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On the reduced Hermite and reduced Schur–Cohn matrix relationships†

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In this paper, a transformation which connects the matrices of the Hermite and Schur–Cohn criteria for root distribution is obtained. Based on this transformation, the relationship between the reduced Hermite and reduced Schur–Cohn criteria is also obtained. Furthermore, the transformation which connects the Liénard–Chipart stability criterion and the simplified determinantal stability criterion for the unit circle is derived. Finally, the connection between the inner form of the Hermite and Schur–Cohn criteria is established.

The importance of the various transformations lies in the fact that their existence implies that a proof of one left-half-plane stability criterion immediately yields a proof of a corresponding unit circle criterion, and conversely. The various matrix transformations are obtained from the matrix transformation linking vectors of the coefficients of two polynomials whose respective root distributions are studied and which are related by a bilinear transformation of the underlying variables. The symmetry and skew symmetry of the matrix transformation are utilized to obtain the various transformations derived in this paper.

1. Introduction

It is well known that the Hermite (1854) and Schur–Cohn (1922) symmetric matrices describe the root distribution of polynomials (real or complex) with respect to the imaginary or $j\omega$-axis in the s-plane and to the unit circle in the z-plane respectively. Also it is known that by using the bilinear transformation on the polynomial variables we extend either one of these criteria to obtain the root-distribution with respect to both regions. To our knowledge, it is however not known how the symmetric Hermite matrix H of one polynomial is related to the symmetric Schur–Cohn matrix S of a second polynomial, obtained from the first by bilinear transformation of the variable. In the following discussion, such a relationship is established.

Recently some discussions of the reduced Hermite criterion (Jury and Ahn 1971, Anderson 1972) for the stability of real polynomials have been published. Also almost simultaneously a corresponding reduction in the Schur–Cohn criterion of stability within the unit circle has been obtained (Anderson and Jury 1973). In the following discussion, we will establish the connection between the reduced Hermite and reduced Schur–Cohn criteria. The importance of this relationship is in obtaining a simple proof of the reduced Schur–Cohn criterion from the reduced Hermite criterion using the bilinear transformation. It may be noted that the dimension of the matrix of the reduced criterion is about one-half of the original one. Based on this reduced relationship, we will also obtain the connection between the Liénard–Chipart determinantal

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criterion (Liénard and Chipart 1914) in terms of inners for the left half-plane and the simplified determinantal criterion (Jury 1971) in terms of inners for the inside of the unit circle. Also, the connection between the Hermite and Schur–Cohn criteria in terms of innerwise matrices will be established in this paper.

The results of this paper are achieved because of the recent studies of what one may term the bilinear transformation matrix by Duffin (1969), Power (1967) and more recently by one of the authors (Jury 1973). In particular, the symmetric and skew symmetric properties of the bilinear transformation matrix Γ or Q (discussed below) are most useful in the following discussions.

2. Schur–Cohn–Hermite transformation

In this transformation we will relate the root distribution relative to the unit circle with the root distribution relative to the imaginary axis. Let

$$\phi(z)=\sum_{i=0}^{n} a_i z^i$$

be a polynomial of n-degree with real or complex coefficients, of which we are interested in its zero distribution relative to the unit circle. By means of the bilinear transformation

$$z=\frac{s+1}{s-1} \quad \text{and} \quad s=\frac{z+1}{z-1} \tag{1}$$

we obtain the n-degree polynomial $f(s)=\sum_{i=0}^{n} b_i s^i$ from $\phi(z)$. We are interested in the zero distribution of $f(s)$ relative to the imaginary axis. The Möbius mapping of eqn. (1) leads to the following relationships (Duffin 1969) between the polynomials $f(s)$ and $\phi(z)$:

$$f(s)=2^{-n/2}(s-1)^n\phi\left(\frac{s+1}{s-1}\right) \tag{2}$$

$$f(-s)=(-1)^n 2^{-n/2}(s+1)^n\phi\left(\frac{s-1}{s+1}\right) \tag{3}$$

The relation between the coefficients of $\phi(z)$ and $f(s)$ is given by (Duffin 1969)

$$b'=a'\Gamma^{(n+1)} \tag{4}$$

where

$$b'=[b_0 b_1 \dots b_n] \tag{5}$$

$$a'=[a_0 a_1 \dots a_n] \tag{6}$$

and $\Gamma^{(n+1)}$ is an $(n+1)\times(n+1)$ matrix of arrays Γ_{ij} and is related to the Q matrix (Power 1967, Jury 1973) by the following relationship :

$$2^{n/2}\Gamma^{(n+1)}=R_{n+1}Q'R_{n+1} \tag{7}$$

where R_{n+1} is a matrix of $(n+1)$ rows and columns which when used as a premultiplier reverses the ordering of rows and when used as a postmultiplier reverses the ordering of columns of the multiplied matrix. Thus

$$R_{n+1}=\begin{bmatrix} 0 & 0 \ldots 0 & 1 \\ 0 & 0 \ldots 1 & 0 \\ \vdots & \iddots & \\ 0 & & \\ & 1 & \\ 1 & 0 \ldots \ldots & 0 \end{bmatrix} \tag{8}$$

The matrix Q' denotes the transpose of Q.

