Recursive Algorithm for Spectral Factorization

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Abstract—**This paper describes a recursive computational algorithm for computing spectral factors of continuous-time and discrete-time power spectrum matrices. The matrices need not be positive definite. Convergence rates of the recursive equation are studied.**

I. Introduction

A STANDARD problem of system theory is that of spectral factorization; in its commonest form, one is given a matrix $\Psi(s)$ of real rational functions of the complex variable s, with $\Psi(s)$ parahermitian, i.e., $\Psi'(-s) = \Psi(s)$, and with $\Psi(j\omega) \geq 0$ for all real ω. Normally (and we shall assume this henceforth), all entries of $\Psi(s)$ are analytic for all $s = j\omega$, ω real; if not, $\Psi(j\omega) \geq 0$ is required to hold almost everywhere. One is required to find a spectral factor $W(s)$, that is, a matrix $W(s)$ satisfying $W'(-s)W(s) = \Psi(s)$. Usually $W(s)$ is required to be real rational and to have entries analytic in Re $[s] \geq 0$; frequently too, $W(s)$ is required to have constant rank in Re $[s] > 0$ equal to the rank (almost everywhere) of $\Psi(s)$. Such a spectral factor is then termed minimum phase.

The above spectral factorization problem arises in some continuous-time optimization and filtering problems. The corresponding discrete-time problems generate another spectral factorization problem, as follows. One is given a matrix $\Psi(z)$ of real rational functions of a complex variable z, with $\Psi'(z^{-1}) = \Psi(z)$ and $\Psi(e^{j\omega}) \geq 0$ for all real ω. One is required to find a $W(z)$ such that $W'(z^{-1})W(z) = \Psi(z)$, with $W(z)$ real and rational, of entries analytic in $|z| \geq 1$, and possibly with $W(z)$ of constant rank in $|z| > 1$, equal to the rank almost everywhere of $\Psi(z)$. The discrete-time and continuous-time problems are roughly equivalent under the bilinear transformation $s = (z - 1)(z + 1)^{-1}$, as is reasonably well known.

Spectral factorization problems also occur in network synthesis. Here, there arise matrices of rational functions of s satisfying a property called positive realness, and one can associate with a positive real matrix spectral factors, knowledge of which can effectively solve certain synthesis problems. The nature of the association will be given subsequently. This paper represents another approach to tackling the various spectral factorization problems stated previously.

Techniques for spectral factorization are legion. The majority, including those of [1]–[8], rely on frequency domain manipulations in which the problem of factoring a matrix of real rational functions is reduced to the problem of factoring a scalar even polynomial $p(s)$ as $w(-s)w(s)$, or a self-inversive polynomial $p(z)$ as $z^n w(z^{-1})w(z)$, with $w(\cdot)$ polynomial in each case. The actual polynomial factorization is not usually discussed; of course, this can be a very real computational problem.

References [9]–[14] discuss various other approaches. Reference [9] shows that the continuous-time problem can be solved if $\Phi(\infty)$ is nonsingular, provided a quadratic matrix equation can be solved. Reference [10] shows that this quadratic matrix equation has one solution which is the limiting solution of a Riccati differential equation, and moreover, from this solution, a minimum phase spectral factor can be obtained.

In another direction, Bauer [11] tackled the polynomial factorization problem. After making the preliminary observation that the task of factorizing an even polynomial $p(s)$ as $w(-s)w(s)$ is equivalent to the task of factorizing a self-inversive $p(z)$ as $z^n w(z^{-1})w(z)$, he gave an iterative procedure for determining the coefficients of $w(z)$ (actually by performing a Cholesky decomposition of an infinite banded matrix). A straightforward generalization to the factoring of a matrix $\sum_{i=-m}^{m} P_i z^i$ which is positive definite for $|z| = 1$ and for which $P_i = P_i'$ is carried out in [12]. Actually, it is also done in [13], but the relationship with [11] in this case is not obvious. [Note that what is being done in [12] and [13] is not a factorization of an arbitrary spectrum matrix, but rather the z-plane equivalent of the factorization of a matrix of even polynomials in s, nonnegative for $s = j\omega$]. The connection between [11], [12], and [13] will be described in more detail in Section V of this paper, where the procedures of the three references are shown to be special cases of the procedure of this paper.

Finally, we mention the work of [14] in which an interesting Newton–Raphson technique can be used to solve the factorization problem. It seems to have no direct link with any of the other methods.

A recent doctoral thesis [15] also contains a number of the results of this paper. In particular, there is use of a Riccati difference equation to achieve a spectral factorization. The discussion however is not as complete; for example, the possibility of a singular power spectrum matrix is not raised, and there is no study of the convergence rate of the equation. This thesis also references the earlier doctoral thesis of one of the authors [16] where results for nonsingular power spectra were obtained.

Having spelled out in rough terms the nature of the earlier contributions, we can now make precise the contribution of our paper. First, we are basically concerned with the z-plane factorization; however, by giving a state-space interpretation to the bilinear mapping between the s and z

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planes, an elegant procedure evolves for tackling an s-plane problem via the z plane; such a phenomenon has also been observed in linear-quadratic regulator problems, and yields a competitive procedure for finding the limiting solution of the Riccati differential equation arising in the regulator problem [17]. The second point is that the method is not restricted to factoring polynomials or matrices of polynomials. Third, the method is recursive in nature, most like that of [11]–[13] in fact, involving the recursive solution of a Riccati difference equation. Fourth, there is no restriction requiring positive definiteness of the spectral matrix, either for special values of the variable, or for all values. This represents an advance over the methods of [9]–[13].

A comparison between the method of [10] and that of this paper is then interesting to make. First, that of [10] requires $\Psi(s)$ to be nonsingular almost everywhere, and in particular, $\Psi^{-1}(\infty)$ actually appears in the Riccati differential equation, and spectral factorization proceeds by finding the limiting solution of the differential equation. The method here replaces the Riccati differential equation by a Riccati difference equation, and provides a technique where none was directly possible in case $\Psi(\infty)$ is singular. The limiting solutions of both equations are the same.

We now outline the layout of the paper. In Section II, definitions and background results are stated. The spectral factorization problem is posed in terms of positive real matrices rather than spectral matrices. In Section III, the full extent of the s-plane to z-plane transformation is explored; principally, the interesting ideas are those associated with state-space descriptions of matrices.

