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Stability of Multidimensional Digital Filters

BRIAN D. O. ANDERSON, MEMBER, IEEE, AND E. I. JURY, FELLOW, IEEE

***Abstract*—Conditions are obtained for a digital filter in three or more variables to be stable, and computational techniques for checking the stability are examined.**

I. INTRODUCTION

TWO-DIMENSIONAL digital filtering has found application in a number of areas, for example image enhancement in optical systems and processing of seismic data. In view of the fact that the dimensionality of two often arises from the fact that the data are spatial, it is reasonable to foreshadow the use of three dimensional, and even higher dimensional, filters.

In the theory of two-dimensional digital filtering, it is often of interest to know whether a prescribed polynomial $A(z_1, z_2)$ in two complex variables has any zeros in the region $|z_1| \leqslant 1 \cap |z_2| \leqslant 1$. (See [1] and [2] for a discussion of the relevance of this condition, which is a stability condition.) In this paper, we consider the case of a multivariable polynomial $A(z_1, z_2, \cdots, z_n)$ in n variables. We derive conditions for the polynomial to be nonzero in the region $\cap_{i=1}^{n} |z_i| \leqslant 1$ which simplify the task of checking this property, and we also indicate in outline how the necessary computations could be carried out, at least for the cases $n = 3$, and $n = 4$. (For larger n, the computational problem appears very severe.)

The paper is structured as follows. In Section II, we interpret the condition

$$A(z_1, z_2, \cdots, z_n) \neq 0, \qquad \bigcap_{i=1}^{n} |z_i| \leqslant 1 \tag{1}$$

for a multivariable polynomial $A(\cdot, \cdot, \cdots, \cdot)$ as a stability condition, and we give reformulations of (1) which, in essence, show that with the polynomial nonzero over a subset of $\cap_{i=1}^{n} |z_i| \leqslant 1$, then it is nonzero in the entire region. These reformulations make the checking of (1) easier, in effect, by reducing the dimension of the region over which the nonzero property must be checked.

In Section III, we exhibit a further reduction of the problem of checking (1), essentially to one of verifying the positivity of various multivariable polynomials on $\cap_{i=1}^{n} |z_i| = 1$. Section IV considers the case of three-variable polynomials, and an example is given in Section V. Extensions, including a generalization of results for two-variable polynomials $H(s_1, s_2)$ nonzero in the region $\text{Re}\,[s_1] \geqslant 0 \cap \text{Re}\,[s_2] \geqslant 0$, are noted in Section VI, and concluding remarks appear in Section VII.

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B. D. O. Anderson is with the Department of Electrical Engineering, University of Newcastle, New South Wales, Australia.
E. I. Jury is with the Department of Electrical Engineering and Computer Science, University of California, Berkeley, Calif. 94720.

II. A Stability Condition and its Reformulation

Many n-dimensional filters can be described by a multivariable transfer function of the form

$$F(z_1, z_2, \cdots, z_n) = \frac{B(z_1, z_2, \cdots, z_n)}{A(z_1, z_2, \cdots, z_n)} \tag{2}$$

where $A(\cdot, \cdot, \cdots, \cdot)$ and $B(\cdot, \cdot, \cdots, \cdot)$ are n-variable polynomials, normally real. One can associate with the transfer function F a multivariable impulse response f which is an n-indexed sequence $\{f_{i_1, i_2, \cdots, i_n}\}$ where $i_j = 0, 1, \cdots$ for $j = 1, 2, \cdots, n$.[1] According to our definition, the impulse response is termed stable if and only if the result of convolving f with an absolutely summable n-indexed sequence is another absolutely summable n-indexed sequence.

A generalization of arguments in [3] for the two-variable case shows that a causal f is stable if and only if with $B(\cdot, \cdot, \cdots, \cdot)$ equal to a nonzero constant, (1) holds. If $B(\cdot, \cdot, \cdots, \cdot)$ is arbitrary, and (1) holds, then f is stable, but arbitrary $B(\cdot, \cdot, \cdots, \cdot)$ and stability of f may only imply (1) when one denies $A(\cdot, \cdot, \cdots, \cdot)$ and $B(\cdot, \cdot, \cdots, \cdot)$ the possibility of having common zeros.

The details of such an argument can be found in work of Justice and Shanks [4], where, actually, stability interpretations are given to a more general condition than $\cap_{i=1}^{n} |z_i| \leq 1$, involving replacement of some of the regions $|z_i| \leq 1$ by either $|z_i| = 1$ or $|z_i| \geq 1$.

