When do the Manley–Rowe relations really hold?

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Synopsis

Difficulties arising because of a standard assumption in the use of the Manley–Rowe relations, that generator frequencies are incommensurable, are discussed from a physical and mathematical viewpoint, and relations are developed which are valid for any generator frequencies, by making realistic assumptions concerning the generators.

The use of these relations is shown to be equivalent to the use of the standard relations when the assumption is dropped that filters used in the circuitry have infinitely narrow bandwidths.

1 Introduction

The Manley–Rowe relations,[1] now with us for about ten years, have been tested in their application by parametric amplifiers, convertors, and the like. Since the original derivation, other derivations[2,3] have appeared, and, to the best of the author's knowledge, all these derivations require certain frequencies to be incommensurable; i.e. they cannot be expressed as the ratio of two integers.

At first glance, such a requirement seems, *per se*, not entirely in accord with physical reality. The question whether incommensurability is a valid assumption was raised by the author in Reference 4; in this paper I suggest a different requirement that has a more physical motivation, and rederive the relations to incorporate the modification.

While signals at incommensurate frequencies are inevitably (phase) incoherent, this may or may not be the case with waveforms at commensurate frequencies. We propose here that the incommensurability requirement of Manley and Rowe can more accurately be thought of as a requirement that all generators in the circuit should be incoherent, without any restriction on the frequencies of the generators.

To be sure, this implies, in the case of harmonically related generators, that their instantaneous frequencies should be 'almost always' incommensurable (as the irrational numbers are infinitely denser than the rational numbers). But it seems we cannot conclude immediately that the Manley–Rowe relations are valid, since they were derived in terms of steady-state frequencies.

Accordingly, the approach will be to argue on physical grounds that the incommensurability condition should be replaced by an assumption of the incoherency of the generators (Section 2); in Section 3 a modified version of the original proof of the relations is presented, and in Section 4 the practical application of the proof is discussed, resolving points that seem to have been inadequately covered in the literature.

2 Physical considerations

This Section will primarily be concerned with what might be termed the 'causative' elements in a circuit to which it is desired to apply the Manley–Rowe relations, the causative elements being the sources or generators connected to the network.

Sinusoidal generators are common enough devices in practical and theoretical engineering work; commonly their physical behaviour may be modelled by assuming they can be represented by an impedance in series with a voltage source $e = E_0 \sin \omega t$. For most applications this model permits a mathematical analysis, the results of which correspond to physical reality; i.e. the model is perfectly satisfactory. But it is a model, and, in a given situation, the model may therefore not be perfectly satisfactory, because it is not an accurate enough description of the physical object. For example, it may be important to take into consideration the fact that we cannot in practice produce a discrete line spectrum from a generator; the generator must have been constructed and then switched on at some finite time, and accordingly its frequency spectrum will be nondiscrete. As a second example, and one that will concern us more here, we note that it is impossible to construct a generator of completely stable (i.e. constant) frequency. To say, as we shall, that a generator has frequency ω is really to say that its instantaneous output frequency is approximately ω, and averages out at ω. We could then very reasonably choose to model the generator voltage by $e = \mathrm{Re}\,(E_0 e^{j\phi} e^{j\omega t})$, where ϕ is a phase angle that varies slowly in time, with a mean zero.

If we consider several generators operating simultaneously, at frequencies $\omega_1, \omega_2, \ldots \omega_p$, where there are no restrictions on the incommensurability or otherwise of ω_i, and if all the generators are generating incoherently, i.e. there is no electrical connection between them that serves as a synchronising device, it is clear that we could model the generators using voltages $e_i = \mathrm{Re}\,(E_{0i} e^{j\phi_i} e^{j\omega_i t})$, where the ϕ_i could be assumed to vary independently. Thus, in addition to $\overline{\phi_i} = 0$ for each i, we have, for example, $\overline{\phi_i \phi_j} = 0$ for each i, j with $i \neq j$, where the averaging is assumed to take place over a sufficiently long time. It is, moreover, reasonable to assume that the random processes described by ϕ_i are ergodic; i.e. ensemble and time averages are the same.

