Synthesis of Linear Time-Varying Passive Networks

BRIAN D. ANDERSON, MEMBER, IEEE, AND PETER J. MOYLAN, MEMBER, IEEE

Abstract—**A state-space synthesis procedure is given for linear time-varying passive impedance matrices. The synthesis uses only passive components.**

I. Introduction

IN THIS PAPER, we consider linear lumped finite networks composed of interconnections of time-variable and passive resistances, capacitances, inductances, gyrators, and transformers. The main problem considered is to pass from an input–output, or port, description of the network (in terms of its impedance matrix) to an internal description (in terms of a set of element values, and a scheme for interconnection).

Amidst prior work on this and related problems, we note especially the work of Spaulding, e.g., [1], [2], who obtained some necessary conditions for a prescribed impedance matrix to be passive, and necessary and sufficient conditions for a prescribed impedance matrix to be the impedance of a network containing all lossless elements. He also obtained a synthesis procedure for this latter class of impedances. Another impedance synthesis procedure for lossless element networks was derived by Saeks [3], paralleling the Cauer synthesis, while [4] presents a lossless synthesis based on the scattering matrix.

Further necessary conditions on an impedance matrix for it to be associated with a passive element network were derived in [5], and more recently, some characterizations of passivity, using a state-space description of a prescribed impedance, were obtained in [6] and [7].

The material closest to that presented in this paper is, however, [8]. In [8], synthesis procedures were obtained given the validity of a certain conjecture; this conjecture was known to be true for a limited class of impedances, and the claim of the conjecture was its truth for all passive impedances. Much of this paper, in effect, amounts to an examination of this conjecture and a delineation of when it is true.

The broad structure of the paper is as follows. We assume there is given the state-space equations

$$\dot{x} = F(t)x + G(t)u \qquad v = H'(t)x + J(t)u \tag{1}$$

of a time-varying impedance matrix $Z(t,\tau)$, which is related to $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, and $J(\cdot)$ by

$$Z(t,\tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau). \tag{2}$$

In (1), $v(\cdot)$ and $u(\cdot)$ denote, respectively, the port voltage and current vectors of some network with impedance $Z(\cdot,\cdot)$. In (2), $\delta(\cdot)$ and $1(\cdot)$ are, respectively, the unit impulse and the unit step function, and $\Phi(\cdot,\cdot)$ is the transition matrix of $F(\cdot)$. The synthesis problem is one of passing from $Z(\cdot,\cdot)$ to a network with impedance $Z(\cdot,\cdot)$. (Slightly more complex $Z(\cdot,\cdot)$ will be considered in the sequel; for this discussion, though, (2) will be adequate.) If $Z(\cdot,\cdot)$ alone is known, rather than $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, and $J(\cdot)$ separately, it is a standard procedure of linear system theory to find $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, and $J(\cdot)$ from $Z(\cdot,\cdot)$; therefore, we shall assume such matrices are all known *a priori*.

The impedance $Z(\cdot,\cdot)$ corresponds to a passive network if one has for all times t_0 and t_1, $t_0 < t_1$ and all $u(\cdot)$:

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} u'(t)Z(t,\tau)u(\tau)\,dt\,d\tau \geq 0. \tag{3}$$

(The double integral on the left is the energy supplied to the network over $[t_0,t_1]$ with the network initially in the zero state.)

The first major step is to use the passivity property (3) to conclude the existence of at least one nonnegative definite symmetric matrix $P(\cdot)$ such that

$$\begin{bmatrix} -PF - F'P - \dot{P} & H - PG \\ (H-PG)' & J + J' \end{bmatrix} \geq 0. \tag{4}$$

A complete controllability condition on $[F,G]$ must be assumed, and existence of $P(\cdot)$ is actually guaranteed in the first instance if $J + J'$ is nonsingular. In case $J + J'$ is singular, we show that $P(\cdot)$ satisfies a slightly weaker condition than (4), but may satisfy (4) also. In case $[F,H]$ is completely observable, $P(\cdot)$ is nonsingular.

Following proof of the existence of $P(\cdot)$, we show how to compute such a $P(\cdot)$. In so doing, we give conditions applying to the case of $J + J'$ singular for $P(\cdot)$ to satisfy (4) rather than the slightly weaker condition.

The matrix $P(\cdot)$ is now used to define a coordinate basis transformation; the new state-space equations then allow both reactance extraction and resistance extraction syntheses. The idea of these syntheses in the time-invariant case is discussed in [9]–[11], and their use in time-varying problems appears in [8] and [12].

The computation of $P(\cdot)$ when $J + J'$ is singular can, if desired, proceed via a series of steps which, from the syn-

Manuscript received June 29, 1973; revised November 9, 1973. This work was supported by the Australian Research Grants Committee.
The authors are with the Department of Electrical Engineering, University of Newcastle, N.S.W., Australia.

thesis point of view, correspond to extraction of series inductor and shunt capacitor elements, such as occurs in the "preambles" of many classical time-invariant synthesis procedures. This interpretation will be made clear subsequently.

II. Necessary and Sufficient Conditions for the Passivity Property

Suppose N is a multiport network comprising a finite number of passive resistors, capacitors, inductors, transformers and gyrators, any of which may be time-variable.[1] Suppose also that the variations are smooth, and the network is such that there exists an impulse response matrix $Z(\cdot,\cdot)$ mapping port current vectors $u(\cdot)$ into port voltage vectors $v(\cdot)$. Then $Z(\cdot,\cdot)$ is the impedance matrix of N and necessarily has the form

$$Z(t,\tau) = \sum_{i=0}^{N} Z_i(t)\delta^{(i)}(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau). \tag{5}$$

Passivity of the network components implies, by, for example, Tellegen's theorem, passivity of the impedance matrix in the sense that

$$E(t_0,t_1,u(\cdot)) = \int_{t_0}^{t_1}\int_{t_0}^{t_1} u'(t)Z(t,\tau)u(\tau)\,dt\,d\tau \ge 0 \tag{6}$$

for all t_0 and t_1 with $t_0 < t_1$, and all sufficiently smooth $u(\cdot)$. This passivity property constrains the structure of $Z(t,\tau)$ more than (5) would indicate.

Lemma 1: With $Z(\cdot,\cdot)$ as defined previously and satisfying (6), $Z(\cdot,\cdot)$ can be rewritten as

$$Z(t,\tau) = T'(t)\delta^{(1)}(t-\tau)T(\tau) + J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{7}$$

and both $T'(t)\delta^{(1)}(t-\tau)T(\tau)$ and

$$\hat{Z}(t,\tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau) \tag{8}$$

individually satisfy (6).

For a proof, based on [1] and [5], see Appendix II.

The significance of Lemma 1 is that it reduces the problem of synthesizing an arbitrary impedance $Z(\cdot,\cdot)$ to the problem of synthesizing an impedance of the form of (8): an arbitrary impedance has the form (7), and can be synthesized as a series connection of a synthesis of $\hat{Z}(t,\tau)$, and transformer-coupled inductors. The transformer-coupled inductors, of course, synthesize the term $T'(t)\delta^{(1)}(t-\tau)T(\tau)$, actually by terminating the secondary ports of a time-varying transformer of turns-ratio matrix $T(\cdot)$ in unit inductors. Similar ideas are well known for time-invariant synthesis [13].