Having settled the significance of (1), we now present the first of two theorems which provide helpful simplifications of the condition. The theorem is a generalization of the following two variable result, see [2]: for a two-variable polynomial $A(z_1, z_2)$,

$$A(z_1, z_2) \neq 0, \quad |z_1| \leq 1 \cap |z_2| \leq 1 \tag{3}$$

is equivalent to

$$A(z_1, 0) \neq 0, \quad |z_1| \leq 1 \tag{4}$$

and

$$A(z_1, z_2) \neq 0, \quad |z_1| = 1, |z_2| \leq 1. \tag{5}$$

Theorem 1: Let $A(z_1, z_2, \cdots, z_n)$ be a polynomial in n variables. Then (1) is equivalent to

$$A(z_1, z_2, \cdots, z_{n-1}, 0) \neq 0, \quad \bigcap_{i=1}^{n-1} |z_i| \leq 1 \tag{6}$$

and

$$A(z_1, z_2, \cdots, z_n) \neq 0, \quad \left\{\bigcap_{i=1}^{n-1} |z_i| = 1\right\} \cap \{|z_n| \leq 1\}. \tag{7}$$

[1] In fact, $F(z_1, z_2, \cdots, z_n) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f_{i_1, i_2, \cdots, i_n} \cdot z_1^{i_1}, z_2^{i_2}, \cdots, z_n^{i_n}$. This definition of the z-transform has z where z^{-1} is more commonly encountered. However, our definition is consistent with the definition used in those earlier papers most closely linked to our work. In the one-variable case, this means that for causal sequences f_i (where $f_i = 0$ for $i < 0$), instability is associated with poles of $F(z)$ *inside* the unit circle. Note also that the definition could be extended to noncausal $\{f_{i_1 i_2 \cdots i_n}\}$, i.e., those for which $f_{i_1 i_2 \cdots i_n}$ may be nonzero with one or more of the i_j negative.

Proof: Obviously, (6) and (7) are implied by (1), since (6) and (7) require $A(\cdot, \cdot, \cdots, \cdot)$ to be nonzero over a region contained in that defined by (1). It remains to prove the converse, which we shall do by induction.

The result is valid for $n = 2$, as noted above. Assume it is true for $n = 3, 4, \cdots, m - 1$. We shall prove it for $n = m$. Fix $z_1 = \xi_1$, for some ξ_1 with $|\xi_1| \leq 1$. Then, by the inductive argument applied to the $(m - 1)$-variable polynomial, $A(\xi_1, z_2, z_3, \cdots, z_m)$, it follows that

$$A(\xi_1, z_2, \cdots, z_{m-1}, 0) \neq 0, \quad \bigcap_{i=2}^{m-1} |z_i| \leq 1$$

and

$$A(\xi_1, z_2, \cdots, z_{m-1}, z_m) \neq 0, \quad \left\{\bigcap_{i=2}^{m-1} |z_i| = 1\right\} \cap \{|z_m| \leq 1\}$$

imply

$$A(\xi_1, z_2, \cdots, z_m) \neq 0, \quad \bigcap_{i=2}^{m} |z_i| \leq 1.$$

Consequently, noting the arbitrary nature of ξ_1, we see that

$$A(z_1, z_2, \cdots, z_{m-1}, 0) \neq 0, \quad \bigcap_{i=1}^{m-1} |z_i| \leq 1 \tag{8}$$

and

$$A(z_1, z_2, \cdots, z_{m-1}, z_m) \neq 0, \quad \{|z_1| \leq 1\} \cap \left\{\bigcap_{i=2}^{m-1} |z_i| = 1\right\} \cap \{|z_m| \leq 1\} \tag{9}$$

imply

$$A(z_1, z_2, \cdots, z_m) \neq 0, \quad \bigcap_{i=1}^{m} |z_i| \leq 1. \tag{10}$$

