A Simple Test for Zeros of a Complex Polynomial in a Sector

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Abstract—A simple proof of a recent result for determining whether the zeros of a real polynomial lie within a sector is first given. Secondly, this result is used in a procedure given for confirming whether or not the zeros of an arbitrary complex polynomial lie in a similar sector.

INTRODUCTION

Recently [1], with a view towards application in problems of relative stability, a test was given for determining whether or not all the roots of a polynomial having real coefficients lie within the sector shown in Fig. 1. In this article, first, a simple proof of the same result is given using Kronecker product of matrices. Second, a procedure is given to determine whether all the roots of a polynomial having complex coefficients lie inside the sector of Fig. 1, making use of the foregoing result. This appears to be simpler than the procedure existing in the mathematical literature [2]. Moreover, there are scopes for applying the results of this article in engineering problems. It is known, for example, that in certain cases of nonlinear system analysis in which a multivalued nonlinear characteristic is present, the characteristic equation of the corresponding linearized system has complex coefficients [3]. Root locations of such polynomials in a sector could be of interest.

MAIN RESULTS

A simple proof of a theorem given in [1] is included.

Theorem 1: Given a polynomial $P(s)$ of degree $n$ with real coefficients, a necessary and sufficient condition for the roots of $P(s)$ to lie inside the shaded region of Fig. 1 is that the eigenvalues of the matrix

$$A_1 = \begin{bmatrix}
A \cos \delta & -A \sin \delta \\
A \sin \delta & A \cos \delta
\end{bmatrix} \tag{1}
$$

have negative real part, where $A$ is the companion matrix associated with $P(s)$. (The matrix $A_1$ is of order $2n$.)

Proof: It is well known [4], that if the eigenvalues of $A$ (which happen to be the roots of $P(s)$) are $\lambda_i$ for $i = 1, 2, \ldots, n$ then the eigenvalues of

$$A_1 = A \otimes \begin{bmatrix}
\cos \delta & -\sin \delta \\
\sin \delta & \cos \delta
\end{bmatrix}
$$

are $\lambda_i(\cos \delta \pm j \sin \delta), i = 1, 2, \ldots, n$, where "$\otimes$" denotes "Kronecker

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product,” and vice-versa. Consequently, the eigenvalues of $A$ lie in the given sector if and only if the eigenvalues of $A_1$ have negative real parts.

Next, consider a polynomial $Q(s)$ having complex coefficients and of degree $n$. It is simple to show that the $2n$ degree real polynomial

$$Q_1(s) = Q(s)Q^*(s),$$

(2)

(where $Q^*(s)$ is obtained from $Q(s)$ by replacing the coefficients by their complex conjugates) has roots within the shaded sector of Fig. 1 if and only if the $Q(s)$ also has its roots there. The validity of the fact that $Q_1(s)$ in (2) has real coefficients can be simply justified by noting that if $Q(s) = \sum_{k=0}^{n} b_k s^k$, then

$$Q_1(s) = \begin{bmatrix} q_1 & 1 \\ q_2 & s \\ \vdots & \vdots \\ q_{2n} & s^{2n} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2n} \end{bmatrix},$$

(3)

where the “bar” denotes “complex conjugate.” From (3)

$$Q_1(s) = \begin{bmatrix} q_1 & 1 \\ q_2 & s \\ \vdots & \vdots \\ q_{2n} & s^{2n} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2n} \end{bmatrix},$$

(4)

is evidently a polynomial of degree $2n$ with real coefficients. (If $Q(s)$ has a root at $s = re^{j\theta}$, where $r$ and $\theta$ are the polar coordinates, then $Q^*(s)$ has a root at $s = re^{-j\theta}$ and vice-versa.) Suppose that the companion matrix associated with $Q_1(s)$ is $B$. Obviously, the elements of $B$ are real. Then, using Theorem 1, Theorem 2 follows.

**Theorem 2**: An arbitrary polynomial $Q(s)$, of degree $n$, having complex coefficients has its roots within the shaded sector of Fig. 1 if and only if the eigenvalues of the matrix $B_1$ of order $4n,$

$$B_1 = \begin{bmatrix} B \cos \delta & -B \sin \delta \\ B \sin \delta & B \cos \delta \end{bmatrix}$$

(5)

have negative real parts, where $B$ is the companion matrix associated with $Q_1(s)$ in (2). Elements of $B$ and $B_1$ are real.

**Conclusions**

First, a simple proof of the result in [1] has been given. This has been used in the procedure given to obtain a necessary and sufficient condition for an arbitrary polynomial $Q(s)$, of degree $n$, having complex coefficients, to have its roots within the shaded sector of Fig. 1. The test for this condition involves determining of whether the eigenvalues of a matrix $B_1$ of order $4n$, have negative real parts. Of course, this can be implemented using standard procedures [5].

It may be noted that in Theorem 1, the condition for the eigenvalues of a matrix $A$ to lie in a sector in terms of the eigenvalues of the matrix $A_1$ to lie in the left half plane, is also valid over the field of complex numbers. Therefore, to test for the eigenvalues of a complex matrix in a sector, one can, using the above artifice, apply the standard Lyapunov test [5] to the generated complex matrix.

Two other comments are relevant. From (2) it is readily seen that the polynomial $Q(s)$ with complex coefficients and of degree $n$ has all zeroes within the unit circle, $|s| < 1$, if and only if the polynomial $Q_1(s)$ with real coefficients and of degree $2n$ has all zeroes in $|s| < 1$. This answers the question posed in [7] regarding finding a real version of the table test [2, p. 197], [9] for zero location of complex polynomials in a unit circle. Finally, it may be noted that the eigenvalues of an arbitrary $n \times n$ complex constant matrix, $A + JB$ (where $A$ and $B$ are real matrices) are in the sector shown in Fig. 1, if and only if the eigenvalues of the $2n \times 2n$ real matrix, $[\begin{bmatrix} A & B \\ -B & A \end{bmatrix}]$ are in the same sector, and $AB = BA$.

This follows from the observations:

$$\det \begin{bmatrix} \lambda I + A - JB \\ 0 \end{bmatrix} = \det (\lambda I - A)^2 + B^2$$

and (note that the eigenvalues of $A + JB$ are the complex conjugates of those of $A - jB$)

$$\det \begin{bmatrix} \lambda I - A \\ B \end{bmatrix} = \det (\lambda I - A)^2 + B^2$$

if and only if $AB = BA$ [8].

**References**


*It is to be recognized that the test on $Q_1(s)$ could also be handled by other means [6].