Of course, if there is synchronisation between the generators, this model is invalid. For example, if two generators could be described by $e_1 = \mathrm{Re}\,(E_1 e^{j\phi_1} e^{j\omega_1 t})$ and $e_2 = \mathrm{Re}\,(E_2 e^{j2\phi_1} e^{j2\omega_1 t})$, so that $\phi_2 = 2\phi_1$, the expected value of $\phi_1 \phi_2$ is evidently nonzero.

Taking another step toward a consideration of the basic problem, let us now consider a circuit containing a number of sources, sinks and a nonlinear reactive element, without loss of generality assumed capacitive. We shall assume that the sources are a set of incoherent generators of frequencies $\omega_i (i = 1, 2, \ldots, p)$. The voltages and currents present in the circuit, and the charge on the capacitor, can be regarded not merely as the superposition of various harmonics, but as the superposition of various harmonics which are somehow dependent on ϕ_i.

If merely one generator is present, we can suppose that the current i through the reactive element is given by

$$i = \sum_{m_1=-\infty}^{+\infty} I_{m_1} e^{jm_1\phi} e^{jm_1\omega_1 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here m_1 runs through all positive and negative integers, and I_{m1} is a complex number giving the magnitude of the m_1th harmonic when the dependence on the phase ϕ, which is a random variable, is separated out. It is not hard to see that the dependence on ϕ is precisely that given, by performing a simple Fourier expansion and recalling that ϕ is assumed to be slowly varying.

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If two generators of the same frequency are present (where by same frequency is really meant the same nominal frequency, but independent, slowly time-varying phases) it is intuitively clear that, for example, the fundamental component of i will certainly contain two components, one depending on ϕ_1 and one depending on ϕ_2. Harmonics may similarly be broken down into subcomponents, and we may, in general, write

$$i = \sum_{m_1=-\infty}^{+\infty} \sum_{m_2=-\infty}^{+\infty} I_{m_1m_2} e^{j(m_1\phi_1+m_2\phi_2)} e^{j(m_1+m_2)\omega_1 t} \quad . \quad . \quad (2)$$

The coefficients $I_{m_1m_2}$ in eqn. 2 can be calculated in terms of i in a manner stated explicitly in the next Section. A similar expansion applies for v, the voltage across the nonlinear reactance.

Consider now power flow at, for example, the second-harmonic frequency. This power flow can be computed as the sum of a number of component power flows. For example, the component of i associated with $e^{+2j\phi_1}$, namely $I_{20}e^{2j\phi_1}$, and the component of v associated with $e^{+2j\phi_1}$, namely $V_{20}e^{2j\phi_1}$, will combine together to give a contribution $\mathrm{Re}(I_{20}^* V_{20})$; the component of i associated with $e^{j(\phi_1+\phi_2)}$, namely $I_{11}e^{j(\phi_1+\phi_2)}$, and the similar component of v, namely $V_{11}e^{j(\phi_1+\phi_2)}$, give a contribution $\mathrm{Re}(I_{11}^* V_{11})$, and so on. There will, however, be no power flow associated with 'mixed' components of v and i. Consider, for example, the $e^{2j\phi_1}$ component of v and the $e^{j(\phi_1+\phi_2)}$ component of i; the power associated with these will be $\mathrm{Re}(I_{11}^* V_{20} e^{j(\phi_1-\phi_2)})$ averaged over the ensemble of ϕ_1 and ϕ_2. Since ϕ_1 and ϕ_2 are independent, however, $\overline{\exp j(\phi_1 - \phi_2)} = \overline{\exp j\phi_1}\ \overline{\exp(-j\phi_2)} = 0$.

More generally, the power flow at the rth harmonic frequency may be considered to be the sum of a number of different power flows, one associated with each distinct pair of integers m_1, m_2 so that $m_1 + m_2 = r$.