Section IV contains the main meat of the paper. A quadratic variational problem is posed, the solution of which solves the spectral factorization problem. In this section, the exact iterative equation solving the spectral factorization problem is stated, and convergence properties are examined.

In Section V, factorization of a spectrum matrix is discussed, and the relation of the method of this paper to those of [11]–[13] is discussed. These methods prove to be special cases of the method of this paper. Section VI contains concluding remarks.

II. Definitions and Background Results

In this section, we recall the concepts of positive real matrices, the Positive Real Lemma, spectral factors, and minimum phase spectral factors.

Definition 1_c [10]

An $m \times m$ matrix $Z_c(s)$ of real rational functions of the complex variable s is termed *positive real* if in $\mathrm{Re}\,[s] > 0$, elements of $Z_c(s)$ are analytic, and $Z_c(s) + Z_c'(s^*)$ is nonnegative definite.

Definition 1_d [18]

An $m \times m$ matrix $Z_d(z)$ of real rational functions of the complex variable z is termed *discrete positive real* if in $|z| > 1$, elements of $Z_d(z)$ are analytic, and $Z_d(z) + Z_d'(z^*)$ is nonnegative definite.

In case $Z_c(\infty)$ has finite entries, it is well known that there exist matrix quadruples $\{F,G,H,J\}$ such that

$$Z_c(s) = J + H'(sI - F)^{-1}G. \tag{1}$$

Further, one can assume that F is of minimal dimension, so that $[F,G]$ and $[F,H]$ are, respectively, completely controllable and completely observable. The following is now a standard result characterizing the positive real property in terms of $\{F,G,H,J\}$.

The Positive Real Lemma [5, 10]

Let $Z_c(s)$ be an $m \times m$ matrix of real rational functions of the complex variable s with $Z_c(\infty)$ finite and with minimal realization $\{F,G,H,J\}$. Then $Z_c(s)$ is positive real if and only if there exists a symmetric negative definite X such that

$$M_c = \begin{bmatrix} XF + F'X & XG + H \\ (XG + H)' & J + J' \end{bmatrix} \geq 0. \tag{2}$$

The corresponding result for discrete positive real matrices is less well known, although versions have appeared (see [18], [19]).

Discrete Positive Real Lemma

Let $Z_d(z)$ be an $m \times m$ matrix of real rational functions of the complex variable z with $Z_d(\infty)$ finite and with minimal realization $\{A,B,C,D\}$. Then $Z_d(z)$ is discrete positive real if and only if there exists a symmetric negative definite Y such that

$$M_d = \begin{bmatrix} A'YA - Y & A'YB + C \\ (A'YB + C)' & B'YB + D + D' \end{bmatrix} \geq 0. \tag{3}$$

Definition 2_c

A *spectral factor* $W_c(s)$ associated with a positive real $Z_c(s)$ is a matrix such that $Z_c'(-s) + Z_c(s) = W_c'(-s)W_c(s)$; it is termed *minimum phase* if $W_c(s)$ is $r \times m$ where rank $[Z_c'(-s) + Z_c(s)] = r$ almost everywhere, if $W_c(s)$ has entries analytic in $\mathrm{Re}\,[s] \geq 0$, and if $W_c(s)$ has rank r everywhere in $\mathrm{Re}\,[s] > 0$.

Definition 2_d

A spectral factor $W_d(z)$ associated with a discrete positive real $Z_d(z)$ is a matrix such that $Z_d'(z^{-1}) + Z_d(z) = W_d'(z^{-1})W_d(z)$; it is termed *minimum phase* if $W_d(z)$ is $r \times m$ where rank $[Z_d'(z^{-1}) + Z_d(z)] = r$ almost everywhere, if $W_d(z)$ has entries analytic in $|z| \geq 1$, including infinity, and $W_d(z)$ has rank r everywhere in $|z| > 1$, including infinity.

As shown in [2], minimum phase spectral factors are unique to within left multiplication by an arbitrary real constant orthogonal matrix, and are real rational as a result of $Z_c(s)$ and $Z_d(z)$ being real rational.

If matrices X and Y are known which yield (2) or (3), spectral factors can be found. In case $M_c \geq 0$, one can factor M_c (in many ways) as

$$M_c = \begin{bmatrix} L \\ W_0' \end{bmatrix} \begin{bmatrix} L' & W_0 \end{bmatrix} \tag{4}$$

(where the partitioning conforms with that of M_c). Then, $W_0 + L'(sI - F)^{-1}G$ is a spectral factor $W_c(s)$ associated with $Z_c(s)$ [10]. Similarly, if

$$M_d = \begin{bmatrix} K \\ V_0' \end{bmatrix} [K' \quad V_0] \tag{5}$$

then $V_0 + K'(zI - A)^{-1}B$ is a spectral factor $W_d(z)$ associated with $Z_d(z)$.

Minimum phase $W_c(s)$ are associated with the maximum X satisfying (2). (That the set of matrices X satisfying (2) has a maximum and, actually, a minimum, is nontrivial.) In case M_c is defined by the maximum X, a minimum phase $W_c(s)$ is defined by factoring M_c as in (4), with $[L' \; W_0]$ possessing the minimum number of rows, viz. M_c. These facts are established in various places, e.g., [10], [20]–[22]. (Actually, [21] and [22] require $J + J'$ nonsingular, but a limiting argument as contained in [20] can be used easily to remove this restriction.) The analogous discrete result does not appear to have been formally stated, but, as we show in the next section, it is an easy consequence of the continuous result.

III. Connections Between Discrete-Time and Continuous-Time Results

The main goal of this paper is to provide a new computational procedure for finding minimum phase spectral factors of positive real $Z_c(s)$. This will be done by solving an equivalent discrete-time problem, and by using the connection between minimum phase spectral factors and the matrices X and Y solving the two Positive Real Lemma inequalities. In this section, we set up a number of connections between discrete- and continuous-time results.

Theorem 1

Consider the bilinear transformation $s = [\alpha(z - 1)]/(z + 1)$ for a positive constant α. Then the following statements hold.

1) A positive real $Z_c(s)$ with $Z_c(\infty) < \infty$ transforms into a discrete positive real $Z_d(z)$ with $Z_d(-1) < \infty$.