Let us now study passivity properties for $\hat{Z}(\cdot,\cdot)$; we drop the superscript hat. Setting $R = J + J'$, and taking cognizance of the state-space equations (1) for $Z(\cdot,\cdot)$, the passivity condition (6) can be rewritten as

$$\int_{t_0}^{t_1} (2x'Hu + u'Ru)\,dt \ge 0 \tag{9}$$

which holds for all $u(\cdot)$ and t_1, given $x(t_0) = 0$. Obviously, one necessary condition for this is $R \ge 0$. Various other necessity and sufficiency conditions for (9) are also known, see, e.g., [6], [7], and the survey [14]. Here, we shall use a condition that is both necessary and sufficient in case $R > 0$, and two slightly distinct conditions, one necessary and one sufficient, in case R is singular (there does not appear to be a single necessary and sufficient condition). We require the following assumptions.

Assumption 1: For all t, there exists $t_0 < t$ such that (1) is completely controllable on $[t_0,t]$, i.e., every state at time t is reachable from the zero state at time t_0.

Assumption 2: The matrices F, G, H, and J have continuously differentiable entries.

Nonsingular Problem—Necessity Conditions: This is available in [7]. Under Assumptions 1 and 2 and (9), there exists a nonnegative definite symmetric matrix $P(t)$ defined by

$$-\dot{P} = PF + F'P + (PG - H)R^{-1}(PG - H)' \tag{10}$$

satisfying the limiting boundary condition[2] $\lim_{t_1\to\infty} P(t_1) = 0$. We remark that $P(t)$ yields the solution of a minimization problem for (1):

$$-x'(t)P(t)x(t) = \inf_{u(\cdot)} \int_t^{\infty} (2x'Hu + u'Ru)\,dt. \tag{11}$$

Notice also that with $R > 0$, (10) implies

$$M(P) = \begin{bmatrix} -PF - F'P - \dot{P} & H - PG \\ (H - PG)' & R \end{bmatrix} \ge 0. \tag{12}$$

If (12) holds for some nonnegative definite symmetric $P(t)$, it does not follow that this $P(t)$ satisfies (10). However, in a roundabout sense, (12) does imply (10) for some other $P(\cdot)$, because it implies the passivity property as shown below, and this in turn implies (10).

Nonsingular Problems—Sufficiency: With $R > 0$, existence of some nonnegative definite symmetric $P(\cdot)$ satisfying (12) is sufficient to guarantee (9), for simple manipulation yields

$$\begin{aligned}\int_{t_0}^{t_1} &(2x'Hu + u'Ru)\,dt \\ &= \int_{t_0}^{t_1} (2x'Hu + u'Ru + x'P(\dot{x} - Fx - Gu))\,dt \\ &= \int_{t_0}^{t_1} [x' \;\; u'] \begin{bmatrix} -PF - F'P - \dot{P} & H - PG \\ (H - PG)' & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt \\ &\quad + x'(t_1)P(t_1)x(t_1).\end{aligned}$$

[1] For definitions of these quantities, see Appendix I.

[2] More precisely, let $\Pi_{t_1}(\cdot)$ be the solution of $-\dot{\Pi} = \Pi F + F'\Pi - (\Pi G - H)R^{-1}(\Pi G - H)'$ satisfying $\Pi_{t_1}(t_1) = 0$. Then $P(t) = \lim_{t_1\to\infty} \Pi_{t_1}(t)$. Existence of the limit is one of the technical questions taken up in [7].

(This line of argument is due to Jacobson, see, e.g., [14].) Notice that there may be many nonnegative symmetric $P(\cdot)$ satisfying (12), and that they do not necessarily satisfy (10), let alone (10) with its limiting boundary condition.

Nonsingular Problem—Reformulation of Conditions: The preceding arguments show that a necessary and sufficient condition for (9), given that R is nonsingular, is that there should exist a nonnegative symmetric $P(t)$ satisfying (12). One such matrix $P(t)$ can be computed via (10), using the associated limiting boundary condition.

Singular Problem—Necessity Conditions: Certainly $R \geq 0$ is a necessary condition. But now (10) cannot be formed. We proceed by following ideas of [15] and [16]. Let $P_\varepsilon(t)$ be defined for arbitrary $\varepsilon > 0$ by

$$-x'(t)P_\varepsilon(t)x(t) = \inf_{u(\cdot)} \int_t^\infty (2x'Hu + u'Ru + \varepsilon u'u)\, dt. \tag{13}$$

Existence of a nonnegative definite symmetric $P_\varepsilon(\cdot)$ follows as in [7], and P_ε satisfies (10) with P replaced by P_ε and R replaced by $R + \varepsilon I$. Now from (13), it is easily seen that $P_\varepsilon(t)$ for fixed t increases monotonically as $\varepsilon \to 0$, while it is bounded above for all $\varepsilon <$ some ε_1 by a simple controllability argument, such as is used in [7] in study of the nonsingular R case. Hence $\lim_{\varepsilon\to 0} P_\varepsilon(t) = P(t)$ exists and is nonnegative definite symmetric.

One cannot conclude that (12) then holds for this $P(\cdot)$, since $P(\cdot)$ does not necessarily inherit the differentiability of $P_\varepsilon(\cdot)$. However, we can get something very close. Since (12) holds for $P_\varepsilon(\cdot)$, we have for all t_0 and t_1 and continuous $w(\cdot)$ positivity of the following Stieltjes integral

$$\int_{t_0}^{t_1} w'(t) \begin{bmatrix} -(P_\varepsilon F + F'P_\varepsilon)\, dt - dP_\varepsilon & (H - P_\varepsilon G)\, dt \\ (H - P_\varepsilon G)'\, dt & R\, dt \end{bmatrix} w(t).$$

By the Helly convergence theorem [17], the evident bounded variation property of $P_\varepsilon(t)$ is inherited by $P(t)$ and

$$\int_{t_0}^{t_1} w'(t) \begin{bmatrix} -(PF + F'P)\, dt - dP & (H - PG)\, dt \\ (H - PG)'\, dt & R\, dt \end{bmatrix} w(t) \geq 0 \tag{14}$$

for all t_0, t_1, and continuous $w(\cdot)$. This nonnegativity, coupled with that of $P(t)$ itself, constitutes the necessary condition.

Singular Problem—Sufficiency Condition: The argument used for the nonsingular case follows with no change to show that (9) is implied by the existence of a symmetric nonnegative $P(t)$ of differentiable entries for which (12) holds.[3] (Note that such a $P(t)$ is not necessarily defined by the minimization procedure used in deriving the necessity condition.)

Let us sum up these results.

[3] Of course, nonnegativity of the Stieltjes integral in (14) for some nonnegative definite symmetric $P(t)$ of bounded variation and all continuous $w(\cdot)$ is also sufficient (and, therefore, necessary and sufficient) for (9) to hold. But this form of condition is not helpful for synthesis. Therefore, the stronger condition is introduced.

Lemma 2: A sufficient condition that the passivity inequality (9) should hold for $x(t_0) = 0$, arbitrary t_1, and all $u(\cdot)$ is that there should exist a nonnegative symmetric $P(\cdot)$ such that (12) holds. In case R is nonsingular, this condition is necessary, and one such $P(\cdot)$ can be found from (10) with the limiting boundary condition $\lim_{t_1\to\infty} P(t_1) = 0$. In case R is singular, it is necessary that there exist a nonnegative symmetric $P(\cdot)$ satisfying condition (14).

Bounds on $P(t)$: For bounded F, G, H, and J and a uniformly completely controllable pair $[F,G]$, it is shown in [7] for the case of nonsingular $R(\cdot)$ that the matrix $P(t)$ defined by the variational problem is bounded on $(-\infty,\infty)$. A minor variation in the argument extends the result to the singular case, when the matrix $P(t)$ is determined via $\lim_{\varepsilon\to 0} P_\varepsilon(t)$, with $P_\varepsilon(t)$ as defined previously.

Note that this bound has not been established for any P satisfying (12) or (14). It has only been established for one particular P satisfying (12) or (14), viz., that obtained from consideration of the minimization problem.