If we let S denote the Schur–Cohn matrix and H denotes the Hermite matrix, then following Fujiwara's (1926) transformation we can state the following definitions:

Definition of S: The entries s_{ij} of S are defined by

$$\frac{\phi(z)w^n\phi^*(w^{-1})-\phi(w)z^n\phi^*(z^{-1})}{z-w}=\sum_{i,j=1}^{n} z^{i-1}s_{ij}w^{n-j} \tag{9}$$

Definition of H: The entries h_{ij} are defined by

$$\frac{f(s)f^*(-t)-f(t)f^*(-s)}{s-t}=\sum_{i,j=1}^{n} s^{i-1}h_{ij}(-t)^{j-1} \tag{10}$$

Note that in both definitions the (*) denotes conjugation of the coefficients. Using eqns. (1), (2) and (3) and setting $w=(t+1)/(t-1)$ we obtain for the left-hand side of eqn. (9), the following:

$$\frac{\phi(z)w^n\phi^*(w^{-1})-\phi(w)z^n\phi^*(z^{-1})}{z-w}$$

$$=\frac{\phi\left(\dfrac{s+1}{s-1}\right)\left(\dfrac{t+1}{t-1}\right)^n\phi^*\left(\dfrac{t-1}{t+1}\right)-\phi\left(\dfrac{t+1}{t-1}\right)\left(\dfrac{s+1}{s-1}\right)^n\phi^*\left(\dfrac{s-1}{s+1}\right)}{\dfrac{s+1}{s-1}-\dfrac{t+1}{t-1}}$$

$$=\frac{(-1)^{n-1}2^{n-1}}{(s-1)^{n-1}(t-1)^{n-1}}\,\frac{f(s)f^*(-t)-f(t)f^*(-s)}{s-t} \tag{11}$$

Noting the right-hand side of eqn. (10), we obtain from eqn. (11) the following:

$$\frac{\phi(z)w^n\phi^*(w^{-1})-\phi(w)z^n\phi^*(z^{-1})}{z-w}=\frac{(-1)^{n-1}2^{n-1}}{(s-1)^{n-1}(t-1)^{n-1}}\sum_{k,l=1}^{n} s^{k-1}h_{kl}(-t)^{l-1} \tag{12}$$

The right-hand side of eqn. (9) yields :

$$\sum_{i,j=1}^{n} z^{i-1}s_{ij}w^{n-j}$$
$$=\sum_{i,j=1}^{n}\left(\frac{s+1}{s-1}\right)^{i-1}s_{ij}\left(\frac{t+1}{t-1}\right)^{n-j}$$
$$=\frac{1}{(s-1)^{n-1}(t-1)^{n-1}}\sum_{i,j=1}^{n}(s+1)^{i-1}(s-1)^{n-i}s_{ij}(t+1)^{n-j}(t-1)^{j-1} \tag{13}$$

Let $\Gamma^{(n)}$ be the $n\times n$ matrix of arrays Γ_{ij}, mapping coefficients of $(n-1)$-degree polynomials into one another analogously to eqn. (4). We also note that matrix Γ_{ij} is defined by (Duffin 1969)

$$2^{-n/2}(w+1)^{i-1}(w-1)^{n-i+1}=\sum_{j=1}^{n+1}\Gamma_{ij}{}^{(n+1)}w^{j-1},\quad i=1,2,\ldots,n+1 \tag{14}$$

Noting eqn. (14), we may write

$$2^{-(n-1)/2}(s-1)^{n-i}(s+1)^{i-1}=\sum_{k=1}^{n}\Gamma_{ik}{}^{(n)}s^{k-1},\quad i=1,2,\ldots,n \tag{15}$$

and also

$$2^{-(n-1)/2}(t+1)^{n-j}(t-1)^{j-1}=2^{-(n-1)/2}(-1)^{n-1}[(-t)-1]^{n-j}[(-t+1)]^{j-1}$$
$$=(-1)^{n-1}\sum_{l=1}^{n}\Gamma_{jl}{}^{(n)}(-t)^{l-1},\quad j=1,2,\ldots,n \tag{16}$$