At this stage let us observe that, in the classic Manley–Rowe derivation, for two generators, the generator frequencies ω_1 and ω_2 are assumed incommensurable, in order that the various harmonics $m_1\omega_1 + m_2\omega_2$ will be distinct (and thus the power flows associated with each will be distinct) for distinct integer pairs m_1, m_2. We have shown that, for identical ω_1 and ω_2, and it follows for general ω_1 and ω_2 as long as we maintain our incoherence assumption, associated with a distinct integer pair m_1, m_2 there is defined a power flow at frequency $m_1\omega_1 + m_2\omega_2$, irrespective of whether other power flows are possible at this frequency. The requirement of a unique power flow associated with a pair m_1, m_2 is automatically fulfilled in the incommensurable case (as only one pair of m_1, m_2 can define any one frequency). Thus the incoherency assumption is principally for dealing with the general case, where generator frequencies are not necessarily incommensurable.

Accordingly the preceding analysis suggests the following. The incommensurability of generator frequencies in the Manley–Rowe relation is more meaningfully described by saying that the generators are incoherent, i.e. their phases are independent random variables. In the next Section we give a derivation, essentially a modification of Manley and Rowe's proof, of the relations under the incoherence, rather than the incommensurability, assumption.

3 Derivation of the relations

Consider a circuit containing a nonlinear capacitance, together with time-invariant passive elements and incoherent generators at frequencies $\omega_1, \omega_2, \ldots, \omega_p$. Suppose the phases of the generators are $\phi_1, \phi_2, \ldots, \phi_p$. These generators will establish various values of network variables throughout the circuit at frequencies $\sum_i m_i\omega_i$ (m_i integers). In particular, we may write for the instantaneous charge q on the nonlinear capacitor

$$q = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \ldots \sum_{m_p=-\infty}^{\infty} Q_{m_1m_2 \ldots m_p} e^{jm_1\phi_1} e^{jm_2\phi_2} \ldots e^{jm_p\phi_p} e^{j(m_1\omega_1+m_2\omega_2+\ldots+m_p\omega_p)t} \quad . \quad (3)$$

or, in briefer notation,

$$q = \sum_{\{m_i\}} Q_{\{m_i\}} e^{j\Sigma m_i\phi_i} e^{j\Sigma m_i\omega_i t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $Q_{\{m_i\}}$ is determined from q through

$$Q_{\{m_i\}} = \frac{1}{(2\pi)^p} \underbrace{\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi}}_{p} \overline{q e^{-j\Sigma m_i\phi_i} e^{-j\Sigma m_i\omega_i t}}\, d(\omega_1 t)d(\omega_2 t)\ldots d(\omega_p t) \quad . \quad (5)$$

In order to avoid difficulties associated with time-averaging, we assume an ensemble average over ϕ_i here, as in the discussion of power flows in Section 2.

If, for example, $p = 2$ and $\omega_1 = \omega_2$, then Q_{02}, Q_{11} and Q_{20} define three of infinitely many distinct components of q, all associated with the single frequency $2\omega_1$.

Note that, since q is real, we require

$$Q_{\{m_i\}} = Q^*_{\{-m_i\}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

We may perform similar expansions on i and v, the current through and the voltage across the capacitance, and obtain

$$i = \sum_{\{m_i\}} I_{\{m_i\}} e^{j\Sigma m_i\phi_i} e^{j\Sigma m_i\omega_i t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where, since $i = dq/dt$, using the slowness of the ϕ_i variation,

$$I_{\{m_i\}} = jQ_{\{m_i\}} \sum_i m_i\omega_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8a)$$

Also

$$V_{\{m_i\}} = \frac{1}{(2\pi)^p} \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} \overline{v e^{-j\Sigma m_i\phi_i} e^{-j\Sigma m_i\omega_i t}}\, d(\omega_1 t)d(\omega_2 t)\ldots d(\omega_p t) \quad . \quad (8b)$$