Nonsingularity of $P(t)$: In the sequel, it proves we shall require that $P(t)$ be nonsingular. In the following lemma, we give a reasonable condition for *any* nonnegative symmetric $P(t)$ satisfying the sufficiency condition (12) to be nonsingular.

Lemma 3: Suppose $[F,H]$ is observable for all time, in the sense that if for any τ, one has

$$H'(t)\Phi(t,\tau)x_0 = 0$$

for all t, then $x_0 = 0$. Then any nonnegative symmetric $P(t)$ satisfying (12) is nonsingular.

Proof: Suppose $P(\tau)x_0 = 0$ for some τ and some $x_0 \neq 0$. Set $X(t) = \Phi'(t,\tau)P(t)\Phi(t,\tau)$ so that $\dot{X}(t) = \Phi'(t,\tau)[PF + F'P + \dot{P}]\Phi(t,\tau) \leq 0$ by (12). Now $P(\tau)x_0 = 0$ implies $X(\tau)x_0 = 0$, which in turn implies $X(t)x_0 = 0$ for all $t \geq \tau$, because $\dot{X}(t) \leq 0$. Then also $\dot{X}(t)x_0 = 0$ for all $t \geq \tau$. Hence

$$P(t)\Phi(t,\tau)x_0 = 0$$

$$[PF + F'P + \dot{P}]\Phi(t,\tau)x_0 = 0.$$

The nonnegativity of (12) then implies $H'(t)\Phi(t,\tau)x_0 = 0$, and so $x_0 = 0$, which is a contradiction.[4]

Lower Bound on $P(t)$: A variation on the preceding argument will show that if F, G, H, and R are bounded and $[F,H]$ is uniformly completely observable, then $P(t)$ is bounded below for all t, i.e., $P^{-1}(t)$ is bounded. (Arguments for the case of nonsingular R can be found in [7].)

For a summary of the results of this section, see Fig. 1.

III. Computation of $P(t)$

In this section, our aim is to suggest a computational procedure for obtaining a nonnegative symmetric matrix $P(t)$ satisfying

$$M(P) = \begin{bmatrix} -PF - F'P - \dot{P} & H - PG \\ (H - PG)' & R \end{bmatrix} \geq 0. \tag{12}$$

[4] A minor variation in the argument extends the conclusion to $P(t)$ satisfying (14) rather than (12), but this fact will not be used.

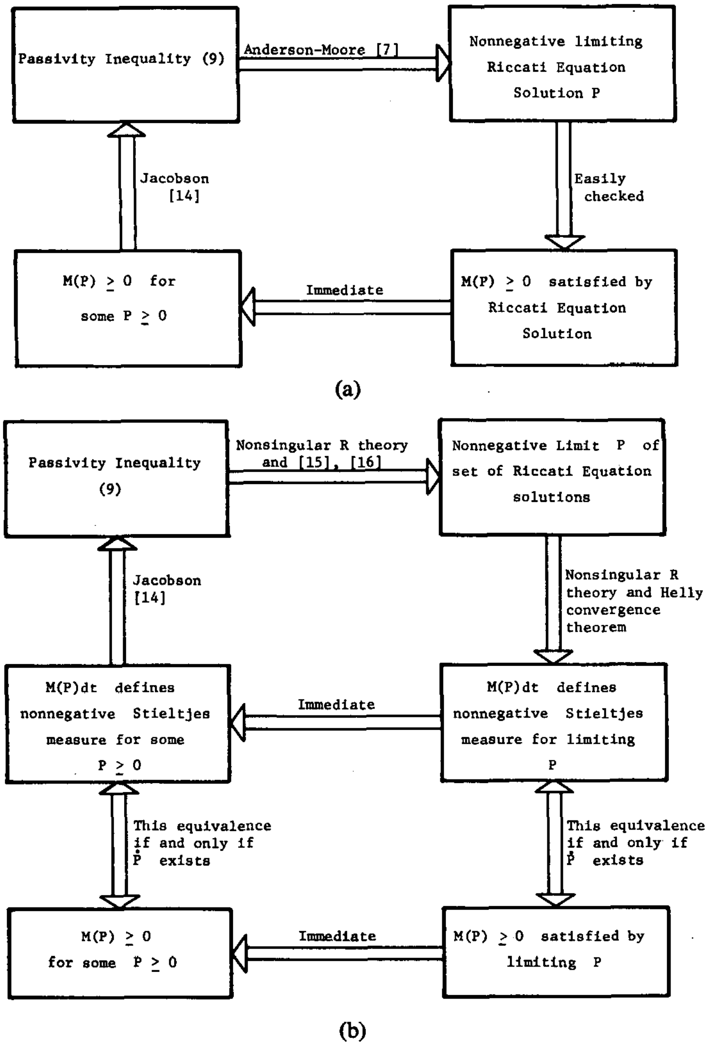

Fig. 1. Passivity conditions. (a) R nonsingular. (b) R singular.

Of course, if R is nonsingular, one such procedure is known. Our real interest is therefore in the case of singular R. Two points are then relevant. First, there may be no $P(\cdot)$ satisfying (12). In this section, we shall introduce extra conditions that will guarantee existence of such a $P(\cdot)$. The extra conditions in the main amount to demanding constancy of rank of certain quantities. Second, one could think of finding $P(\cdot)$ as the limit of the matrices P_ε defined earlier; while this is perfectly true, the procedure is obviously computationally unwieldy, it leaves untackled the question of whether the limiting $P(\cdot)$ satisfies (12) or merely the weaker condition (14), and it leaves open the question of whether, should the limiting $P(\cdot)$ not satisfy (12), there are other $P(\cdot)$ that do satisfy (12).

Let us therefore assume that $R(\cdot)$ is singular. In what follows, we shall present a recursive procedure which at each step of the recursion replaces the problem of finding a P satisfying (12) by one of finding a P satisfying a condition like (12), save that *either* the dimension of F is reduced, *or* the number of columns of G is reduced, *or* R is nonsingular. The recursion must terminate when either F shrinks to zero dimension, or G and H shrink to having zero columns, or one encounters a nonsingular R. (Actually, the preservation of complete controllability at each step of the recursion rules out the second possibility.) In case the recursion terminates with F of zero dimension, P has zero dimension and so the problem of finding it is vacuous. In case a nonsingular R is encountered, one can find a $P(\cdot)$ by the procedures already given.

Let us make several preliminary observations.

1) The problem of finding a nonnegative symmetric P to satisfy (12) is equivalent to the problem of finding P to satisfy (12) with $R(t)$, $G(t)$, and $H(t)$ replaced by, respectively, $\hat{R}(t) = S'(t)R(t)S(t)$, $\hat{G}(t) = G(t)S(t)$, and $\hat{H}(t) = H(t)S(t)$ for any nonsingular matrix $S(t)$ of differentiable entries. With obvious definition of $\hat{M}(P)$, clearly $\hat{M}(P) \geq 0$ if and only if $M(P) \geq 0$, and a similar statement holds in respect of the integral inequality (14). Notice too that if and only if the passivity condition (9) holds for (1), then the same condition with $\hat{H}$ replacing H and $\hat{R}$ replacing R holds for (1) with $\hat{G}$ replacing G (and, actually, $\hat{u} = S^{-1}u$ replacing u); also, Assumptions 1 and 2 are equivalent for the two sets of quantities.