2) The matrices $\{F,G,H,J\}$ of a state-space realization of $Z_c(s)$ transform into matrices $\{A,B,C,D\}$ of a state-space realization of $Z_d(z)$ with

$$\begin{aligned} A &= (\alpha I - F)^{-1}(\alpha I + F) \\ B &= \sqrt{2\alpha}\,(\alpha I - F)^{-1}G \\ C &= \sqrt{2\alpha}\,(\alpha I - F')^{-1}H \\ D &= J + H'(\alpha I - F)^{-1}G \end{aligned} \tag{6}$$

and $\{A,B,C,D\}$ is minimal if $\{F,G,H,J\}$ is minimal.

3) Spectral factors $W_c(s)$ associated with $Z_c(s)$ transform into spectral factors $W_d(z)$ associated with $Z_d(z)$, and $W_d(z)$ is minimum phase if $W_c(s)$ is minimum phase.

4) Given the relation (6) between $\{F,G,H,J\}$ and $\{A,B,C,D\}$, the matrices M_c of (2) and M_d of (3) are related by

$$X = Y; \quad \begin{bmatrix} \frac{1}{\sqrt{2\alpha}}(A' + I) & 0 \\ \frac{1}{\sqrt{2\alpha}}B' & I \end{bmatrix} M_c \cdot \begin{bmatrix} \frac{1}{\sqrt{2\alpha}}(A + I) & \frac{1}{\sqrt{2\alpha}}B \\ 0 & I \end{bmatrix} = M_d. \tag{7}$$

5) If $W_c(s) = W_0 + L'(sI - F)^{-1}G$ is a spectral factor associated with $Z_c(s)$ defined by the factorization (4) of M_c, the associated $W_d(z)$ is defined by $V_0 + K'(zI - A)^{-1}B$ where $K = \sqrt{2\alpha}\,(\alpha I - F)^{-1}L$, $V_0 = W_0 + L'(\alpha I - F)^{-1}G$, and K and V_0 satisfy the factorization (5) of M_d, with M_d and M_c related as in (7).

6) The bilinear transformation is invertible, and all obvious converse statements hold.

The theorem is straightforward to prove by direct calculation, and a full proof will not be attempted here. The proof may essentially be found in [6] (see Appendix J) and [18]. The frequency domain content of the theorem is certainly not novel; the bilinear transformation maps the half plane Re $[s] > 0$ into the exterior $|z| > 1$ of the unit circle, and this accounts for claims 1) and 3). Claim 2) follows by manipulating the expression $J + H'[\alpha(z - 1)\cdot(z + 1)^{-1} - F]^{-1}G$ into the form $D + C'(zI - A)^{-1}B$. Claim 4) follows by direct calculation; the important part of this claim is the fact that the same matrices X or Y solve the linear matrix inequalities of the two positive real lemmas. Claim 5) constitutes a refinement of claim 3). An allied theorem is used to tackle the standard linear-quadratic regulator problem in [17].

For the purposes of this paper, the most important consequences of the theorem are those associated with minimum phase spectral factors. Since a minimum phase $W_c(s)$ is determined by the maximum $X = Y$ satisfying $M_c \geq 0$, it follows from (7) that this Y is also the maximum Y satisfying $M_d \geq 0$ [and, by claims 3) and 5), that this maximum Y also determines a minimum phase $W_d(z)$].

In case $(J + J')$ is nonsingular, the maximum X can be found as the limiting solution of a Riccati differential equation [10], [20]. In case $(J + J')$ is singular, a Riccati equation can still be used in determining X but the situation is a good deal more complicated, requiring either a computationally awkward limiting operation [20] or a number of preliminary coordinate basis changes [10]. In the next section, we show that the maximum X can be found as the limiting solution of a Riccati difference equation irrespective of the singularity or otherwise of $J + J'$.

IV. A Discrete-Time Linear-Quadratic Variational Problem

We start in this section with a discrete positive real $Z_d(z)$ with minimal realization $\{A,B,C,D\}$. We shall set up a discrete-time linear-quadratic variational problem, and show that in solving this problem we obtain the maximum X satisfying the inequality $M_d \geq 0$ via a convenient algorithm.

The Variational Problem

Consider the following minimization problem: for the system

$$x(i+1) = Ax(i) + Bu(i); \qquad x(0) = x_0 \text{ known} \quad (8)$$

minimize for a prescribed symmetric matrix Q and any integer $n \geq 1$ the performance index

$$I[x_0,u(\cdot),Q,0,n] = \sum_{i=0}^{n-1} [u'(i)(D + D')u(i) + 2x'(i)Cu(i)] + x'(n)Qx(n). \quad (9)$$

The analogous continuous-time problem has been studied in [10], [20], and [23]. By analogy with continuous-time results, it is easy to establish the following property.

Lemma 1

Assuming $[A,B]$ is completely controllable and that $Q = 0$, a necessary and sufficient condition for the existence of a lower bound on I, for all x_0, $u(\cdot)$ and n, is that $D + C'(zI - A)^{-1}B$ be discrete positive real.

We shall omit the proof of this result. The minimum in (9) can be found as described in the following lemma; again the proof will be omitted, as it follows by a dynamic programming argument that is standard for quadratic loss problems.

Lemma 2

In case $Q = 0$, the minimum value of I is $x_0'\phi(n)x_0$, where $\phi(\cdot)$ is determined recursively by

$$\phi(i+1) = A'\phi(i)A - [A'\phi(i)B + C] \cdot [B'\phi(i)B + D + D']^{\#}[A'\phi(i)B + C]' \quad (10)$$

initialized by $\phi(0) = 0$. Here, $Z^{\#}$ denotes the Moore–Penrose pseudo-inverse of Z. Further, $B'\phi(i)B + D + D' \geq 0$ and $N[B'\phi(i)B + D + D'] \subset N[A'\phi(i)B + C]$, where $N[Z]$ denotes the nullspace of Z. In case $Q \neq 0$, the minimum value of I is still given by $x_0'\phi(n)x_0$ provided that, now, the initialization is $\phi(0) = Q$, and provided that for $i = 0, 1, \cdots, n-1$ one has $B'\phi(i)B + D + D' \geq 0$ and $N[B'\phi(i)B + D + D'] \subset N[A'\phi(i)B + C]$.[1] If either condition fails, the minimum is minus infinity.

