions on the even dimension leading principal minors of $m\times m$ matrix

$$\begin{bmatrix} a_{m-1} & a_{m-3} & a_{m-5} & \cdots \\ a_m & a_{m-2} & a_{m-4} & \cdots \\ 0 & a_{m-1} & a_{m-3} & \cdots \\ 0 & a_m & a_{m-2} & \cdots \\ 0 & 0 & a_{m-1} & \cdots \\ 0 & 0 & a_m & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \qquad (a_k=0 \quad \text{for} \quad k<0)$$

for m even, and with minor variation when m is odd, as noted in [1, pp. 248–250].

The same conditions together with the sign conditions on the a_i guarantee by the Liénard–Chipart criterion that $F_0(z)$ is Hurwitz, [1, p. 221]. The idea is not hard to extend to $F_1(z)$.

7) A second proof will follow using the reduced Hermite test as described in, for example, [2]. Direct calculation will show that the Hermite matrices for $F_1(z)$ and $F_2(z)$ are essentially the same as the two reduced Hermite matrices for $F_0(z)$. The sign constraints on the a_i and positive definiteness of either reduced Hermite matrix imply $F_0(z)$ is Hurwitz, while positive definiteness of the Hermite matrix of $F_1(z)$ guarantees the Hurwitz property for that polynomial and likewise for $F_2(z)$. In this way, Theorem 1 follows.

8) A third way of establishing Theorem 1 is to count the zeros of $F_0(z)$, $F_1(z)$, and $F_2(z)$ in $\text{Re}[z]<0$ by observing the change in argument of these polynomials as z moves around a contour comprising the imaginary axis, and a semicircle of arbitrarily large radius extending in the left-half plane. The Cauchy index may be used to compute this change of argument, as is standard for this sort of problem, see [1, ch. XV]; the relations (4) and (5) prove crucial in relating the zeros of $F_0(z)$, $F_1(z)$, and $F_2(z)$. Actually, the argument closely follows that of [1, p. 221–225] used in establishing the Liénard–Chipart stability criterion from the Hurwitz criterion.

III. Aperiodicity Conditions

A polynomial $f_0(z)$ is termed aperiodic if all its zeros are distinct and negative real. Tests for aperiodicity appear in [3]–[5].

Here, we link up these tests and the theorem of Section I.

Theorem 2: Suppose

$$f_0(z)=\sum_{i=0}^{n}\alpha_i z^i$$

and

$$F_0(z)=f_0(z^2)+z\frac{df_0(z^2)}{d(z^2)}$$

$$F_1(z)=f_0(jz)+j\frac{df_0(jz)}{d(jz)}$$

$$F_2(z)=f_0(-jz)+z\frac{df_0(-jz)}{d(-jz)}.$$

Suppose also that $\alpha_i>0$, $i=0,1,\cdots,n$. Then aperiodicity of $f_0(z)$ implies and is implied by the Hurwitz nature of $F_0(z)$, $F_1(z)$, or $F_2(z)$.

Remarks

1) $F_0(z)$ is a real $2n$th degree polynomial, and $F_1(z)$ and $F_2(z)$ are complex nth degree polynomials. Moreover, $F_0(z)$, $F_1(z)$, and $F_2(z)$ are related in the same way as they are in Theorem 1. Also, positivity of the α_i implies and is implied by positivity of a_0, a_{2n}, and a_{2i} or a_{2i+1} for $i=1,2,\cdots,(n-1)$, where $F_0(z)=\sum_{i=0}^{2n}a_iz^i$.

2) Aperiodicity of $f_0(z)$ and the Hurwitz property of $F_0(z)$ are shown to be equivalent in [3] and [4]. A speedy proof is given below. Aperiodicity of $f_0(z)$ and the Hurwitz property of $F_1(z)$ are shown to be equivalent in [5]. Therefore, Theorem 1 relates the results of [3], [4], and [5], and also introduces $F_2(z)$ into the aperiodicity picture.

3) To prove Theorem 2, we recall an easily proved and fairly well known result, see [1, p. 228]; the polynomial $f(z)=h(z^2)+zg(z^2)$ is Hurwitz if and only if the zeros of $h(z)$ and $g(z)$ are distinct, negative real, and interlace, and the highest coefficients of $h(z)$ and $g(z)$ are of like sign.