Using eqns. (15) and (16) in the right-hand side of eqn. (13), we obtain the following :

$$\sum_{i,j=1}^{n} z^{j-1}s_{ij}w^{n-i}$$
$$=\frac{2^{n-1}(-1)^{n-1}}{(s-1)^{n-1}(t-1)^{n-1}}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}\Gamma_{ik}{}^{(n)}s^{k-1}\right)s_{ij}\left(\sum_{l=1}^{n}\Gamma_{jl}{}^{(n)}(-t)^{l-1}\right)$$
$$=\frac{2^{n-1}(-1)^{n-1}}{(s-1)^{n-1}(t-1)^{n-1}}\sum_{k,l=1}^{n}s^{k-1}\left[\sum_{i,j=1}^{n}\Gamma_{ik}{}^{(n)}s_{ij}\Gamma_{jl}{}^{(n)}\right](-t)^{l-1} \tag{17}$$

Comparing eqns. (12) and (17) and noting eqn. (9) we have

$$h_{kl}=\sum_{i,j=1}^{n}\Gamma_{ik}{}^{(n)}s_{ij}\Gamma_{jl}{}^{(n)} \tag{18}$$

or

$$H=\Gamma^{(n)\prime}S\Gamma^{(n)} \tag{19}$$

Noting the symmetry of the following expression :

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_i\Gamma_{ij}\Gamma_{jk}=a_k \tag{20}$$

and since the a_i's are arbitrary, it follows that $\Gamma^{(n)}$ is idempotent (Duffin 1969). Therefore eqn. (9) gives

$$S = \Gamma^{(n)\prime} H \Gamma^{(n} \tag{21}$$

Equations (19) and (21) establish the connection between H and S and vice versa. Clearly, if one of these matrices is positive definite so is the other.

The formulae of eqns. (19) and (21) show that any zero distribution property for $f(s)$ which is describable in terms of the signs of the eigenvalues of H gives rise to a parallel property for $\phi(z)$, describable in terms of the signs of the eigenvalues (Fujiwara 1926) of S. In Appendix 1, an example of the application of eqn. (21) is presented.

3. Reduced Schur–Cohn to reduced Hermite relationships

In the following discussion we will assume that the polynomials $f(s)$ and $\phi(z)$ are real. By utilizing the Schur–Cohn to Hermite transformation shown earlier, we will prove that the reduced Schur–Cohn (Anderson and Jury 1973) matrices $B>0$, $A>0$ (also defined below) correspond to the reduced Hermite (Anderson 1972) matrices $C>0$ and $D>0$ (also defined below) and vice versa. Such relationships enable us to represent an alternate simple proof of the reduced Schur–Cohn criterion.

Let $f(s) = \sum_{i=0}^{n} b_i s^i$; be the real polynomial whose zeros in the left half plane are under consideration. The matrix H corresponding to $f(s)$ can be written, following eqn. (10), as follows (Marden 1966):

$$\left.\begin{aligned} h_{ij} &= \sum_{k=1}^{i} b_{k-1} b_{i+j-k} (-1)^{k+i}, \quad && (i \leqslant j,\ i+j \text{ even}) \\ &= 0, && (i+j \text{ odd}) \end{aligned}\right\} \tag{22}$$

Define a matrix M by

$$M' = \left[\begin{array}{cccccc} 1 & 0 & 0 & \cdots\cdots & 0 & \cdots \\ 0 & 0 & 1 & 0 \cdots & 0 & \cdots \\ 0 & 0 & 0 & 0 \quad 1 \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \quad \vdots & \vdots & \\ \hline 0 & 1 & 0 & \cdots\cdots & 0 & \cdots \\ 0 & 0 & 0 & 1 \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}\right] \tag{23}$$

When $n=2m$ the top and bottom halves of M' have n rows each. When $n=2m+1$, the top part has $m+1$ rows, the bottom part m rows. Evidently, premultiplication of an $n \times n$ matrix by M' moves the odd-numbered rows to the first m (or $m+1$) rows, and the even-numbered rows to the last m rows. Post-multiplication by M produces the same operation on the columns. Thus

$$M'HM = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] \tag{24}$$

where D is an $m \times m$ matrix, and C either an $(m+1) \times (m+1)$ matrix when n is odd, or an $m \times m$ matrix when n is even. The matrices C and D are the reduced Hermite matrices (Anderson 1972).