In the corresponding continuous-time problem with $Q = 0$, it is shown that the optimum performance index is monotone decreasing with the length of the optimization interval. An analogous argument here shows that $\phi(n)$ is monotone decreasing with n. Lemma 1 guarantees that the sequence $\phi(n)$ is bounded below, and so

$$\lim_{n\to\infty} \phi(n) \triangleq \Phi$$

exists. It also follows by analogy with the corresponding continuous-time argument that the nullspace of Φ coincides with the set of unobservable states, i.e., those ω for which $C'A^i\omega = 0$ for all i. With $[A,C]$ completely observable then, Φ is nonsingular.

Let us now establish some other properties of Φ.

Property 1

Φ satisfies $M_d \geq 0$, in the sense that $M_d \geq 0$ holds with Y replaced by Φ.

Proof: If a symmetric matrix P has the form

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix}$$

then $P \geq 0$ if and only if $P_3 \geq 0$, $P_1 - P_2P_3^{\#}P_2' \geq 0$ and $N[P_3] \subset N[P_2]$. (Appendix I contains a proof of this moderatively well-known result). Now take

$$P(i) = \begin{bmatrix} A'\phi(i)A - \phi(i+1) & A'\phi(i)B + C \\ [A'\phi(i)B + C]' & B'\phi(i)B + D + D' \end{bmatrix}.$$

Using Lemma 2, we see that $P(i) \geq 0$. Take the limit as $i \to \infty$ to recover $\lim_{i\to\infty} P(i) = M_d \geq 0$ with Y replaced by Φ.

Property 2

For any nonpositive definite Y satisfying[2] $M_d \geq 0$, one has $\Phi \geq Y$, i.e., Φ is the maximum of the matrices satisfying $M_d \geq 0$.

Proof: First observe that if Y satisfies $M_d \geq 0$, one has

$$A'YA - Y - (A'YB + C)(B'YB + D + D')^{\#} \cdot (A'YB + C)' \geq 0$$

or

$$Y \leq A'YA - (A'YB + C)(B'YB + D + D')^{\#} \cdot (A'YB + C)'. \quad (11)$$

Now consider the problem of minimizing for (8) the index

$$I[x(n-i), u(\cdot), Y, n-i, n] = \sum_{j=n-i}^{n-1} [u'(j)(D + D')u(j) + 2x'(j)Cu(j)] + x'(n)Yx(n).$$

The minimum, if it exists, is $x'(n-i)\phi_Y(i)x(n-i)$, where ϕ_Y satisfies the recursive equation (10), with $\phi_Y(0) = Y$. Now observe that $\phi_Y(i) \geq Y$ for all i. We prove this by induction. Obviously it holds for $i = 0$. Now from the principle of optimality,

$$x'(n-i)\phi_Y(i)x(n-i) = \min_{u(n-i)} [u'(n-i)(D + D')u(n-i)$$

[1] The set inclusion is not necessarily strict.

[2] Actually, $M_d \geq 0$ and $Z_d(z)$ positive real together imply $Y \leq 0$.

$$+ 2x'(n-i)Cu(n-i)$$
$$+ x'(n-i+1)\phi_Y(i-1)x(n-i+1)]$$
$$\geq \min_{u(n-i)} [u'(n-i)(D+D')u(n-i)$$
$$+ 2x'(n-i)Cu(n-i)$$
$$+ x'(n-i+1)Yx(n-i+1)]$$

where the inductive hypothesis is used. The minimization is easily carried out, and this leads to

$$x'(n-i)\phi_Y(i)x(n-i)$$
$$\geq x'(n-i)[A'YA - (A'YB + C)(B'YB + D + D')^{\#}$$
$$\cdot (A'YB + C)']x(n-i). \quad (12)$$

The induction is completed on using (11).

We then have, in particular,

$$x'(0)Yx(0) \leq \min I[x(0),u(\cdot),Y,0,n]$$
$$\leq \min I[x(0),u(\cdot),0,0,n]$$
$$= x'(0)\phi(n)(0).$$

(The second inequality follows from the nonpositivity of Y.) Letting $n \to \infty$ yields $Y \leq \Phi$, since $x(0)$ is arbitrary.

Properties 1 and 2 establish the computational algorithm for finding the maximum solution of the inequality $M_d \geq 0$. This is the main result of the section. However, it is interesting to note several other properties, particularly relating to the convergence of the $\phi(i)$ to Φ, which so far has only been noted as being monotonic.

Property 3

Φ satisfies the limiting version of (10), i.e.,

$$\Phi = A'\Phi A - [A'\Phi B + C][B'\Phi B + D + D']^{\#}$$
$$\cdot [A'\Phi B + C]'. \quad (13)$$

Proof: This is trivial if $B'\Phi B + D + D'$ is nonsingular (for then the limit of a monotone sequence of inverses is the inverse of the limit, and the result is immediate from (10).) Otherwise, proceed as follows. Since Φ solves $M_d \geq 0$, evidently

$$A'\Phi A - \Phi - [A'\Phi B + C][B'\Phi B + D + D']^{\#}$$
$$\cdot [A'\Phi B + C]' \geq 0 \quad (14)$$

or, for all $x(0)$, we have

$$x'(0)\Phi x(0) \leq x'(0)\{A'\Phi A - [A'\Phi B + C]$$
$$\cdot [B'\Phi B + D + D']^{\#}[A'\Phi B + C]'\}x(0).$$
$$(15)$$

Assume for some $x(0)$ that strict inequality holds. Now following the argument and notation as used in the proof of Property 2, to yield (12), and identifying Φ with Y, there obtains

$$x'(0)\phi(n)x(0) \geq x'(0)\phi_\infty(n)x(0)$$
$$\geq x'(0)\{A'\Phi A - [A'\Phi B + C]$$
$$\cdot [B'\Phi B + D + D']^{\#}[A'\Phi B + C]'\}x(0).$$

Now let $n \to \infty$, to conclude

$$x'(0)\Phi x(0) \geq x'(0)\{A'\Phi A - [A'\Phi B + C]$$
$$\cdot [B'\Phi B + D + D']^{\#}[A'\Phi B + C]'\}x(0)$$

and this contradicts the assumption that (15) holds with strict inequality. Hence, (15) holds with strict equality for all $x(0)$, i.e., (13) holds.

Observe that any Y for which

$$Y = A'YA - [A'YB + C][B'YB + D + D']^{\#}$$
$$\cdot [A'YB + C]' \quad (16)$$

and for which $B'YB + D + D' \geq 0$ and $N[B'YB + D + D'] \subset N[A'YB + C]$ solves[3] $M_d \geq 0$. Hence as a particular case of Property 2, we see that Φ is that solution of (16) which is maximum among all solutions satisfying the two side constraints.