2) The problem of finding a nonnegative symmetric P to satisfy (12) is equivalent to the problem of finding a nonnegative symmetric $\hat{P}$ to satisfy (12) with F, G, and H replaced, respectively, by $\hat{F} = TFT^{-1} + \dot{T}T^{-1}$, $\hat{G} = TG$, and $\hat{H} = (T^{-1})'H$. Here, T is an arbitrary nonsingular matrix with differentiable entries. If, then, $\hat{M}(\hat{P}) \geq 0$, then $M(P) \geq 0$ and conversely, where $P = T'\hat{P}T$. Again, a similar statement holds in respect of the integral inequality (14). The passivity condition (9) holds for (1) if and only if it holds with $\hat{H}$ replacing H in (9) and $\hat{F}$ and $\hat{G}$ replacing F and G in (1) (and, actually, $\hat{x} = Tx$ replacing x); also, Assumptions 1 and 2 are equivalent for the two sets of quantities. Finally, P is nonsingular if and only if $\hat{P}$ is nonsingular.

Observation 1) is concerned with input (or port) transformations, and observation 2) with coordinate-basis transformations. Interpretations from a synthesis point of view will be given subsequently to some of these transformations, but to avoid clouding the issue, we will postpone such interpretations. The recursive procedure now follows.

Step 1: Select a nonsingular $S(\cdot)$ of differentiable entries so that

$$\hat{R}(t) = S'(t)R(t)S(t) = \begin{bmatrix} \hat{R}_0(t) & 0 \\ 0 & 0_{p_1 \times p_1} \end{bmatrix}$$

with $\hat{R}_0$ nonsingular.

Assumption 3: $R(t)$ has constant rank. Under this assumption, it follows that $S(\cdot)$ exists, being definable by the Lagrange method [18] in terms of the entries of $R(\cdot)$. Set $\hat{G}(t) = G(t)S(t)$, $\hat{H}(t) = H(t)S(t)$, and now seek $P(t)$ satisfying $\hat{M}(P) \geq 0$. Drop the superscript hat.

Step 2—Input or Port Vector Dimension Reduction: Suppose $G(t)$ is of dimension $n \times r$. Partition it as $[G_1(t)\ G_2(t)]$ with $G_2(t)$ of dimension $n \times p_1$. Make the further assumption:

Assumption 4: $G_2(t)$ has constant rank $p \leq p_1$. If $p = p_1$, pass to Step 3. Otherwise, let $S_0(t)$ be a nonsingular

$p_1 \times p_1$ matrix with differentiable entries such that $G_2(t)S_0(t) = [\hat{G}_2(t)\ 0]$ with $\hat{G}_2(t)$ having p columns. Set $S(t) = I \oplus S_0(t)$ with the unit matrix of dimension $(r - p_1)$ and define $\hat{R}(t) = S'(t)R(t)S(t)$, $\hat{H}(t) = H(t)S(t)$, and $\hat{G}(t) = G(t)S(t)$. This yields, dropping the superscript hat again,

$$R(t) = \begin{bmatrix} R_0(t) & 0 \\ 0 & 0_{p_1 \times p_1} \end{bmatrix}$$

$$G(t) = [G_1(t) \quad G_2(t) \quad 0_{n \times (p_1 - p)}]$$

$$H(t) = [H_1(t) \quad H_2(t) \quad H_3(t)].$$

Now observe that we must have $H_3(t) \equiv 0$. Let $P(t)$ be that particular matrix appearing in the necessity condition (14). Writing $w(t) = [w_1'(t)\ w_2'(t)]'$ with $w_2(t)$ of dimension p_1, one obtains from (14)

$$\int_{t_0}^{t_1} w_1'(t)[dY(t)]w_1(t) + 2\int_{t_0}^{t_1} w_1'(t)H_3(t)w_2(t)\,dt \geq 0$$

for a certain $dY(t)$ whose form is inessential. Since the inequality holds for all continuous $w_1(t)$ and $w_2(t)$, one must have $H_3(t) = 0$ almost everywhere. Since $H(\cdot)$ is continuous, $H_3(t) \equiv 0$.

Now define

$$\hat{R}(t) = \begin{bmatrix} R_0(t) & 0 \\ 0 & 0_{p \times p} \end{bmatrix}$$

$$\hat{G}(t) = [G_1(t) \quad G_2(t)]$$

$$\hat{H}(t) = [H_1(t) \quad H_2(t)].$$

Though not covered in our preliminary observations, the effect of this transformation is easily seen. One has $\hat{M}(P) \geq 0$ if and only if $M(P) \geq 0$, and with a similar remark holding for the integral inequality (14); one has the passivity condition linking (1) and (9) for the superscript hat quantities if and only if the same is true for the original quantities, and likewise Assumptions 1 and 2 hold for the superscript hat quantities if and only if they hold for the original quantities.

If now $p = 0$, a matrix $P(t)$ satisfying $\hat{M}(P) \geq 0$ can be immediately found because $\hat{R}$ is nonsingular. Otherwise, proceed to Step 3, again dropping the superscript hat quantities.

Observe that the effect of this step has been to reduce the dimension of the matrices R, G, and H, i.e., the dimension of the vectors u and v appearing in (1) and (9).

Step 3—State-Space Dimension Reduction: Using the fact that $G_2(t)$ has p columns and rank p, select a coordinate-basis change matrix $T(t)$, nonsingular and with differentiable entries, such that

$$\hat{G}(t) = T(t)G(t) = \begin{bmatrix} G_{11}(t) & 0 \\ G_{21}(t) & I_{p \times p} \end{bmatrix}.$$

Define $\hat{F}$ and $\hat{H}$ by the usual formulas, and regard our problem as one of finding $\hat{P}$ such that $\hat{M}(\hat{P}) \geq 0$ (for, as noted earlier, P such that $M(P) \geq 0$ is given by $P = T'\hat{P}T$). Now drop the superscript hats again. Partition P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix}$$

with P_{22} of dimension $p \times p$, and partition H similarly to G. Now observe that the last p rows and columns of R are zero, which means that if P is to satisfy (12), viz., $M(P) \geq 0$, one must have the last p columns of $H - PG$ zero, whence

$$\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} = \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix}. \tag{15}$$

Therefore, the sufficiency condition (12) uniquely identifies three of the submatrices of any, and all, matrices P satisfying $M(P) \geq 0$. Actually, the necessity condition (14), known to be satisfied by a matrix P not necessarily satisfying (12), also yields $P_{12}(t) = H_{12}(t)$ and $P_{22}(t) = H_{22}(t)$ almost everywhere. An argument contained in [15, proof of theorem A1] then shows that the equalities hold everywhere.

In summary, under Assumptions 1–4, there exists one nonnegative symmetric $P(t)$ satisfying (14) and (15), and any nonnegative symmetric $P(t)$ satisfying (12) must also satisfy (15).

Now suppose that $[F(t),H(t)]$ is completely observable, so that any $P(t)$ of interest is nonsingular. In particular, $P_{22}(t) = H_{22}(t)$ must then be nonsingular. Define a further coordinate-basis change

$$T(t) = \begin{bmatrix} I & 0 \\ -H_{22}^{-1}(t)H_{12}'(t) & I \end{bmatrix}.$$

Obviously, $T(t)$ is nonsingular for all t. Now set $\hat{F} = TFT^{-1} + \dot{T}T^{-1}$, etc. One obtains

$$\hat{G} = \begin{bmatrix} G_{11}(t) & 0 \\ \hat{G}_{21}(t) & I \end{bmatrix} \qquad \hat{H}(t) = \begin{bmatrix} \hat{H}_{11}(t) & 0 \\ H_{21}(t) & H_{22}(t) \end{bmatrix}$$

$$\hat{P}(t) = \begin{bmatrix} \hat{P}_{11}(t) & 0 \\ 0 & H_{22}(t) \end{bmatrix}$$

with $\hat{P}_{11} = P_{11} - H_{12}H_{22}^{-1}H_{12}'$, and $\hat{G}_{21}$ and $\hat{H}_{11}$ defined in the obvious manner. [In computing $\hat{P}$, (15) is used.] This transformation of P applies both to the matrix known to satisfy the integral inequality (14), as well as any matrices that may satisfy the sufficiency condition (12). Dropping the superscript hats again, the sufficiency inequality (12) becomes

$$M(P) = \begin{bmatrix} -P_{11}F_{11} - F_{11}'P_{11} - \dot{P}_{11} & -P_{11}F_{12} - F_{21}'H_{22} & H_{11} - P_{11}G_{11} & 0 \\ -F_{12}'P_{11} - H_{22}F_{21} & -H_{22}F_{22} - F_{22}'H_{22} - \dot{H}_{22} & H_{21} - H_{22}G_{21} & 0 \\ H_{11}' - G_{11}'P_{11} & H_{21}' - G_{21}'H_{22} & R_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \geq 0 \tag{16}$$

and the necessity inequality (14) exhibits a similar pattern.