Let $\phi(z) = \sum_{i=0}^{n} a_i z^i$ be a real polynomial whose roots inside the unit circle are to be established. Following eqn. (9), the elements of the Schur–Cohn matrix (Anderson and Jury 1973) are given by

$$s_{ij} = \sum_{p=1}^{\min(i,j)} a_{n-i+p} a_{n-j+p} - a_{i-p} a_{j-p}, \quad (i, j = 1, 2, \ldots, n) \tag{25}$$

If we define R_m as in eqn. (8) and I_m as the $m \times m$ identity matrix, it turns out (Anderson and Jury 1973) that we can write

$$S = \tfrac{1}{2} \begin{bmatrix} I_m & -I_m \\ R_m & R_m \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_m & R_m \\ -I_m & R_m \end{bmatrix}, \quad \text{for } n = 2m \tag{26}$$

and

$$S = \tfrac{1}{2} \begin{bmatrix} I_m & 0 & -I_m \\ 0 & 1 & 0 \\ R_m & 0 & R_m \end{bmatrix} \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \begin{bmatrix} I_m & 0 & R_m \\ 0 & 1 & 0 \\ -I_m & 0 & R_m \end{bmatrix}, \quad \text{for } n = 2m+1 \tag{27}$$

The matrices A and B are the reduced Schur–Cohn matrices (Anderson and Jury 1973). Now the Q matrix (Power 1967, Jury 1973) has certain symmetric and skew symmetric properties which imply via (7) certain related properties for the matrix $\Gamma^{(n)}$. Thus we can write for $\Gamma^{(n)}$ the following partitioned form :

$$2^{(n-1)/2} M' \Gamma^{(n)'} \triangleq \left[\begin{array}{c|c} -X & XR_m \\ \hline Y & YR_m \end{array}\right], \quad \text{for } n = 2m \tag{28}$$

The particular entries of X and Y are irrelevant here. Note though that since M and $\Gamma^{(n)}$ are non-singular, X and Y are also.

Now we are ready to show the connection between the reduced Schur–Cohn and the reduced Hermite matrices as follows :

(a) Even case, $n = 2m$

From eqns. (19) and (24), we can write

$$H = \Gamma^{(n)'} S \Gamma^{(n)} \tag{29}$$

and

$$M'HM = M' \Gamma^{(n)'} S \Gamma^{(n)} M \tag{30}$$

Noting eqns. (24), (26) and (30), we have

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = \tfrac{1}{2} \left[\begin{array}{c|c} -X & XR_m \\ \hline Y & YR_m \end{array}\right] \begin{bmatrix} I_m & -I_m \\ R_m & R_m \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I_m & R_m \\ -I_m & R_m \end{bmatrix} \begin{bmatrix} -X' & Y' \\ R_m X' & R_m Y' \end{bmatrix} \tag{31}$$

or

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = 2 \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & Y' \\ X' & 0 \end{bmatrix} \tag{32}$$

Equation (32) can also be written as

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = \begin{bmatrix} 2XBX' & 0 \\ 0 & 2YAY' \end{bmatrix} \tag{33}$$

Hence,

$$C > 0 \Leftrightarrow B > 0 \tag{34}$$

$$D > 0 \Leftrightarrow A > 0 \tag{35}$$

(*b*) Odd case, $n = 2m + 1$

Using the symmetry of Γ in rows and columns we have

$$M'\Gamma' \triangleq \left[\begin{array}{c|c} X_{m+1} & X_{m+1}\begin{bmatrix} R_m \\ 0 \end{bmatrix} \\ \hline [-Y_m \,\vdots\, 0] & Y_m R_m \end{array}\right] \tag{36}$$

and thus for $n = 2m + 1$

$$M'HM = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = M'\Gamma' S \Gamma M \tag{37}$$

Hence, C is an $(m+1) \times (m+1)$ matrix and D an $m \times m$ matrix. Using eqns. (27) and (36) in eqn. (37) we obtain

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = \left[\begin{array}{c|c} X_{m+1}\begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{2} \end{bmatrix} & 0 \\ \hline 0 & Y_m \end{array}\right] \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \left[\begin{array}{c|c} \begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{2} \end{bmatrix} X_{m+1}' & 0 \\ \hline 0 & Y_m' \end{array}\right] \tag{38}$$

The above equation can also be written as

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = \left[\begin{array}{c|c} X_{m+1}\begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{2} \end{bmatrix} A \begin{bmatrix} I_m & 0 \\ 0 & \frac{1}{2} \end{bmatrix} X_{m+1}' & 0 \\ \hline 0 & Y_m B Y_m' \end{array}\right] \tag{39}$$

The matrices X_{m+1} and Y_{m+1} are non-singular for similar reasons as X_m and Y_m and hence

$$C > 0 \Leftrightarrow A > 0 \tag{40}$$

$$D > 0 \Leftrightarrow B > 0 \tag{41}$$

The formulas (34), (35) and (40), (41) establish the connection between the reduced Schur–Cohn matrices and the reduced Hermite matrices. Therefore by proving one of the reduced Hermite or reduced Schur–Cohn criteria for stability†, the other follows directly. In Appendix 2, an illustrative example is presented.