Identify $F_0(z)$ with $f(z)$, $f_0(z)$ with $h(z)$ and $(df_0(z)/dz)$ with $g(z)$. Then if $F_0(z)$ is Hurwitz, $f_0(z)$ is immediately aperiodic, while if $f_0(z)$ is aperiodic, it is evident that its zeros will interlace those of $(df_0(z)/dz)$; the other requirements of negative realness and sign identity of the highest coefficients can be checked, so that $F_0(z)$ is Hurwitz.

IV. Conclusions

We have been conditioned to expect unit circle parallels of many stability results involving Hurwitz polynomials, and we can now consider briefly unit circle equivalents of the results of earlier sections.

Let $w=(z+1/z-1)$ (thus, also $z=(w+1/w-1)$) and let $G_0(w)$ be an arbitrary degree m real polynomial in w. One can construct the sequence

$$G_0(w)\mapsto F_0(z)\mapsto F_1(z)\mapsto G_1(w)$$

via

$$F_0(z)=(z-1)^nG\left(\frac{z+1}{z-1}\right), \quad G_1(w)=(w-1)^{[n/2]}F_1\left(\frac{w+1}{w-1}\right)$$

to obtain a polynomial $G_1(w)$ of degree $[m/2]$ which has all zeros inside $|z|=1$ if and only if $G_0(w)$ has this property, given also the satisfaction of certain inequalities involving linear combinations of the coefficients of $G_0(w)$. Using results of [6] relating the Schur–Cohn matrix and Hermite matrix of two polynomials related by transformations of the above type, together with results of [7] describing a reduced Schur–Cohn criterion (analogous to the reduced Hermite criterion), one can even show that a reduced Schur–Cohn criterion matrix associated with $G_0(w)$ is (to within an inessential, and coefficient independent, transformation) the same as the Schur–Cohn matrix of $G_1(w)$. [Similar results hold if $F_2(z)$ is used in lieu of $F_1(z)$]. All this seems of little interest, however, because the mapping $G_0(w) \to G_1(w)$ is not one that has a straightforward explicit description. Each coefficient of $G_1(w)$ will be a linear combination of (normally) all the coefficients of $G_0(w)$, weighted via various products of binomial coefficients, so that the aesthetically pleasing simplicity of the relation between $F_0(z)$ and $F_1(z)$ [see (1) and (2)] is lost.

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Stability Analysis of Stochastic Composite Systems

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Abstract—**New results for asymptotic stability and exponential stability with probability one of several classes of continuous parameter and discrete parameter stochastic composite systems are established. In all cases the objective is to analyze composite systems in terms of their lower order subsystems and in terms of their interconnecting structure. The results are applied to three specific examples.**

I. Introduction

In several recent reports the stability of large-scale deterministic systems, also referred to as composite systems or interconnected systems, has been considered (see, e.g., [1]–[11]). In the present paper new stability results for several classes of stochastic composite systems are established. Systems considered include 1) continuous parameter systems (described by Itô differential equations), and 2) discrete parameter systems (discrete independent increment processes). In all cases the objective is the same: to analyze composite systems in terms of their lower order subsystems and in terms of their interconnecting structure. In order to demonstrate the usefulness of the present approach, three specific examples are considered.

II. Notation

Let R^n denote the Euclidean n-space, let $|\cdot|$ denote the Euclidean norm, and let $x'=(x_1,\cdots,x_n)$ denote the transpose of $x \in R^n$. Let $\lambda(A)$ denote an eigenvalue of a square matrix A, and let the norm of a rectangular matrix D induced by the Euclidean norm be denoted by $\|D\| = \sqrt{\lambda_{\max}(D'D)}$, where D' is the transpose of D. Let $T=[0,\infty)$ and let $J=\{0,1,2,\cdots\}$. Let $\{x_t, t\in T\}$ denote a continuous parameter stochastic process and let $\{x_j, j \in J\}$ denote a discrete parameter stochastic process. Henceforth it is assumed that $x_0=x$ is known. Let $E_{x,s}x_t$ denote the expected value of x_t at $t\in T$ if it is known that $x_s=x$ (if $s=0$, $E_x x_t$ is used). A real-valued function $\varphi(r)$ is said to belong to *class K* (i.e., $\varphi \in K$) if it is defined, continuous, and strictly increasing over $0 \leq r < \infty$, and if it vanishes at $r=0$.