As noted earlier, from Φ, which is the maximum Y satisfying $M_d \geq 0$, a minimum phase spectral factor can be constructed. Let N be such that

$$B'\Phi B + D + D' = N'N$$

with N possessing a number of rows equal to rank $(B'\Phi B + D + D')$. Then, using (13), and the fact that $N[B'\Phi B + D + D'] \subset N(A'\Phi B + C)$, we see that

$$M_d = \begin{bmatrix} A'\Phi A - \Phi & A'\Phi B + C \\ (A'\Phi B + C)' & B'\Phi B + D + D' \end{bmatrix}$$
$$= \begin{bmatrix} (A'\Phi B + C)(B'\Phi B + D + D')^{\#}N' \\ N' \end{bmatrix}$$
$$\cdot \begin{bmatrix} (A'\Phi B + C)(B'\Phi B + D + D')^{\#}N' \\ N' \end{bmatrix}'.$$

This means that a spectral factor associated with $Z_d(z)$ is

$$W_d(z) = N[I + (B'\Phi B + D + D')^{\#}(A'\Phi B + C)'$$
$$\cdot (zI - A)^{-1}B]. \quad (17)$$

This $W_d(z)$ has a number of rows equal to the number of rows of N, or the rank of $B'\Phi B + D + D'$. It is straightforward to show that this is also the rank of M_d. So $W_d(z)$ is in fact the minimum phase spectral factor.

For the continuous time problem, suppose the positive real $Z_c(s)$ is given with minimal realization $\{F,G,H,J\}$. A minimum phase spectral factor is found as follows. The quadruple $\{A,B,C,D\}$ is found via (6), Φ is found by the

[3] Such matrices Y coincide with the set of solutions of $M_d \geq 0$ making M_d of minimum rank. This is analogous with the continuous-time situation [10], [21].

Riccati difference equations, and then $W_c(s) = W_0 + L'(sI - F)^{-1}G$ where

$$L = \frac{1}{\sqrt{2\alpha}}(\alpha I - F)(A'\Phi B + C)(B'\Phi B + D + D')^{\#}N'$$

$$W_0 = N - \frac{1}{\sqrt{2\alpha}}N(B'\Phi B + D + D')^{\#}(A'\Phi B + C)'G.$$

Alternatively, one can factor M_c with X replaced by Φ.

Use of the minimum phase property yields more information about Φ, which in turn can be used to study the rate of convergence of $\phi(i)$ to Φ as $i \to \infty$.

Property 4

Define $\Pi = A - B(B'\Phi B + D + D')^{\#}(A'\Phi B + C)'$. Then $|\lambda_i(\Pi)| \leq 1$.

Proof: Since rank $W_d(z)$ must equal the number of rows of N in $|z| > 1$, one must have $\det[I + (B'\Phi B + D + D')^{\#}(A'\Phi B + C)'(zI - A)^{-1}B]$ nonzero in $|z| > 1$. The inverse of the term whose determinant is being taken is of the form $I + E_1(zI - \Pi)^{-1}E_2$, whence $|\lambda_i(\Pi)| \leq 1$.

There is an obvious frequency domain condition for $|\lambda_i(\Pi)| < 1$, as opposed to $|\lambda_i(\Pi)| \leq 1$. Since

$$Z_d'(z^{-1}) + Z_d(z) = W_d'(z^{-1})W_d(z)$$

and since unity modulus eigenvalues of Π are associated with reduction in rank of $W_d(z)$ on $|z| = 1$, evidently $|\lambda_i(\Pi)| < 1$ if $Z_d'(z^{-1}) + Z_d(z)$ has constant rank for all $|z| = 1$. In the s plane, $Z_c'(-s) + Z_c(s)$ must have constant rank for all $s = j\omega$, ω real.

As foreshadowed previously, the matrix Π yields information about the rate of approach of $\phi(i)$ to Φ. Set $\phi_Q(i)$ as the solution of (10) with initial condition $\phi_Q(0) = Q$. (Thus $\phi_0(i) = \phi(i)$ in the earlier notation.) Then with $\Delta_Q(i) = \phi_Q(i) - \Phi$, (10) and (14) for $\phi_Q(i)$ and Φ lead, after some manipulation, to

$$\Delta_Q(i + 1) = \Pi'[\Delta_Q(i) - \Delta_Q(i)B(B'\Phi B + D + D' + B'\Delta_Q(i)B)^{\#}B'\Delta_Q(i)]\Pi. \quad (18)$$

In case $Q = 0$ or in fact $Q \geq \Phi$, $\Delta_Q(i) \geq 0$ for all i and

$$\Delta_Q(i + 1) \leq \Pi'\Delta_Q(i)\Pi$$

from which it is clear that $\Delta(i) \to 0$ at least as fast as $(\max |\lambda_i(\Pi)|)^{2i} \to 0$. The authors have verified such convergence numerically. Further, because the convergence works for all $Q \geq \Phi$, this gives the iterative procedure some numerical stability.

In case Π has eigenvalues of unity modulus, the situation is a little more complicated. If $Z_d(z)$ is lossless discrete positive real, i.e., $Z_d(z^{-1}) + Z_d(z) = 0$, it turns out that $\Pi = A$ and all eigenvalues are of unity modulus; it is also not hard to show (see Appendix II) that $\Delta(i)$ converges to zero in a finite number of steps. If $Z_d(z)$ is the sum of a lossless discrete positive real matrix and another positive real matrix $\hat{Z}_d(z)$ with $\hat{Z}_d(z^{-1}) + \hat{Z}_d(z)$ positive definite on $|z| = 1$, an extension of this idea will show convergence of $\Delta(i)$ as fast as $(\max |\lambda_i(\Pi)|)^{2i} \to 0$ where the maximum is over those eigenvalues which are not of unity modulus. Finally, in case $Z_d(z^{-1}) + Z_d(z)$ drops in rank on $|z| = 1$, convergence of $\Delta(i)$ to zero occurs at least as fast as i^{-1}, as shown in Appendix III. (This result has been experimentally observed by the authors, and also experimentally for a special case in [13].)