4. Liénard–Chipart and the simplified determinantal criteria connections

Similarly to the reduced Hermite and reduced Schur–Cohn connections, we can also establish the connection between the Liénard–Chipart determinantal criterion with the corresponding simplified determinantal criterion for stability within the unit circle. This can be achieved by noting that the inner form of the Liénard–Chipart criterion can be obtained from the reduced Hermite criterion by matrix multiplication. Similarly, the inner form of the simplified determinantal criterion for the discrete case can also be obtained from the reduced Schur–Cohn criterion by matrix multiplication. By utilizing the result of § 3, we can readily obtain the connections as discussed below.

Letting n be even, the inner form of the Liénard–Chipart matrix is (Jury and Ahn 1971)

$$[\Delta_n]=\begin{bmatrix} a_{n-1} & & a_{n-3} & a_{n-5} & 0 & 0 \\ 0 & 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots\, 0 \\ \vdots & 0 & a_n & a_{n-2} & \cdots & a_0 \\ 0 & & & & & \\ a_n & a_{n-2} & & a_{n-4} & \cdots & 0 \end{bmatrix} \tag{42}$$

The innerwise matrix $[\Delta_n]$ of eqn. (42) is related to the reduced Hermite criterion matrix D as follows (Jury and Ahn 1971):

$$T_D[\Delta_n]=[\tilde{D}:0] \tag{43}$$

where the premultiplying T_D is given as

$$T_D=\begin{bmatrix} a_{n-2} & a_{n-4} & \cdots\cdots & -a_{n-3} & -a_{n-3} \\ a_{n-4} & a_{n-6} & \cdots\cdots & -a_{n-7} & -a_{n-5} \\ \vdots & & & & \vdots \\ a_2 & a_0 & 0\ldots & 0 & -a_1 \\ a_0 & 0 & 0\ldots & 0 & 0 \end{bmatrix} \tag{44}$$

and the matrix $\tilde{D}$ is related to D in eqns. (24) and (33) as follows:

$$M'\tilde{H}M=\left[\begin{array}{c|c} \tilde{C} & 0 \\ \hline 0 & \tilde{D} \end{array}\right]=\left[\begin{array}{c|c} R_mYAY'R_m & 0 \\ \hline 0 & R_mXBX'R_m \end{array}\right] \tag{45}$$

and $\tilde{H}$ is related to H in eqn. (24) as follows

$$\tilde{H}=R_nHR_n$$

† It may be noted that for stability we also require additional conditions on the coefficients of the polynomial (Anderson 1972, Anderson and Jury 1973).

From eqn. (45) it is evident that

$$\tilde{D}>0 \Leftrightarrow B>0 \tag{46}$$

The connection between the reduced Schur–Cohn matrix B and the innerwise matrix for the simplified determinantal criterion can be readily verified similarly to eqn. (43) as follows:

$$P_B[\Delta_n^-]=[0:B] \tag{47}$$

where the premultiplying matrix P_B of dimension $[n/2]\times n$ is given as

$$P_B=\begin{bmatrix} & & & a_0 & a_n & & \\ \bigcirc & & a_0 & a_1 & a_{n-1} & a_n & \bigcirc \\ & \cdot^{\cdot^{\cdot}} & & & & & \ddots \\ a_0 & & & & & & a_n \end{bmatrix} \tag{48}$$

and the innerwise matrix $[\Delta_n^-]$ is given by (Jury 1971)

$$[\Delta_n^-]=X_n-Y_n$$

where

$$X_n=\begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 \\ & \ddots & & a_2 \\ & & \ddots & \vdots \\ \bigcirc & & & a_n \end{bmatrix}, \quad Y_n=\begin{bmatrix} \bigcirc & & & a_0 \\ & & \cdot^{\cdot^{\cdot}} & \vdots \\ & \cdot^{\cdot^{\cdot}} & & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix} \tag{49}$$

