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A Simple Test for Zeros of a Complex Polynomial in a Sector

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***Abstract*—A simple proof of a recent result for determining whether the zeros of a real polynomial lie within a sector is first given. Secondly, this result is used in a procedure given for confirming whether or not the zeros of an arbitrary complex polynomial lie in a similar sector.**

Introduction

Recently [1], with a view towards application in problems of relative stability, a test was given for determining whether or not all the roots of a polynomial having real coefficients lie within the sector shown in Fig. 1. In this article, first, a simple proof of the same result is given using Kronecker product of matrices. Second, a procedure is given to determine whether all the roots of a polynomial having complex coefficients lie inside the sector of Fig. 1, making use of the foregoing result. This appears to be simpler than the procedure existing in the mathematical literature [2]. Moreover, there are scopes for applying the results of this article in engineering problems. It is known, for example, that in certain cases of nonlinear system analysis in which a multivalued nonlinear characteristic is present, the characteristic equation of the corresponding linearized system has complex coefficients [3]. Root locations of such polynomials in a sector could be of interest.

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[1] The proof given is independent of the form of A, so long as $\det(sI - A) = P(s)$. Also, the results are applicable to root distribution studies.

Main Results

A simple proof of a theorem given in [1] is included.

Theorem 1: Given a polynomial $P(s)$ of degree n with real coefficients, a necessary and sufficient condition for the roots of $P(s)$ to lie inside the shaded region of Fig. 1 is that the eigenvalues of the matrix

$$A_1 = \begin{bmatrix} A \cos \delta & -A \sin \delta \\ A \sin \delta & A \cos \delta \end{bmatrix} \tag{1}$$

have negative real part, where A is the companion matrix[1] associated with $P(s)$. (The matrix A_1 is of order $2n$.)

Proof: It is well known [4], that if the eigenvalues of A (which happen to be the roots of $P(s)$) are λ_i, for $i = 1,2,\cdots,n$ then the eigenvalues of

$$A_1 = A \otimes \begin{bmatrix} \cos \delta & -\sin \delta \\ \sin \delta & \cos \delta \end{bmatrix}$$

are $\lambda_i(\cos \delta \pm j \sin \delta)$, $i = 1,2,\cdots,n$, where "$\otimes$" denotes "Kronecker

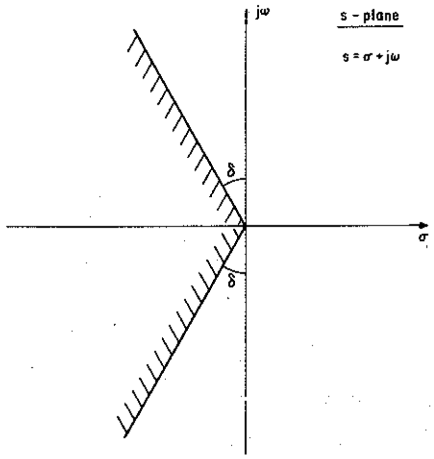

Fig. 1.

product," and vice-versa. Consequently, the eigenvalues of A lie in the given sector if and only if the eigenvalues of A_1 have negative real parts.

Next, consider a polynomial $Q(s)$ having complex coefficients and of degree n. It is simple to show that the $2n$ degree real polynomial

$$Q_1(s) = Q(s)Q^*(s), \tag{2}$$

(where $Q^*(s)$ is obtained from $Q(s)$ by replacing the coefficients by their complex conjugates) has roots within the shaded sector of Fig. 1 if and only if the $Q(s)$ also has its roots there. The validity of the fact that $Q_1(s)$ in (2) has real coefficients can be simply justified by noting that if $Q(s) = \Sigma_{k=0}^{n} b_k s^k$, then

$$Q_1(s) = [b_0 b_1 \cdots b_n] \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^n \end{bmatrix} [1 \; s \; \cdots \; s^n] \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{b}_n \end{bmatrix} \tag{3}$$

where the "bar" denotes "complex conjugate." From (3)

$$Q_1(s) = [b_0 b_1 \cdots b_n] \begin{bmatrix} 1 & s & \cdots & s^n \\ s & s^2 & \cdots & s^{n+1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ s^n & & \cdots & s^{2n} \end{bmatrix} \begin{bmatrix} \bar{b}_0 \\ \bar{b}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{b}_n \end{bmatrix} \tag{4}$$

is evidently a polynomial of degree $2n$ with real coefficients. (If $Q(s)$ has a root at $s = re^{j\theta}$, where r,θ are the polar coordinates, then $Q^*(s)$ has a root at $s = re^{-j\theta}$ and vice-versa.) Suppose that the companion matrix associated with $Q_1(s)$ is B. Obviously, the elements of B are real. Then, using Theorem 1, Theorem 2 follows.

Theorem 2: An arbitrary polynomial $Q(s)$, of degree n, having complex coefficients has its roots within the shaded sector of Fig. 1 if and only if the eigenvalues of the matrix B_1 of order $4n$,

$$B_1 = \begin{bmatrix} B \cos \delta & -B \sin \delta \\ B \sin \delta & B \cos \delta \end{bmatrix} \tag{5}$$

have negative real parts, where B is the companion matrix associated with $Q_1(s)$ in (2).[2] Elements of B and B_1 are real.

Conclusions

First, a simple proof of the result in [1] has been given. This has been used in the procedure given to obtain a necessary and sufficient condition for an arbitrary polynomial $Q(s)$, of degree n, having complex coefficients, to have its roots within the shaded sector of Fig. 1. The test for this condition involves determining of whether the eigenvalues of a matrix B_1 of order $4n$, have negative real parts. Of course, this can be implemented using standard procedures [5].

It may be noted that in Theorem 1, the condition for the eigenvalues of a matrix A to lie in a sector in terms of the eigenvalues of the matrix A_1 to lie in the left half plane, is also valid over the field of complex numbers. Therefore, to test for the eigenvalues of a complex matrix in a sector, one can, using the above artifice, apply the standard Lyapunov test [5] to the generated complex matrix.

Two other comments are relevant. From (2) it is readily seen that the polynomial $Q(s)$ with complex coefficients and of degree n has all zeroes within the unit circle, $|s| < 1$, if and only if the polynomial $Q_1(s)$ with real coefficients and of degree $2n$ has all zeroes in $|s| < 1$. This answers the question posed in [7] regarding finding a real version of the table test [2, p. 197], [9] for zero location of complex polynomials in a unit circle. Finally, it may be noted that the eigenvalues of an arbitrary $n \times n$ complex constant matrix, $A + jB$ (where A and B are real matrices) are in the sector shown in Fig. 1, if and only if the eigenvalues of the $2n \times 2n$ real matrix, $\begin{bmatrix} A & B \\ -B & A \end{bmatrix}$, are in the same sector, and $AB = BA$.

This follows from the observations.

$$\det \begin{bmatrix} \lambda I - A - jB & 0 \\ 0 & \lambda I - A + jB \end{bmatrix} = \det [(\lambda I - A)^2 + B^2],$$

if and only if $AB = BA$

and (note that the eigenvalues of $A + jB$ are the complex conjugates of those of $A - jB$)

$$\det \begin{bmatrix} \lambda I - A & -B \\ B & \lambda I - A \end{bmatrix} = \det [(\lambda I - A)^2 + B^2], \text{ if and only if}$$

$AB = BA$ [8].

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[2] It is to be recognized that the test on $Q_1(s)$ could also be handled by other means [6].