Another aspect of computational interest arises in considering the use of the pseudo-inverse in (10) and (14). (It is well known that pseudo-inverse occurrence raises potentially awkward problems.) Fortunately, there is some sort of stability here too. This comes about as follows. Let $\phi_\varepsilon(n)$ and Φ_ε denote the quantities obtainable from (10) when $D + D'$ is replaced by $D + D' + \varepsilon I$ for some positive ε. Examination of the performance index shows

$$\lim_{\varepsilon \to 0} \phi_\varepsilon(n) = \phi(n)$$

and since the convergence of $\phi_\varepsilon(n)$ as both $\varepsilon \to 0$ and $n \to \infty$ is monotonic it follows [24, p. 414] that

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon = \lim_{\varepsilon \to 0}\lim_{n \to \infty} \phi_\varepsilon(n) = \lim_{n \to \infty}\lim_{\varepsilon \to 0} \phi_\varepsilon(n) = \Phi.$$

Now Φ_ε satisfies (14) with $D + D'$ replaced by $D + D' + \varepsilon I$ and the pseudo inverse will be replaced by an inverse. Then, even if $B'\Phi B + D + D'$ is singular, so that

$$\lim_{\varepsilon \to 0}(B'\Phi B + D + D' + \varepsilon I)^{\#} \neq (B'\Phi B + D + D')^{\#}$$

replacement of $(B'\Phi B + D + D')^{\#}$ by $(B'\Phi B + D + D' + \varepsilon I)^{\#}$ for small enough ε will not cause a catastrophic error in (14) because $\lim_{\varepsilon \to 0} \Phi_\varepsilon = \Phi$. Similar remarks hold for (10). In numerical simulations, the authors found no difficulty with pseudo inverses.

One further computational point is the following. With $\phi(i)$ in (10) initialized by $\phi(0) = 0$, it can be shown that $\phi(i + 1) - \phi(i)$ has rank at most equal to that of B. This allows one to write down different iterative equations which can involve less computation in obtaining the minimum phase spectral factor. With $M(i) = \phi(i)B$, one has

$$\begin{aligned} M(i + 1) &= M(i) - Q(i)R(i)B \\ Q(i) &= [A' - (A'M(i) + C) \\ &\qquad \cdot (B'M(i) + D + D')^{\#}B']Q(i - 1) \\ R(i)' &= [A' - (A'M(i - 1) + C) \\ &\qquad \cdot (B'M(i - 1) + D + D')^{\#}B']R(i - 1)' \end{aligned}$$

with the equations initialized by

$$M(0) = 0; \qquad Q(0) = R'(0) = C'[(D + D')^{1/2}]^{\#}.$$

The limiting $M(i)$ is ΦB, whence the spectral factor can be found.

Though in the preceding analysis we have been assuming minimality of $\{A,B,C,D\}$, it turns out that the complete observability constraint can be relaxed, except when we require Φ nonsingular, provided that unobservable states are asymptotically stable. (Otherwise, computational problems arise in solving the Riccati equation.) A result where observability is definitely needed however is the following.

It is certainly of academic interest to consider what is the *minimum* Y satisfying the matrix inequality $M_d \geq 0$. This may be obtained in the following way. The inequality $M_c \geq 0$ implies and is implied by

$$\begin{bmatrix} \hat{X}F' + F\hat{X} & G + \hat{X}H \\ (G + \hat{X}H)' & J + J' \end{bmatrix} \geq 0 \tag{19}$$

where $\hat{X} = X^{-1}$. Now this is the matrix inequality associated with the positive real matrix $Z'(s) = J' + G'(sI - F')^{-1}H$. The minimum X satisfying $M_c \geq 0$ is therefore the inverse of the maximum $\hat{X}$ satisfying (19). Making the s-plane to z-plane transformation, it follows that the minimum Y satisfying $M_d \geq 0$ is also the inverse of the maximum $\hat{Y}$, call it $\hat{\Phi}$, satisfying

$$\begin{bmatrix} A\hat{Y}A' - \hat{Y} & A\hat{Y}C + B \\ (A\hat{Y}C + B)' & C'\hat{Y}C + D + D' \end{bmatrix} \geq 0.$$

(This fact can also be shown directly, via some lengthy manipulations.) Of course, $\hat{\Phi} = \lim_{i \to \infty} \hat{\phi}(i)$ where

$$\hat{\phi}(i+1) = A\hat{\phi}(i)A' - [A\hat{\phi}(i)C + B] \cdot [C'\hat{\phi}(i)C + D + D']^{\#}[A\hat{\phi}(i)C + B]'$$

and $\hat{\phi}(0) = 0$. Note that observability of $[A,C]$ is needed to guarantee existence of $\hat{\phi}(\cdot)$ and controllability of $[A,B]$ gives the desired invertibility.

V. Spectral Factorization

In this section, we shall discuss the factorization of self-inversive and even scalar polynomials and polynomial matrices, and then the problem of rational matrix spectral factorization.

Consider first the self-inversive "polynomial"

$$p(z) = 2d + c_n z + \cdots + c_1 z^n + c_n z^{-1} + \cdots + c_1 z^{-n}; \quad d > 0. \tag{20}$$

Let us assume that $p(z) \geq 0$ for all z on $|z| = 1$. Then a spectral factorization problem is well defined. To apply the methods of this paper towards finding a polynomial $w(z)$ such that $p(z) = w(z^{-1})w(z)$, we can set up a discrete positive real function; the simplest way of doing this is to write

$$Z_d(z) = d + c_n z^{-1} + \cdots + c_1 z^{-n}.$$

For then $Z_d(z) + Z_d(z^{-1}) = p(z)$, and the nonnegativity of $p(z)$ on $|z| = 1$ and analyticity of $Z_d(z)$ in $|z| > 1$ will guarantee $Z_d(z)$ is discrete positive real.