With reference to both inequalities, make the following definitions:

$$\hat{P} = P_{11} \qquad \hat{F} = F_{11}$$
$$\hat{G} = [F_{12} \quad G_{11}] \qquad \hat{H} = [-F_{21}'H_{22} \quad H_{11}]$$
$$\hat{R} = \begin{bmatrix} -H_{22}F_{22} - F_{22}'H_{22} - \dot{H}_{22} & H_{21} - H_{22}G_{21} \\ H_{21}' - G_{21}'H_{22} & R_0 \end{bmatrix}. \tag{17}$$

Then the necessity condition for P translates into a condition that there necessarily exists a nonnegative symmetric $\hat{P}$ (nonsingular with P nonsingular) such that

$$\int_{t_0}^{t_1} \hat{w}'(t) \begin{bmatrix} (-\hat{P}\hat{F} - \hat{F}'\hat{P})\,dt - d\hat{P} & (\hat{H} - \hat{P}\hat{G})\,dt \\ (\hat{H} - \hat{P}\hat{G})'\,dt & \hat{R}\,dt \end{bmatrix} \hat{w}(t) \geq 0$$

for all continuous $\hat{w}(\cdot)$, and the problem of finding a positive definite symmetric P satisfying the sufficiency condition (16) becomes a problem of finding a positive definite symmetric $\hat{P}$ such that

$$\hat{M}(\hat{P}) = \begin{bmatrix} -\hat{P}\hat{F} - \hat{F}'\hat{P} - \dot{\hat{P}} & \hat{H} - \hat{P}\hat{G} \\ \hat{H}' - \hat{G}'\hat{P} & \hat{R} \end{bmatrix} \geq 0.$$

The crucial point is the reduction in state-space dimension achieved. If $\hat{R}$ is nonsingular, $\hat{P}$ can be obtained immediately. If not, then the stage is set for further applications of Steps 1–3, provided that Assumptions 1–4 continue to be satisfied. [Actually, Assumptions 1 and 2 will always continue to be satisfied. Assumption 1 certainly carries through to the penultimate part of Step 3, and it is in fact not hard to show that if the pair

$$\begin{bmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t) \end{bmatrix} \begin{bmatrix} G_{11}(t) & 0 \\ G_{21}(t) & I \end{bmatrix}$$

has the controllability property, so does the pair

$$[F_{11}] \qquad [F_{12} \quad G_{11}].$$

Carry through of the observability property follows similarly. Hence Assumption 1 remains in force. That Assumption 2 remains in force is also easily seen. On the other hand, Assumptions 3 and 4 need to be reinvoked each time one cycles through Steps 1–3.] A diagrammatic representation of the algorithm is shown in Fig. 2.

Of course, the repeated cycling through Steps 1–3 must end; either a nonsingular R matrix is encountered, or a dimension shrinks to zero (as when, for example, in Step 3, the matrix P becomes identical with P_{22}). In case a nonsingular R is never encountered, the whole of the P matrix ends up becoming identified as in Step 3. Now, as pointed out in Step 3, the P matrix satisfying the integral inequality of the necessity condition is identified in this step; it must therefore be the case that the matrix $P(t)$ obtained in this way is also that obtained by the minimization procedure leading to the necessity condition. On the other hand, if a nonsingular R is encountered, the P matrix satisfying the sufficiency condition need not be that which solved the minimization problem, although the P matrix that is most easily found is precisely that satisfying the Riccati equation (10) and therefore the necessity condition. Either way then, the P matrix found by the procedures of this section can be made to coincide with that P defined by the minimization problem stated earlier.

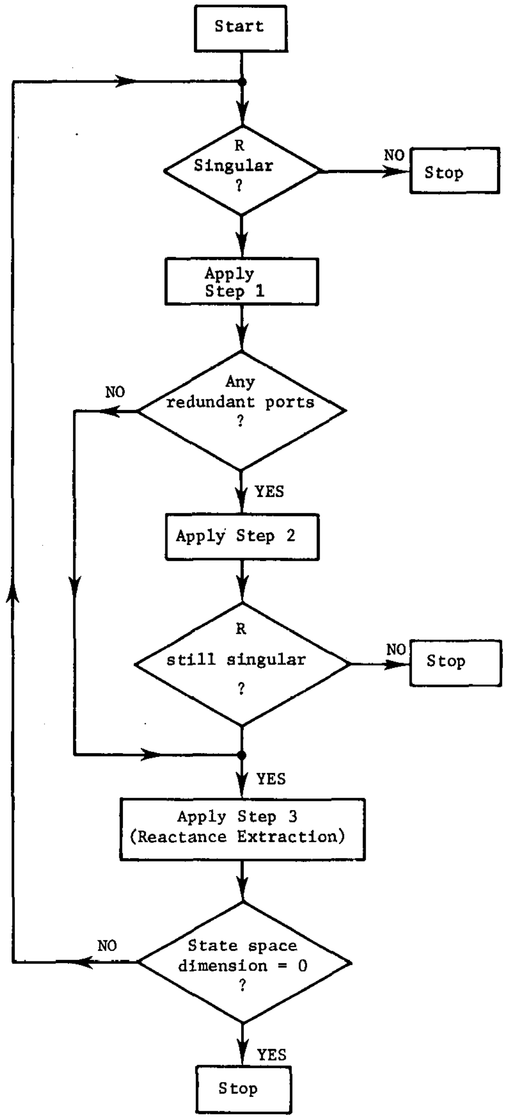

Fig. 2. Flow chart for algorithm of Section III.

Note, incidentally, that all the transformation matrices $T(t)$ and $S(t)$ introduced in the algorithm are nonsingular (for all t) and therefore rank preserving. It is not hard to deduce from this observation that the satisfaction or otherwise of Assumptions 3 and 4 depends only on the original $Z(t,\tau)$ and is independent of the particular method used to choose $T(t)$ and $S(t)$ at each step of the algorithm. Unfortunately, there does not appear to be a simple check in terms of the matrices $F(t)$, $G(t)$, $H(t)$, and $J(t)$ defining $Z(t,\tau)$ on whether Assumptions 3 and 4 are in fact satisfied, other than by attempting to apply the algorithm of this section.

Network theoretic interpretations of these steps will be given in Section V. In the next section, we indicate the synthesis procedure.