Systems are considered which may be represented by Itô differential equations

$$dx = f(x)\,dt + \sigma(x)\,dz \tag{1}$$

where $x\in R^n$, $t \in T$, $f: R^n \to R^n$, $\sigma: R^n \to R^{n\cdot m}$, and where $\{z_t, t\in T\}$ is a normalized m-dimensional Wiener process. It is assumed that $f(\cdot)$ and $\sigma(\cdot)$ fulfill all conditions required to insure for every $x_0=x$ the existence and uniqueness of a solution, $\{x_t, t\in T\}$, *with probability one* (wp1) (see [12]–[14]). This solution is called an Itô process. It is also assumed that $\{x_t=0, t\in T\}$ is the only equilibrium of (1).

For various definitions of stability wp1 of the equilibrium of (1), refer to [13] and [14]. Stability results wp1 for (1) involve the existence of Lyapunov-type functions $V: R^n \to R^1$. Henceforth it is assumed that all V-functions are such that the sets

$$Q_m = \{x \in R^n : V(x) < m, \quad m \text{ a positive constant}\} \tag{2}$$

are bounded. Furthermore, it is assumed that $V(x)$ is bounded on Q_m and possesses continuous and bounded first- and second-order partials in x over Q_m. Stability results wp1 for (1) also require that $V(x)$ be in the domain of the weak infinitesimal operator $\tilde{A}_{Q_m}$ for (1) (see [12]–[14]). For (1), $\tilde{A}_{Q_m}$ is determined by the differential generator

$$\mathfrak{L}_{(1)} = \sum_i f_i(x)' \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} S_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \tag{3}$$

where the $S_{ij}(x)$ are defined by the matrix $((S_{ij}(x))) = \sigma(x)\sigma(x)'$.

Also considered are systems described by

$$x_{k+1} = f(x_k) + \sigma(x_k) z_k \tag{4}$$

where $k\in J$, $x \in R^n$, $f: R^n \to R^n$, $\sigma: R^n \to R^{n\cdot m}$, and z_k is an m-dimensional normalized, discrete, independent increment process. It is assumed that (2) possesses a unique solution $\{x_k, k\in J\}$ for every $x_0=x$, and moreover, that $\{x_k=0, k\in J\}$ is the only equilibrium of (4). For definitions and results of stability wp1 of the equilibrium of (4) refer to [14].

III. Composite Systems Considered

Composite systems are considered which may be described by

$$dw_i = f_i(w_i)\,dt + \sigma_i(w_i)\,dz_i + g_i(w_1,\cdots,w_l)\,dt \tag{5}$$

where $w_i \in R^{n_i}$, $f_i: R^{n_i} \to R^{n_i}$, $\sigma_i: R^{n_i} \to R^{n_i\cdot m_i}$, $f_i(w_i)=0$ and $\sigma_i(w_i)=0$, if and only if $w_i=0$, $\{z_{i_t}, t \in T\}$ is an m_i-dimensional Wiener process, and $g_i: R^{n_1} \times \cdots \times R^{n_l} \to R^{n_i}$. Letting $\sum_{j=1}^{l} n_j = n$, $x'=(w_1',\cdots,w_l') \in R^n$, $f(x)' = [f_1(w_1)',\cdots,f_l(w_l)']$, $z'=(z_1',\cdots,z_l')$, $g(x)'=[g_1(w_1,\cdots,w_l)',\cdots,g_l(w_1,\cdots,w_l)'] \triangleq [g_1(x)',\cdots,g_l(x)']$, and letting

$$\sigma(x) = \begin{bmatrix} \sigma_1(w_1) & 0 & \cdots & 0 \\ 0 & \sigma_2(w_2) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_l(w_l) \end{bmatrix}$$

one can represent (5) equivalently as

$$dx = f(x)\,dt + \sigma(x)\,dz + g(x)\,dt \triangleq F(x)\,dt + \sigma(x)\,dz \tag{6}$$

where $f: R^n \to R^n$, $\sigma: R^n \to R^{n\cdot m}$, $g: R^n \to R^n$, and $\{z_t, t\in T\}$ is an m-dimensional Wiener process (i.e., $\sum_{j=1}^{l} m_j = m$). System (6), which is of

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