It can be readily verified from eqn. (47) that

$$B_{\text{p.d.}} \Leftrightarrow [\Delta_n^-]_{\text{p.i.}} \tag{50}$$

where $_{\text{p.d.}}$ indicates positive definite and p.i. indicates positive innerwise. Equations (43) and (44) can be also written as

$$\tilde{D}=[\tilde{D}:0]\begin{bmatrix} I_m \\ 0_m \end{bmatrix}, \quad B=[0:B]\begin{bmatrix} 0_m \\ I_m \end{bmatrix} \tag{51}$$

Noting eqns. (43), (45) and (47) we finally obtain the following relationship which connects $[\Delta_n]$ with $[\Delta_n^-]$:

$$T_D[\Delta_n]\begin{bmatrix} I_m \\ 0_m \end{bmatrix}=R_m X P_B[\Delta_n^-]\begin{bmatrix} 0_m \\ I_m \end{bmatrix} X' R_m \tag{52}$$

A similar matrix relationship can also be obtained for n odd. It is of interest to note that although eqn. (52) relates the Liénard–Chipart innerwise matrix $[\Delta_n]$ generated from the coefficients of $f(s)$ to the innerwise matrix for the simplified determinantal criterion $[\Delta_n^-]$, generated from the coefficients $\phi(z)$, it does not allow one to find $[\Delta]$ from $[\Delta_n^-]$ (or vice versa), in view of the

$\begin{bmatrix} I_m \\ 0_m \end{bmatrix}$ matrix multiplication on each side of the equation. Furthermore, from eqn. (52) it is evident that $[\Delta_n]_{\text{p.i.}} \Leftrightarrow [\Delta_n^-]_{\text{p.i.}}$

5. Hermite and Schur–Cohn innerwise matrix relationship

Similarly to the discussion of the preceding section, we can also obtain the innerwise matrix relationship between the Hermite and Schur–Cohn matrices. This relationship is obtained by using the results shown in § 2 for the symmetric forms.

The innerwise matrix $[\Delta_{2n}]$ is related to the Hermite symmetric matrix H as follows :

$$V[\Delta_{2n}] = [H' : 0] \tag{53}$$

where the premultiplying matrix V of dimension $[n \times 2n]$ is given as follows :

$$V = \begin{bmatrix} \cdots & a_{n-2} & -a_{n-1} & a_n & (-1)^{n+1}\bar{a}_n & (-1)^{n+2}\bar{a}_{n-1} \cdots \\ \cdots & -a_{n-1} & a_n & 0 & 0 & (-1)^{n+2}\bar{a}_n \ \cdots \\ \cdots & a_n & 0 & 0 & 0 & 0 \ \ \cdots \\ & \vdots & & \vdots & \vdots & \vdots \end{bmatrix} \tag{54}$$

Equation (53) can be readily verified using a similar matrix multiplication procedure indicated by Parks (1967). The matrix $[\Delta_{2n}]$ presented below represents rearrangement of the Bilharz $[2n \times 2n]$ matrix in terms of complex elements (Parks 1967, Marden 1966).

$$[\Delta_{2n}] = \begin{bmatrix} \bar{a}_0 & -a_1 & a_2 & -a_3 & \cdots & (-1)^n \bar{a}_n & \cdots & 0 \\ 0 & & & & & & & \\ \vdots & -\bar{a}_0 & \bar{a}_1 & -\bar{a}_2 & \bar{a}_3 & \cdots & & \\ \vdots & 0 & \bar{a}_0 & -\bar{a}_1 & \bar{a}_2 & \cdots & & \\ \vdots & 0 & a_0 & a_1 & a_2 & \cdots & & \\ \vdots & a_0 & a_1 & a_2 & a_3 & \cdots & & \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{n-1} & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & \cdots & a_{n-1} & \cdots & 0 \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_n & \cdots & 0 \end{bmatrix} \tag{55}$$

The matrix H' is the transpose of H in eqn. (10). From eqn. (53) it is noted that $[\Delta_{2n}]_{\text{p.i.}} \Leftrightarrow H'_{\text{p.d.}}$

Similarly the connection between the innerwise matrix $[\Delta]$ of the Schur–Cohn matrix and the symmetric form S is given by

$$\tilde{S} = Q[\Delta] \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{56}$$

where.

$$S = \bar{S}' \tag{57}$$

The innerwise matrix $[\Delta]$, as well as the premultiplying matrix are given respectively in eqns. (59) and (60) of Jury and Ahn (1972).