IV. Synthesis Procedure

In this section, we return to the problem of synthesizing

$$Z(t,\tau) = J(t)\delta(t-\tau) + H'(t)\Phi(t,\tau)G(\tau)1(t-\tau). \quad (2)$$

We shall suppose that the various assumptions given previously have allowed the derivation of a nonnegative symmetric $P(t)$ such that

$$\begin{bmatrix} -PF - F'P - \dot{P} & H - PG \\ (H - PG)' & J + J' \end{bmatrix} \geq 0 \quad (12)$$

and we shall suppose further that $[F,H]$ is completely observable for all time, so that $P(t)$ is nonsingular, by Lemma 3. Since $P(\cdot)$ is differentiable, we can find a nonsingular $T(t)$ of differentiable entries (by, for example, the Lagrange method) such that $P(t) = T'(t)T(t)$. Now set $\hat{F} = TFT^{-1} + \dot{T}T^{-1}$, $\hat{G} = TG$, and $\hat{H} = (T^{-1})'H$. Then (3) becomes

$$\begin{bmatrix} -(\hat{F} + \hat{F}') & \hat{H} - \hat{G} \\ (\hat{H} - \hat{G})' & J + J' \end{bmatrix} \geq 0. \quad (18)$$

Synthesis is now immediate, by either the reactance extraction [9], [11] or resistance extraction [10]–[12] approach. Since a time-varying version of the resistance extraction procedure is to be found in [12], we shall summarize only the reactance extraction procedure, which is almost the same as for the time-invariant case. Equation (18) implies that the matrix

$$N = \begin{bmatrix} -\hat{F} & \hat{H} \\ -\hat{G} & \hat{J} \end{bmatrix} \quad (19)$$

has a symmetric part that is nonnegative definite. This means that the time-varying impedance matrix

$$Z_c(t,\tau) = \begin{bmatrix} -\hat{F}(t) & \hat{H}(t) \\ -\hat{G}(t) & \hat{J}(t) \end{bmatrix} \delta(t-\tau) \quad (20)$$

satisfies the passivity condition. It is easily synthesized as the series connection of syntheses of $\frac{1}{2}(N + N')\delta(t-\tau)$ and $\frac{1}{2}(N - N')\delta(t-\tau)$, which are achievable, respectively, with transformer-coupled resistors and transformer-coupled gyrators. The only difference from the time-invariant case [9], [11] is that the transformers are time varying.

As for the time-invariant case, termination of the first $n = \dim \hat{F}$ ports of the network synthesizing $Z_c(t,\tau)$ in unit inductors yields a synthesis of $Z(t,\tau)$.

As remarked in Section III, the matrix $P(t)$ found by the procedures of Section III can be taken to be the solution of the minimization problem introduced in Section II. If this is the case, if F, G, H, and J are bounded, and if $[F,G]$ and $[F,H]$ are, respectively, uniformly completely controllable and uniformly completely observable, $P(t)$ and $P^{-1}(t)$ are bounded. If also $\dot{P}$ is bounded,[5] then $T(t)$, $\hat{F}(t)$, $\hat{G}(t)$, and $\hat{H}(t)$ will be bounded. The result is that a synthesis using bounded components follows, as argued in [12].

[5] In the event that $P(t)$ is defined by the Riccati equation (10) and $R^{-1}(t)$ is bounded, a bound on $\dot{P}$ follows from bounds on F,G,H, and P. If $R(t)$ is singular, boundedness of $\dot{P}(t)$ does not follow without various boundedness conditions on the state coordinate basis and port transformation matrices introduced in Section III.

V. Alternative Synthesis Procedure

In Section IV, a synthesis of $Z(t,\tau)$ was achieved after solving for the matrix $P(t)$ appearing in the passivity conditions; as was shown in Section III, this matrix may actually be computed by a sequence of transformations on the various matrices defining state-space equations for $Z(t,\tau)$. It will now be shown that *each of the transformations of Section III has a simple physical interpretation in terms of the network to be realized.* Using this observation, it is possible to carry out the synthesis in parallel with the computation of $P(t)$. Although the resulting synthesis does not differ substantially from that of Section IV, the procedure to be described does illustrate vividly the motivation for and implications of each step used in computing $P(t)$.

Suppose, as in earlier sections, that the network to be synthesized is described by the equations

$$\dot{x} = F(t)x + G(t)u$$
$$v = H'(t)x + J(t)u.$$

The earlier computations have, in effect, been described in terms of transformations on the matrices $F(t)$, $G(t)$, $H(t)$, and $J(t)$. However, one can also interpret them as transformations on the vectors x, u, and v. Basically, there are three classes of transformations used.

Class A—Port Vector Transformations: Here, a new set of port vectors $\hat{u}$ and $\hat{v}$ are defined via

$$\hat{v} = S'(t)v \qquad \hat{u} = S^{-1}(t)u$$

for some nonsingular matrix $S(t)$. These equations may be written more compactly as

$$\begin{bmatrix} \hat{v} \\ u \end{bmatrix} = \begin{bmatrix} 0 & S'(t) \\ -S(t) & 0 \end{bmatrix} \begin{bmatrix} -\hat{u} \\ v \end{bmatrix}.$$

Clearly, the new port variables may be derived from the original u and v via a transformer of turns-ratio matrix $S^{-1}(t)$. The state equations are now of the form

$$\dot{x} = \hat{F}(t)x + \hat{G}(t)\hat{u}$$
$$\hat{v} = \hat{H}'(t)x + \hat{J}(t)\hat{u}$$

where

$$\hat{F} = F \quad \hat{G} = GS \quad \hat{H} = HS \quad \text{and} \quad \hat{J} = S'JS.$$

Class B—State Vector Transformations: By setting $\hat{x} = T(t)x$, for some nonsingular $T(t)$, the appropriate changes are now $\hat{F} = \dot{T}T^{-1} + TFT^{-1}$, $\hat{G} = TG$, $\hat{H} = (T^{-1})'H$, and $\hat{J} = J$. Note, however, that this merely implies a change in the internal description of the network. The port vectors are not affected, so that no corresponding synthesis step is implied.

Class C—Partitioning of the State Vector: If x is partitioned so that $x' = [x_1' \; x_2']$, it may be that the network can be viewed as the interconnection of two smaller networks, with state vectors x_1 and x_2.

The synthesis procedure will now be described. Step 0 has no direct equivalent in terms of the earlier calculations, since these deal only with the symmetric part of J. The remaining steps parallel the corresponding steps in Section III.

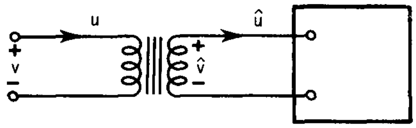

Fig. 3. Port transformer extraction.

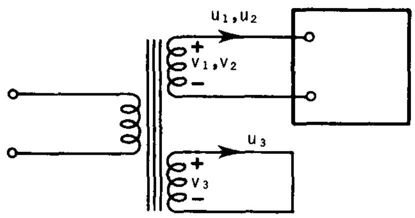

Fig. 4. Elimination of redundant ports.

Step 0—Gyrator Extraction: Set

$$\hat{v} = H'(t)x + \hat{J}(t)u$$

where

$$\hat{J}(t) = \tfrac{1}{2}[J(t) + J'(t)].$$

Then $v = \hat{v} + \tfrac{1}{2}[J(t) - J'(t)]u$, where the second term may be realized as a multiport transformer terminated in gyrators [5], as for the time-invariant case [13].

Step 1: This is simply a transformation of Class A given previously. A transformer is extracted from the network, as shown in Fig. 3.

Step 2: Again, a transformer is used. At this point, the new state equations reduce—with a suitable partitioning of u and v—to the form

$$\dot{x} = F(t)x + [G_1(t) \quad G_2(t) \quad 0]\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} H_1'(t) \\ H_2'(t) \\ 0 \end{bmatrix} x + \begin{bmatrix} J_0(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

where u_3 and v_3 each have $(p_1 - p)$ entries. Clearly, u_3 does not affect the equations, and v_3 is identically zero. The last $(p_1 - p)$ ports of the transformer should therefore be terminated in short circuits, as shown in Fig. 4.