Now we may follow Manley and Rowe and multiply both sides of eqn. 8b by $jm_1Q^*_{\{m_i\}}$. Thus, assuming interchangeability of summation with integration and the averaging operation,

$$\sum_{\{m_i\}} jm_1Q^*_{\{m_i\}} V_{\{m_i\}} = \frac{1}{(2\pi)^p} \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} \overline{v \sum_{\{m_i\}} jm_1Q^*_{\{m_i\}} e^{-j\Sigma m_i\phi_i} e^{-j\Sigma m_i\omega_i t}}\, d(\omega_1 t)d(\omega_2 t)\ldots d(\omega_p t) \quad . \quad (9)$$

Using eqn. 3, the reality of q, and eqn. 6, the integrand on the right-hand side of eqn. 9 simplifies so that, with interchange of integration and averaging,

$$\sum_{\{m_i\}} jm_1Q^*_{\{m_i\}} V_{\{m_i\}} = -\frac{1}{(2\pi)^p} \overline{\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} v \frac{\partial q}{\partial(\omega_1 t)} d(\omega_1 t)d(\omega_2 t)\ldots d(\omega_p t)} \quad . \quad . \quad . \quad . \quad (10)$$

The derivation is now identical to that of Reference 1. The integral on the right-hand side is readily found to be zero, while the use of eqn. 8a and some easy manipulations on the left-hand side yield

$$\sum_{\{m_i\}} jm_1Q^*_{\{m_i\}} V_{\{m_i\}} = -\sum_{\{m_i\}} \frac{m_1 V_{\{m_i\}} I^*_{\{m_i\}}}{\sum_i m_i\omega_i}$$

$$= -\sum_{\{m_i\}} \frac{m_1 P_{\{m_i\}}}{\sum_i m_i\omega_i}$$

where $P_{\{m_i\}}$ is the power associated with the components of

v and i that are mutiplied by $e^{+j\sum m_i\phi_i}$ in the Fourier decompositions of these quantities.

Thus from eqn. 10 we obtain

$$\sum_{\{m_i\}} \frac{m_1 P_{\{m_i\}}}{\sum_i m_i \omega_i} = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

More generally, it is clear that

$$\sum_{\{m_i\}} \frac{m_k P_{\{m_i\}}}{\sum_i m_i \omega_i} = 0 \; (k = 1, 2, \ldots, p) \quad . \; . \; . \; . \quad (12)$$

These are the Manley–Rowe relations as originally derived, provided $P_{\{m_i\}}$ is suitably interpreted in the case where some of the ω_i are not incommensurable.

4 Practical considerations

Consider a circuit where the generator frequencies are nominally not incommensurable. For example, we may have a generator at frequency ω_1, and one at frequency $\omega_2 = 2\omega_1$, and we may be interested in the power in a sink at frequency $\omega_3 = 3\omega_1$. The relations in eqn. 12 apply, if we recognise that the power at frequency ω_3 is made up of an infinite set of subcomponents, e.g. P_{30}, P_{11}, P_{5-1} etc. In general, we will be unable to find all these components individually, or their sum, by using eqn. 12. How then can eqn. 12 be used in a practical situation?

The answer lies in the fact that even with incommensurable frequencies ω_1, ω_2 a difficulty analogous to that described above can arise, and the elimination of the difficulty, though not always possible, can be carried out in roughly corresponding fashions both when ω_1 and ω_2 are incommensurable and when they are not. Accordingly we shall consider the difficulty that arises in the incommensurable case. The extension to more than two generators is simple, and thus for convenience we shall restrict discussion to the two-generator case.

It is customary, in the usual Manley–Rowe derivations, to predicate the use of ideal filters—ideal in the sense that one and only one frequency is passed. The incoherence of the generated voltages will imply a finite spectral width for each source, and this in turn will require the use of nonideal filters. Moreover, in practice, any filter used will have a finite bandwidth, under any sensible definition of bandwidth, and accordingly we are led to drop the ideal filter assumption. But now, even if ω_1 and ω_2 are incommensurable, any filter that is designed to pass, for example, $\omega_1 + \omega_2$ must also pass components at frequency $m_1\omega_1 + m_2\omega_2$, where m_1 and m_2 are such that $m_1\omega_1 + m_2\omega_2$ and $\omega_1 + \omega_2$ are approximately equal.