From eqns. (19) and (53) we can write

$$H = \left(V[\Delta_{2n}] \begin{bmatrix} I \\ 0 \end{bmatrix} \right)' = \Gamma^{(n)'} S \Gamma^{(n)} \tag{58}$$

Taking the transpose of eqn. (58) and noting eqns. (56) and (57), we finally establish the following relationship :

$$V[\Delta_{2n}] \begin{bmatrix} I \\ 0 \end{bmatrix} = \Gamma^{(n)'} Q[\Delta] \begin{bmatrix} I \\ 0 \end{bmatrix} \Gamma^{(n)} \tag{59}$$

The same remark indicated for eqn. (47) also applies to the above equation. Also note that $[\Delta_{2n}]$ is generated from the complex polynomial $f(s)$ and $[\Delta]$ is generated from the complex polynomial $\phi(z)$. Furthermore, $[\Delta_{2n}]_{p.i.} \Leftrightarrow [\Delta]_{p.i.}$

6. Conclusion

In this paper four types of matrix transformations are derived. The first transformation relates the Hermite $n \times n$ symmetric matrix to the $n \times n$ Schur–Cohn symmetric matrix. The importance of this transformation is to relate the zero distribution properties of two polynomials related by the bilinear transformation. The second transformation relates the reduced Hermite matrix of dimension $(n/2) \times (n/2)$ to the reduced Schur–Cohn matrix of the same dimension. The importance of this transformation is to obtain a simplified proof of the reduced Schur–Cohn criterion for stability within the unit circle. The third transformation relates the innerwise form of the Liénard–Chipart matrix of dimension $n \times n$ to the innerwise stability matrix of the same dimension for the discrete case. Finally, the fourth transformation relates the innerwise Hermite matrix for complex coefficients of dimension $2n \times 2n$ to the innerwise Schur–Cohn matrix of the same dimension. The last two transformations are of importance in studying the stability and zero distribution properties of polynomials.

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Appendix 1

To explain the various relationships discussed in § 2, we present the following example for $n = 4$.

Let,

$$\phi(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \tag{A 1}$$

From eqn. (2), we have

$$f(s)=2^{-2}\phi\left(\frac{s+1}{s-1}\right)(s-1)^4 \tag{A 2}$$

From eqn. (3), we have

$$[b_0b_1b_2b_3b_4]=[a_0a_1a_2a_3a_4]\Gamma^{(5)} \tag{A 3}$$

The matrix $\Gamma^{(5)}$ is obtained from eqn. (3) and the Q matrix (Jury 1973) to give :

$$\Gamma^{(5)}=2^{-2}[R_5Q'R_5]$$

$$=2^{-2}\begin{bmatrix} & & & & 1\\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & & 0\\ 1 & & & & \end{bmatrix}\begin{bmatrix} 1 & 4 & 6 & 4 & 1\\ 1 & 2 & 0 & -2 & -1\\ 1 & 0 & -2 & 0 & 1\\ 1 & -2 & 0 & 2 & -1\\ 1 & -4 & 6 & -4 & 1\end{bmatrix}\begin{bmatrix} & & & & 1\\ 0 & & & 1 & \\ & & 1 & & \\ & 1 & & & 0\\ 1 & & & & \end{bmatrix}$$

$$=2^{-2}\left[\begin{array}{ccc|cc} 1 & -4 & 6 & -4 & 1\\ -1 & 2 & 0 & -2 & 1\\ 1 & 0 & -2 & 0 & 1\\ \hline -1 & -2 & 0 & 2 & 1\\ 1 & 4 & 6 & 4 & 1\end{array}\right] \tag{A 4}$$

Furthermore the b_i's are obtained from (A 3) and (A 4) to give :

$$\left.\begin{aligned} b_0&=a_0-a_1+a_2-a_3+a_4\\ b_1&=-4a_0+2a_1-2a_3+4a_4\\ b_2&=6a_0-2a_2+6a_4\\ b_3&=-4a_0-2a_1+2a_3+4a_4\\ b_4&=a_0+a_1+a_2+a_3+a_4\end{aligned}\right\} \tag{A 4a}$$

For the polynomial $f(s)=\sum_{i=0}^{4} b_is^i$ of eqn. (A 2), we have from eqn. (22) the following :

$$H=2\begin{bmatrix} b_0b_1 & 0 & b_0b_3 & 0\\ 0 & -b_0b_3+b_1b_2 & 0 & b_1b_4\\ b_0b_3 & 0 & -b_1b_4+b_2b_3 & 0\\ 0 & b_1b_4 & 0 & b_3b_4\end{bmatrix} \tag{A 5}$$