The multiport transformer of this step is actually cascaded with the transformer produced in Step 1 and, moreover, some of the turns are made redundant by the short circuits. These effects could, of course, be combined to give a single multiport transformer.

Step 3: This step begins with two coordinate-basis changes, which as discussed previously do not affect the external behavior of the network. With the obvious partitioning of x, u, and v, the state equations are then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11}(t) & F_{12}(t) \\ F_{21}(t) & F_{22}(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_{11}(t) & 0 \\ G_{21}(t) & I \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} H_{11}'(t) & H_{21}'(t) \\ 0 & H_{22}'(t) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} J_0(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

with $H_{22}(t)$ nonnegative definite symmetric for all t.

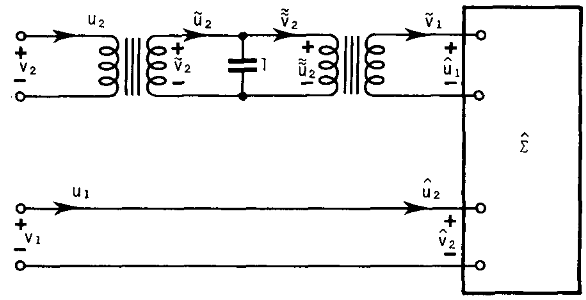

Fig. 5. Reactance extraction at ports. ($\tilde{\tilde{u}}^2$ and $\tilde{\tilde{v}}^2$ are represented by $\boldsymbol{\tilde{u}}_2$ and $\boldsymbol{\tilde{v}}_2$ in text.)

Now let $H_{22}(t)$ be factored as $S(t)S'(t)$, and define the following sequence of transformations:

i) $\tilde{u}_2 = S'u_2$ and $v_2 = S\tilde{v}_2$
ii) $\boldsymbol{\tilde{u}}_2 = \tilde{v}_2$ and $\tilde{v}_2 = \tilde{u}_2 - \dot{\boldsymbol{\tilde{v}}}_2$
iii) $\hat{v}_1 = S\boldsymbol{\tilde{v}}_2$ and $\boldsymbol{\tilde{u}}_2 = S'\hat{u}_1$
iv) $\hat{v}_2 = v_1$ and $\hat{u}_2 = u_1$.

Transformations i) and iii) represent transformers of turns ratio $S'(t)$ and $S(t)$, respectively, while iv) is a simple renumbering of the ports. The second transformation represents the insertion of 1-F shunt capacitors. At the same time, the roles of u and v have been exchanged: $\boldsymbol{\tilde{u}}_2$ is a vector of voltages, and $\boldsymbol{\tilde{v}}_2$ a vector of currents. In effect, the "x_2" part of the network has been inverted, which allows the extraction of a shunt capacitor. The resulting network is shown in Fig. 5.

To find the state equations of the remainder of the network ($\hat{\Sigma}$ in Fig. 5), recall that $H_{22}(t)$ is nonsingular. Then it is readily shown that $\hat{u}_1 = x_2$, and thence that

$$\dot{x}_1 = \hat{F}x_1 + \hat{G}\hat{u}$$
$$\hat{v}_1 = \hat{H}'x_1 + \hat{J}\hat{u}$$

where

$$\hat{u}' = [\hat{u}_1' \quad \hat{u}_2'] \qquad \hat{v}' = [\hat{v}_1' \quad \hat{v}_2']$$

and

$$\hat{F} = [F_{11}]$$
$$\hat{G} = [F_{12} \quad G_{11}]$$
$$\hat{H} = [-F_{21}'H_{22} \quad H_{11}]$$
$$\hat{J} = \begin{bmatrix} -\dot{S}S' + H_{22}'F_{22} & -H_{22}'G_{21} \\ H_{21}' & J_0 \end{bmatrix}.$$

After setting $\hat{R} = \hat{J} + \hat{J}'$, these definitions agree with those of Section III.

The matrices $\hat{F}$, $\hat{G}$, $\hat{H}$, and $\hat{J}$ now define state-space equations of a hybrid matrix, which in general will not be the same as an impedance or admittance matrix. Accordingly, further cycles of the synthesis procedure involve interpretations different in detail to those given previously. We shall not pursue this point further here.

VI. Conclusions

The primary aim of this paper has been to derive synthesis techniques for a certain class of time-varying impedance matrices; two such methods have been described, in Sections IV and V. However, one also gains considerable insight into the structure of the class of matrices $P(t)$ that appear in the necessary and sufficient conditions of Section II. In general, there will be a number of such matrices, but an important point established in Section III is that—in an appropriate coordinate basis—one can identify certain submatrices of *any and all* matrices $P(t)$ satisfying the necessary (and, of course, the sufficient) conditions. In other words, the set of all solutions $P(t)$ lies in a linear manifold. This property only holds, of course, for what we have termed the "singular" problem.

It is of some interest to examine the significance of the constancy of rank assumptions of Section III. An examination of those parts of the algorithm depending on them shows that the assumptions are, in fact, sufficient conditions for existence of matrices $P(t)$ satisfying both the necessary conditions and also the slightly weaker sufficiency conditions for passivity. Equivalently, the assumptions imply that there exists at least one $P(t)$, satisfying the Stieltjes integral inequality (14), that is also differentiable. Whether these assumptions are also necessary for this to occur is still an open question. A further open question is whether the algorithm could be made to work if the constancy of rank assumptions failed to hold. It would appear that one could break up the time interval under consideration into a number of subintervals in which the assumptions did hold, and apply the algorithm in each of these time intervals separately; the final result should be a piecewise continuous $P(t)$. Corresponding to each rank change there would be a finite discontinuity in $P(t)$, which would in turn correspond to a structural change (for example, the switching in of a new capacitor) in the final synthesis. The major difficulty in this approach is, of course, the matching of boundary conditions at times where jumps in $P(t)$ occur.

It is important to appreciate that such discontinuous solutions may also occur even when the assumptions of this paper are in force. In practice, however, one tends to ignore these solutions on the grounds that the resulting synthesis would be unworkable, both computationally and in terms of actual construction (this should be clear from Section IV). On the other hand, all continuous solutions $P(t)$ are realistic candidates for use in the procedures outlined or referenced in Section IV. Since it has been shown that the restrictions imposed on $P(t)$ actually apply to all possible solutions, it follows that the algorithm is actually capable of generating all solutions—none have been discarded at any step of the procedure. This is important in a practical sense, at least for time-invariant problems, for one would actually choose different solutions $P(t)$ depending on whether one is interested in, for example, a reciprocal synthesis or one using the minimum possible number of resistors.

Appendix I
Definitions of Time-Variable Components

With $v(t)$ and $u(t)$ denoting voltage and current, respectively, define the linear time-varying resistor, capacitor, and inductor by

$$v(t) = r(t)u(t) \qquad u(t) = \frac{d}{dt}(c(t)v(t)) \qquad v(t) = \frac{d}{dt}(l(t)u(t)).$$

Passivity is guaranteed by $r(t) \geq 0$; $c(t) \geq 0$, $\dot{c}(t) \geq 0$; $l(t) \geq 0$, $\dot{l}(t) \geq 0$.

In terms of primary and secondary port voltage vectors $v_1(t)$ and $v_2(t)$ and associated port current vectors, define the time-variable transformer by $v_1(t) = T'(t)v_2(t)$, $u_2(t) = -T(t)u_1(t)$, the turns-ratio matrix being $T(t)$. The time-variable transformer is lossless.

The time-varying gyrator is a two-port device defined as the obvious generalization of the time-invariant gyrator: $v_1(t) = \gamma(t)i_2(t)$, $v_2(t) = -\gamma(t)i_1(t)$. It is lossless.