Practically, these other components will act to all intents and purposes like noise perturbing the filter output signal at frequency $\omega_1 + \omega_2$. Practically also, the narrower the filter passband, the higher will be the values of m_1 and m_2 which enable $m_1\omega_1 + m_2\omega_2$ to fall within the passband of the filter, and thus also the smaller the magnitude of the perturbing signal. Thus, when we write the Manley–Rowe relations applying, for example, to an upconvertor, and assume that power flow occurs at only three discrete frequencies, we are inherently neglecting power flows at frequencies close to the three considered, on the grounds that the magnitudes of these power flows are negligible.

Of course, this may not be the case in a given situation, but in theory we can certainly analyse a given situation and estimate whether this is the case. Note, however, that the exact nature of the reactance nonlinearity will now be important in such an estimation.

Having dealt with the case of incommensurable ω_1 and ω_2, it is now relatively easy to see what happens when ω_1 and ω_2 are commensurable. Under these conditions there will be infinitely many components of power at frequency $\omega_1 + \omega_2$, with smaller magnitudes the larger m_1 and m_2 are, where $m_1\omega_1 + m_2\omega_2 = \omega_1 + \omega_2$. Now suppose ω_2 is changed, by an extremely small amount, to a frequency ω_2' incommensurable with ω_1. Then, with the same m_1, m_2 pairs as above, $m_1\omega_1 + m_2\omega_2'$ will approximately equal $\omega_1 + \omega_2$, differing more from $\omega_1 + \omega_2$ for the larger values of m_1 and m_2. In any case, a filter whose passband is centred at $\omega_1 + \omega_2$ will pass as many significant components in the incommensurable case as there are significant components in the commensurable case if ω_2' is chosen sufficiently close to ω_2.

Our conclusion is therefore that the problem associated with the commensurable frequency case, namely that we cannot determine all components of power at a given frequency, is paralleled by a problem in the incommensurable frequency case, namely that we cannot determine all components of power in a given frequency band.

In both cases we are forced to make an engineering approximation in order to perform calculations, namely that the magnitudes of the embarrassing components are negligible. The validity of this assumption will in part depend on the nature of the reactance nonlinearity and, of course, on the actual frequencies involved.

For example, if we had a signal at frequency ω_1, a pump at frequency $\omega_2 = 9/8\omega_1$, and desired to examine the output at frequency $\omega_3 = 1/8\omega_1$, in terms of the preceding notation, we see that output power is made up of components including, for example, P_{-11}, P_{8-7} etc., with all possible components save P_{-11} corresponding to large m_1 and/or m_2. Accordingly we might assume that the only significant component was P_{-11}, and then apply eqn. 12. If ω_2 were not $9/8\omega_1$, but a frequency extremely close to $9/8\omega_1$, so that ω_1 and ω_2 were incommensurable, then we should have to make precisely the same assumption in this case, if the output filter bandwidth were such that it passed, for example, P_{8-7}.

5 Conclusions

To summarise, we observe that the incommensurability condition of the Manley–Rowe relations is more accurately described as a condition on the sources that they be 'incoherent' generators. The use of solely incommensurable frequencies in earlier derivations meant that this was automatically the case; by explicitly postulating the requirement, commensurable frequencies can be considered too. With this assumption the derivation of Manley and Rowe can be modified to yield the result (eqn. 12). The difficulties associated with the existence of an infinite number of power components at certain frequencies, when the exciting frequencies are not incommensurable, highlight an approximation normally made in the application of the relations, namely that certain frequency components are negligible in magnitude. Moreover, this approximation has been masked by the assumption of filters with infinitely narrow passbands.

The validity of the approximation will, in part, depend on the nature of the nonlinearity of the reactive element; qualitatively the approximation amounts to disregarding perturbations of important frequency components by 'noise-like' signals.

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