A minimal realization of $Z_d(z)$ is provided by

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & & & \vdots \\ & & & & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}, \qquad D = d$$

as is easily seen. Then one can solve the discrete-time Riccati difference equations to find a limiting solution Φ. Let Φ_{ij} denote the i, j entry of Φ, and d_i the ith entry of a vector d. Then, the minimum phase spectral factor is

$$\sqrt{2d + B'\Phi B}\,[1 + (2d + B'\Phi B)^{-1} \cdot (A'\Phi B + C)'(zI - A)^{-1}B].$$

Observe that $(2d + B'\Phi B) = 2d + \Phi_{nn}$, and $(zI - A)^{-1}B = z^{-n}[1 \;\; z \;\; z^2 \;\; \cdots \;\; z^{n-1}]$. Also,

$$A'\Phi B + C = \begin{bmatrix} c_1 \\ c_2 + \Phi_{1n} \\ c_3 + \Phi_{2n} \\ \vdots \\ c_n + \Phi_{n-1n} \end{bmatrix}$$

noting the special form of A and B. Thus with $w_0 = (2d + \Phi_{nn})^{-1/2}$

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = (2d + \Phi_{nn})^{-1/2} \begin{bmatrix} c_1 \\ c_2 + \Phi_{1n} \\ \vdots \\ c_n + \Phi_{n-1n} \end{bmatrix}$$

and the spectral factor $W_d(z)$ of $Z_d(z)$ is $w_0 + w_n z^{-1} + \cdots + w_1 z^{-n}$. Consequently, the spectral factor of $p(z)$ is $w_0 + w_n z + \cdots + w_1 z^n$.

The algorithm is actually precisely that used by Bauer [11]. To explain the connection a little more fully, define

$$\begin{bmatrix} w_1(k) \\ w_2(k) \\ \vdots \\ w_n(k) \end{bmatrix} = [2d + B'\phi(k)B]^{-1/2} \begin{bmatrix} c_1 + (A'\phi(k)B)_1 \\ c_2 + (A'\phi(k)B)_2 \\ \vdots \\ c_n + (A'\phi(k)B)_n \end{bmatrix}$$

$$= [2d + \phi_{nn}(k)]^{-1/2} \begin{bmatrix} c_1 \\ c_2 + \phi_{1n}(k) \\ c_3 + \phi_{2n}(k) \\ \vdots \\ c_n + \phi_{n-1n}(k) \end{bmatrix} \tag{21}$$

on using the special form of A and B. This definition of the $w_i(k)$ taken with the special form of A and B can be used to show that the solution of the Riccati equation can be written in terms of these quantities:

$$\phi(k+1) = -\begin{bmatrix} w_1(k) \\ \vdots \\ w_n(k) \end{bmatrix} [w_1(k) \;\; \cdots \;\; w_n(k)] - \begin{bmatrix} 0 \\ w_1(k-1) \\ \vdots \\ w_{n-1}(k-1) \end{bmatrix} \cdot [0 \;\; w_1(k-1) \;\; \cdots \;\; w_{n-1}(k-1)] - \cdots$$

$$- \begin{bmatrix} 0 \\ \vdots \\ w_1(k-n+1) \end{bmatrix} \cdot [0 \;\; \cdots \;\; 0 \;\; w_1(k-n+1)].$$

When this is used in (21), the following recursion follows for the vector of $w_i(k)$:

$$\begin{bmatrix} w_1(k+1) \\ \vdots \\ w_n(k+1) \end{bmatrix} = [2d - w_n^2(k) - \cdots - w_1^2(k-n+1)]^{-1/2} \cdot \begin{bmatrix} c_1 \\ c_2 - w_1(k)w_n(k) \\ c_3 - w_2(k)w_n(k) - w_1(k-1)w_{n-1}(k-1) \\ \vdots \\ c_n - w_{n-1}(k)w_n(k) - \cdots \\ \quad - w_1(k-n+2)w_2(k-n+2) \end{bmatrix} \tag{22}$$

As $k \to \infty$, $w_i(k) \to w_i$ and $2d - w_n^2(k) - \cdots - w_1^2 \cdot (k-n+1) \to w_0^2$. Equation (22) is precisely the recursive equations of Bauer.

When we turn to the $r \times r$ matrix polynomial

$$\Gamma(z) = (D + D') + C_n z + \cdots + Cz^n + C_n' z^{-1} + \cdots + C_1' z^{-n} \tag{23}$$

assumed nonnegative definite on $|z| = 1$, a similar approach works. Then one recovers the results of [12] extending the Bauer algorithm to the matrix case (apparently restricted in [12] to positive definite $\Gamma(z)$), and also essentially the results of [13], where the algorithm is presented in terms of the discrete-time Riccati equation (again, there is apparent restriction to the positive definite case). One takes A as the direct sum of r copies of the A matrix used for the scalar polynomial, B as an $nr \times r$ matrix with the ith columns of all zeros except for the inth element, which is one. Finally, $C' = [C_1' \; C_2' \; \cdots \; C_n']$. Then, $Z_d(z) = D + C'(zI - A)^{-1}B$ is such that $Z_d'(z^{-1}) + Z_d(z) = \Gamma(z)$. (Actually, [13] has a minor variation on this which allows some dimension reduction of the A, B, and C matrices in case C_1 is not of full rank. The difference is inessential for our description.)

Even scalar polynomials and the corresponding matrix problem can be handled by conversion through the bilinear transformation. Let us then consider the problem of factoring a power spectrum matrix $\Psi(z)$. One can write, obviously, $\Psi(z)$ as

$$\Psi(z) = \frac{\Gamma(z)}{p(z)}$$

where $p(z)$ is a scalar "polynomial," with $p(z) = p(z^{-1})$, $p(0) > 0$, and $\Gamma(z)$ is a matrix of the form of (23). Further, $\Gamma(z)$ and $p(z)$ are nonnegative on $|z| \geq 1$, with $p(z)$ positive if $\Psi(z)$ is analytic there. One can then proceed in two ways. Either one can factor $\Gamma(z)$ and $p(z)$ separately as described previously, getting $p(z) = w(z^{-1})w(z)$ and $\Gamma(z) = W(z^{-1})W(z)$ with $w(z)$ and $W(z)$ nonzero and of constant rank, respectively, in $|z| > 1$. Then a minimum phase spectral factor is $w^{-1}(z)W(z)$. Alternatively, and clearly more conveniently should a factorization of $p(z)$ be already available, one can write

$$\Psi(z) = \frac{\Gamma_0(z)}{w(z)} + \frac{\Gamma_0'(z^{-1})}{w(z^{-1})}$$

with $w^{-1}(z)\Gamma_0(z)$ discrete positive real. Then one can form a state-space realization of this matrix and proceed as outlined in the paper.

When one turns to the factorization of $\Psi(s) = \Psi'(-s)$, with $\Psi(j\omega) \geq 0$ for all real ω, the same sort of remarks hold true.