Appendix II
Proof of Lemma 1

We first show that $N = 1$. Suppose the contrary, and choose

$$u(t) = \left[\frac{(t-t_a)^{N-1}}{(N-1)!}1(t-t_a) + c\frac{(t-t_b)^{N-1}}{(N-1)!}1(t-t_b)\right]\beta$$

where t_a and $t_b > t_a$ are arbitrary, c is an arbitrary constant, and β an arbitrary vector. This yields

$$E(t_a,t_b+,u(\cdot)) = c\frac{(t_b-t_a)^{N-1}}{(N-1)!}\beta' Z_N(t_b)\beta + \text{(terms independent of } c)$$

for all $N > 1$ (but not actually for $N = 1$). Hence $Z_N(t)$ must be skew for all t if (6) always holds. Now take a new $u(\cdot)$ as

$$u_k(t) = \frac{(t-t_a)^{N-1}}{(N-1)!}1(t-t_a)$$

$$u_j(t) = c\frac{(t-t_b)^{N-1}}{(N-1)!}1(t-t_b)$$

$$u_l(t) = 0, \qquad l \neq k,j.$$

This yields

$$E(t_a,t_b+,u(\cdot)) = -c(Z_N)_{jk}(t_b)\frac{(t_b-t_a)^{N-1}}{(N-1)!} + \text{(terms independent of } c).$$

It follows that $Z_N(t) \equiv 0$.

Now consider $Z_1(t)$. Take $u(t) = (t-t_a)1(t-t_a)\beta$ for arbitrary β and t_a. For $t - t_a$ small, $t \geq t_a$, we obtain

$$E(t_a,t,u(\cdot)) = \tfrac{1}{2}\beta' Z_1(t)\beta(t-t_a)^2 + [\text{terms of higher order in } (t-t_a)]$$

which shows that $Z_1(t) + Z_1'(t) \geq 0$. To show that $Z_1(t)$ is symmetric, take

$$u_k(t) = (t-t_a)1(t-t_a)$$

$$u_j(t) = \sin\omega(t-t_a)u_k(t)$$

$$u_l(t) = 0, \qquad l \neq k,j.$$

For $t - t_a$ and $\omega(t-t_a)$ small, we obtain

$$E(t_a,t,u(\cdot)) = \tfrac{1}{2}[z_{kk}(t_a) + z_{jj}(t_a)\sin^2\omega(t - t_a) + (z_{kj}(t_a) + z_{jk}(t_a))\sin\omega(t - t_a)](t - t_a)^2 + \frac{\omega}{6}(z_{kj}(t_a) - z_{jk}(t_a))\cos\omega(t - t_a)(t - t_a)^3 + [\text{terms of order 3 and higher in } (t - t_a), \text{ and bounded in } \omega].$$

Since ω is arbitrary, we obtain $z_{kj}(t_a) = z_{jk}(t_a)$, i.e., $Z_1(t) = Z_1'(t)$.

Now let $T(t)$ be an arbitrary matrix such that $T'(t)T(t) = Z_1(t)$. Use the identity

$$T(t)\delta^{(1)}(t - \tau) = \delta^{(1)}(t - \tau)T(\tau) - T^{(1)}(t)\delta(t - \tau)$$

to write $Z(t,\tau)$ as

$$Z(t,\tau) = T'(t)\delta^{(1)}(t - \tau)T(\tau) + J(t)\delta(t - \tau) + H'(t)\Phi(t,\tau)G(\tau)1(t - \tau)$$

for some $J(t)$. One then has

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} u'(t)T'(t)\delta^{(1)}(t - \tau)T(\tau)u(\tau)\,dt\,d\tau = \tfrac{1}{2}u'(t_1)T'(t_1)T(t_1)u(t_1) \geq 0$$

and

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1} u'(t)[J(t)\delta(t - \tau) + H'(t)\Phi(t,\tau)G(\tau)1(t,\tau)]u(\tau)\,dt\,d\tau = \int_{t_0}^{t_1}\int_{t_0}^{t_1} u'(t)Z(t,\tau)u(\tau)\,dt\,d\tau - \tfrac{1}{2}u'(t_1)T'(t_1)T(t_1)u(t_1).$$

The left-hand side will vary by an arbitrarily small amount when $u(\cdot)$ is perturbed smoothly to yield $u(t_1) = 0$. Then (6) implies the passivity property for $\hat{Z}(\cdot,\cdot)$.

Acknowledgment

The authors wish to thank J. B. Moore and L. M. Silverman for assisting in the development of some of the ideas of this paper.

References

[1] D. A. Spaulding, "Passive time-varying networks," Ph.D. dissertation, Stanford University, Stanford, Calif., 1964.
[2] ——, "Lossless time-varying impedance synthesis," *Electron. Lett.*, vol. 1, pp. 165–167, Aug. 1965.
[3] R. Saeks, "Synthesis of lossless time-variable networks," in *Proc. 2nd Princeton Conf. Circuits and Systems*, 1968, p. 404.
[4] B. D. O. Anderson, "Cascade synthesis of time-varying nondissipative *N*-Ports," Stanford Electron. Lab., Stanford, Calif., Tech. Rep. 6559-3, Mar. 1967.
[5] ——, "Properties of time-varying *N*-port impedance matrices," in *Proc. 8th Midwest Symp. Circuit Theory* (Fort Collins, Colo., June 1965), pp. 8-0–8-13.
[6] R. A. Rohrer, "Lumped network passivity criteria," *IEEE Trans. Circuit Theory*. vol. CT-15, pp. 24–31, Mar. 1968.
[7] B. D. O. Anderson and J. B. Moore, "Extensions of quadratic minimization theory II, infinite time results," *Int. J. Contr.*, vol. 7, May 1968.
[8] ——, "Procedures for time-varying impedance synthesis," *Proc. 11th Midwest Symp. Circuit Theory*, May 1968, pp. 17–26.
[9] B. D. O. Anderson and R. W. Newcomb, "Impedance synthesis via state-space techniques," *Proc. Inst. Elec. Eng.*, vol. 115, pp. 928–936, July 1968.
[10] B. D. O. Anderson and R. W. Brockett, "A multiport state-space Darlington synthesis," *IEEE Trans. Circuit Theory* (Corresp.), vol. CT-14, pp. 336–337, Sept. 1967.
[11] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis—A Modern Systems Theory Approach.* Englewood Cliffs, N.J.: Prentice-Hall, 1973, p. 548.
[12] L. M. Silverman, "Synthesis of impulse response matrices by internally stable and passive realizations," *IEEE Trans. Circuit Theory*, vol. CT-15, pp. 238–245, Sept. 1968.
[13] R. W. Newcomb, *Linear Multiport Synthesis.* New York: McGraw-Hill, 1966.
[14] D. H. Jacobson, "Totally singular quadratic minimization problems," *IEEE Trans. Automat. Contr.* (*Special Issue on Linear-Quadratic-Gaussian Problems*), vol. AC-16, pp. 651–658, Dec. 1971.
[15] D. H. Jacobson and J. L. Speyer, "Necessary and sufficient conditions for optimality for singular control problems: A limit approach," *J. Math. Anal. Appl.*, vol. 34, pp. 239–266, May 1971.
[16] D. H. Jacobson, S. B. Gershwin, and M. M. Lele, "Computation of optimal singular controls," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 67–73, Feb. 1970.
[17] G. E. Shilov and B. L. Gurevich, *Integral, Measure and Derivative: A Unified Approach.* Englewood Cliffs, N.J.: Prentice-Hall, 1966.
[18] F. R. Gantmacher, *The Theory of Matrices.* New York: Chelsea, 1959.