Now to obtain S using eqn. (21) we have to obtain first $\Gamma^{(4)}$. This matrix is obtained similarly to $\Gamma^{(5)}$ except in this case the Q matrix required is as follows (Jury 1973) :

$$Q=\begin{bmatrix} 1 & 1 & 1 & 1\\ 3 & 1 & -1 & -3\\ 3 & -1 & -1 & 3\\ 1 & -1 & 1 & -1\end{bmatrix} \tag{A 6}$$

The corresponding $\Gamma^{(4)}$ yields

$$\Gamma^{(4)} = 2^{-3/2}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 3 & 3 & 1 \end{bmatrix} \tag{A 7}$$

Finally substituting eqns. (A 5) and (A 7) in eqn. (21), we obtain

$$S = \Gamma^{(4)\prime} H \Gamma^{(4)}$$

$$= \begin{bmatrix} a_4^2 - a_0^2 & a_3a_4 - a_1a_0 & a_2a_4 - a_2a_0 & a_1a_4 - a_3a_0 \\ a_3a_4 - a_1a_0 & a_4^2 + a_3^2 - a_1^2 - a_0^2 & \begin{matrix} a_3a_4 + a_2a_3 \\ - a_2a_1 - a_1a_0 \end{matrix} & a_2a_4 - a_2a_0 \\ a_2a_4 - a_2a_0 & \begin{matrix} a_3a_4 + a_2a_3 \\ - a_2a_1 - a_1a_0 \end{matrix} & a_4^2 + a_3^2 - a_1^2 - a_0^2 & a_3a_4 - a_1a_0 \\ a_1a_4 - a_3a_0 & a_2a_4 - a_2a_0 & a_3a_4 - a_1a_0 & a_4^2 - a_0^2 \end{bmatrix} \tag{A 8}$$

Noting eqn. (25), the above matrix is the Schur–Cohn symmetric matrix S *corresponding* to the fourth-degree polynomial $\phi(z)$.

Appendix 2

In this appendix we will illustrate the reduced Hermite to the reduced Schur–Cohn *connection for* $n = 4$. Let $f(s)$ be given :

$$f(s) = \sum_{i=0}^{4} b_i s^i \tag{A 9}$$

From eqns. (24) and (33), we can write :

$$M'HM = \left[\begin{array}{c|c} C & 0 \\ \hline 0 & D \end{array}\right] = 2\left[\begin{array}{c|c} XBX' & 0 \\ \hline 0 & YAY' \end{array}\right] \tag{A 10}$$

The matrix M' for this case is obtained from eqn. (23), as follows :

$$M' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.11}$$

The matrix $\Gamma^{(4)}$ is given in eqn. (A.7). Hence,

$$M'\Gamma^{(4)\prime} = 2^{-3/2}\left[\begin{array}{cc|cc} -1 & 1 & -1 & 1 \\ -3 & -1 & 1 & 3 \\ \hline 3 & -1 & -1 & 3 \\ 1 & 1 & 1 & 1 \end{array}\right] \tag{A.12}$$

From eqn. (28) we have,

$$-X = \begin{bmatrix} -1 & 1 \\ -3 & -1 \end{bmatrix} \tag{A.13}$$

or

$$X=\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \tag{A 14}$$

Now we can check $C=2XBX'$. For $\phi(z)=\sum_{i=0}^{4} a_i z^i$ we have from eqn. (3) of (Anderson and Jury 1973) the following

$$B=\begin{bmatrix} a_4^2-a_4a_1+a_0a_3-a_0^2 & a_4a_3-a_4a_2+a_0a_3-a_0a_2 \\ a_4a_3-a_4a_2+a_0a_2-a_0a_1 & a_4^2-a_4a_3+a_3^2-a_3a_0-a_1^2+a_1a_2-a_0^2+a_0a_1 \end{bmatrix} \tag{A 15}$$

Using eqns. (A 15) and (A 14) we obtain

$$2XBX'=2\begin{bmatrix} b_0b_1 & b_0b_3 \\ b_0b_3 & -b_1b_4+b_2b_3 \end{bmatrix} \tag{A 16}$$

where the b_i's are those given in eqn. (A 4*a*). Noting Anderson (1972), the above is equivalent to $2C$. Hence,

$$B>0 \Leftrightarrow C>0 \tag{A 17}$$

Similarly we can verify from eqn. (33) that

$$D=2YAY'$$

where Y is obtained from eqns. (A 12) and (A 28) as follows:

$$Y=\begin{bmatrix} 3 & -1 \\ -3 & -1 \end{bmatrix}$$

and A is obtained from eqn. (2) of Anderson and Jury (1973) for $\phi(z)=\sum_{i=0}^{4} a_i z^i$.

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