VI. Conclusions

We have discussed a procedure for computing spectral factors of polynomials (scalar and matrix) and power spectrum matrices. The procedure is based on a convenient computational algorithm, and extends to cover the case of factoring nonnegative definite as opposed to positive definite spectra. Rate of convergence results are also given. Further, we have exhibited a tight connection between continuous-time and discrete-time spectral factorization, giving an implicit solution of the spectral factorization problem by a negative definite matrix that is the same for both problems.

The close connection between discrete-time and continuous-time problems can be carried over to general quadratic minimization problems, including differential games. This will be discussed in work currently under preparation.

Appendix I

Suppose

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix}$$

is nonnegative definite. Let $x_2 \in N(P_3)$. Then, $x'Px = (x_1' \; kx_2')P(x_1' \; kx_2')' = x_1'P_1x_1 + 2kx_1'P_2x_2$ for all constant k. If $P_2x_2 \neq 0$, taking $x_1 = P_2x_2$ shows that $x'Px$ will be negative for suitably negative k. Hence, $x_2 \in N(P_2)$ or $N(P_3) \subset N(P_2)$.

By direct calculation, we have

$$\begin{bmatrix} I & -P_2P_3^{\#} \\ 0 & I \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix} \begin{bmatrix} I & 0 \\ -P_3^{\#}P_2' & I \end{bmatrix} = \begin{bmatrix} P_1 - P_2P_3^{\#}P_2' & P_2 - P_2P_3^{\#}P_3 \\ P_2' - P_3P_3^{\#}P_2' & P_3 \end{bmatrix}.$$

With $N(P_3) \subset N(P_2)$, $P_2 - P_2P_3^{\#}P_3 = 0$ and nonnegativity of P implies nonnegativity of $P_1 - P_2P_3^{\#}P_2'$ and P_3. Conversely, if these matrices are nonnegative and $N(P_3) \subset N(P_2)$, the identity implies P is nonnegative.

Appendix II

Suppose $Z_d'(z^{-1}) + Z_d(z) = 0$ and $\{A,B,C,D\}$ is a minimal realization of $Z_d(z)$. Then, $W_d(z) = 0$, and so, $A'\Phi B + C = 0$ and $B'\Phi B + D + D' = 0$. Then, $\Pi = A$

and the equation for $\Delta(i)$ becomes

$$\Delta(i+1) = A'[\Delta(i) - \Delta(i)B(B'\Delta(i)B)^{\#}B'\Delta(i)]A. \quad \text{(A1)}$$

All eigenvalues of A lie on $|z| = 1$, so that A is invertible. Premultiply by $B'(A')^{-1}$ and postmultiply by $A^{-1}B$ to obtain $B'(A')^{-1}\Delta(i+1)A^{-1}B = 0$. Since $\Delta(i+1) \geq 0$, $\Delta(j)A^{-1}B = 0$, $j > 0$. Now premultiply by $B'(A')^{-2}$ and postmultiply by $A^{-2}B$ to obtain $B'(A')^{-2}\Delta(i+1)A^{-2}B = 0$ if $i > 0$ or $\Delta(j)A^{-2}B = 0$ for $j > 1$. Continuation of the argument shows $\Delta(j)A^{-3}B = 0$ for $j > 2$, $\Delta(j)A^{-4}B = 0$ for $j > 3$, etc. Using controllability implies $\Delta(j) = 0$ for $j \geq n$.

Appendix III

In this appendix, we shall show that if $|\lambda_i(\Pi)| = 1$ for some i, then $\Delta(i) \to 0$ as fast as i^{-1} as $i \to \infty$. First, observe without loss of generality that Π can be taken as nonsingular. (Otherwise, change the coordinate basis so that Π is the direct sum of a nonsingular matrix and a matrix of zero eigenvalues, and observe that certain submatrices of $\Delta(i)$ are found to be zero, with the nonzero part depending only on the nonsingular part of Π.)

Consider the problem of minimizing for the system $x(i+1) = \Pi x(i) + Bu(i)$ the performance index

$$I[x(0),u(\cdot),-\Phi,0,n] = \sum_{i=0}^{n-1} [u(i)(B'\Phi B + D + D' + \varepsilon I)u(i)] - x'(n)\Phi x(n)$$

where ε is a nonnegative number. For $\varepsilon = 0$, the minimum is $x'(0)\Delta(n)x(0)$, where $\Delta(n)$ is defined by (18) with $Q = 0$. For nonzero ε, obviously the minimum exists, being $x'(0)\Delta_\varepsilon(n)x(0) \geq x'(0)\Delta(n)x(0) \geq 0$ where

$$\begin{aligned}\Delta_\varepsilon(i+1) &= \Pi'[\Delta_\varepsilon(i) - \Delta_\varepsilon(i)B(B'\Phi B + D + D' \\ &\quad + B'\Delta_\varepsilon(i)B + \varepsilon I)^{-1}B'\Delta_\varepsilon(i)]\Pi \\ &= \Pi'[\Delta_\varepsilon(i) - \Delta_\varepsilon(i)B(N + B'\Delta_\varepsilon(i)B)^{-1}B'\Delta_\varepsilon(i)]\Pi \qquad \text{(A2)}\end{aligned}$$

where N is a certain nonsingular matrix, and $\Delta_\varepsilon(0) = -\Phi$. Equation (A2) may be rewritten as

$$\Delta_\varepsilon^{-1}(i+1) = \Pi^{-1}\Delta_\varepsilon^{-1}(i)(\Pi^{-1})' + \Pi^{-1}BN^{-1}B'(\Pi^{-1})'. \qquad \text{(A3)}$$

Notice that $\Delta_\varepsilon(0) = -\Phi$ is nonsingular if we assume (A,C) is completely observable. Nonsingularity of $\Delta_\varepsilon(i)$ also implies, by (A3), nonsingularity of $\Delta_\varepsilon(i+1)$, so in fact, $\Delta_\varepsilon(i)$ is nonsingular for all i.

Because (A,B) is completely controllable, and because $\Pi = A - BK'$ for some K', (Π,B) is also completely controllable. Then, one can argue using the eigenvalue property of Π that $\sum_{j=1}^{i} \Pi^{-j}BN^{-1}(B'\Pi^{-j})'$ grows at least as fast as i, whence $\Delta_\varepsilon(i) \to 0$ as i^{-1}. Since $\Delta(i) \leq \Delta_\varepsilon(i)$, the convergence of $\Delta(i)$ follows. The proof can actually be extended to the case when (A,C) has unobservable states and $|\lambda_i(A)| \leq 1$, but this is not of great interest.

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