Now, temporarily fix $z_2 = \xi_2, \cdots, z_{m-1} = \xi_{m-1}$ in the region $\cap_{i=2}^{m-1} |z_i| = 1$ so that the two-variable theorem can be applied in (9), thinking of z_1 and z_m as the variables. Then, (9) is implied by

$$A(z_1, \xi_2, \cdots, \xi_{m-1}, 0) \neq 0, \quad |z_1| \leq 1$$

and

$$A(z_1, \xi_2, \cdots, \xi_{m-1}, z_m) \neq 0, \quad |z_1| = 1 \cap |z_m| \leq 1$$

or, using the arbitrary nature of $\xi_2, \cdots, \xi_{m-1}$, (9) is implied by

$$A(z_1, z_2, \cdots, z_{m-1}, 0) \neq 0, \quad \{|z_1| \leq 1\} \cap \left\{\bigcap_{i=2}^{m-1} |z_i| = 1\right\} \tag{11}$$

and

$$A(z_1, z_2, \cdots, z_{m-1}, z_m) \neq 0, \quad \{|z_1| = 1\} \cap \left\{\bigcap_{i=2}^{m-1} |z_i| = 1\right\} \cap \{|z_m| \leq 1\}. \tag{12}$$

Hence (8), (11), and (12) imply (10). However, (8) implies (11), so that (8) and (12) alone imply (10). This verifies the inductive hypothesis for the case $n = m$, and therefore proves the theorem.

The second theorem rests on the observation that $A(z_1, z_2, \cdots, z_{n-1}, 0)$ is an $(n-1)$-variable polynomial, so that (6) may therefore be replaced by

$$A(z_1, z_2, \cdots, z_{n-2}, 0, 0) \neq 0, \quad \bigcap_{i=1}^{n-2} |z_i| \leqslant 1 \tag{13}$$

and

$$A(z_2, z_2, \cdots, z_{n-2}, z_{n-1}, 0) \neq 0, \quad \left\{\bigcap_{i=1}^{n-2} |z_i| = 1\right\} \cap \{|z_{n-1}| \leqslant 1\}$$

by reapplication of Theorem 1. The same idea can then be reapplied to the $(n-2)$-variable polynomial appearing in (13), and so on. In this way, one proves the following.

Theorem 2: Let $A(z_1, z_2, \cdots, z_n)$ be a polynomial in n variables. Then (1) is equivalent to

$$A(z_1, z_2, \cdots, z_n) \neq 0, \quad \bigcap_{i=1}^{n-1} |z_i| = 1 \ \cap \{|z_n| \leqslant 1\} \tag{7}$$

$$A(z_1, z_2, \cdots, z_{n-1}, 0) \neq 0, \quad \left\{\bigcap_{i=1}^{n-2} |z_i| = 1\right\} \cap \{|z_{n-1}| \leqslant 1\} \tag{14}$$

$$A(z_1, z_2, \cdots, z_{n-2}, 0, 0) \neq 0, \quad \left\{\bigcap_{i=1}^{n-3} |z_i| = 1\right\} \cap \{|z_{n-2}| \leqslant 1\} \tag{15}$$

$$\vdots$$

$$A(z_1, z_2, 0, \cdots, 0) \neq 0, \quad \{|z_1| = 1\} \cap \{|z_2| \leqslant 1\}$$
$$A(z_1, 0, 0, \cdots, 0) \neq 0, \quad |z_1| \leqslant 1. \tag{16}$$

Of course, the ordering of the variables z_i is arbitrary, so that there many alternatives to the indexing in the conditions appearing in Theorem 2. Some alternatives may involve conditions which are more easily checked than others.

III. Further Reformulation of the Stability Conditions

According to Theorem 2, the problem of checking the stability condition (1) for an n-variable polynomial is equivalent to the problem of checking conditions

$$A_m(z_1, z_2, \cdots, z_m) \neq 0, \quad \left\{\bigcap_{i=1}^{m-1} |z_i| = 1\right\} \cap \{|z_m| \leqslant 1\} \tag{17}$$

for $m = 1, 2, \cdots, n$ where $A_m(\cdot, \cdot, \cdots, \cdot)$ is $A(\cdot, \cdot, \cdots, \cdot)$ with the last $n - m$ variables set to zero. In this section, we show that for each m, (17) is equivalent to a set of inequalities of the form

$$C(z_1, z_1^*, z_2, z_2^*, \cdots, z_{m-1}, z_{m-1}^*) > 0, \quad \bigcap_{i=1}^{m-1} |z_i| = 1 \tag{18}$$

where C is a polynomial in $2(m-1)$ variables taking real values on $\cap_{i=1}^{m-1} |z_i| = 1$, and the number of such inequalities associated with each $A_m(\cdot, \cdot, \cdots, \cdot)$ depends on the degree of A_m regarded as a polynomial in z_m.

To establish this, we recall the Schur-Cohn criterion, see, e.g., [5]: suppose that $f(z)$ is a pth degree polynomial

$$f(z) = \sum_{i=0}^{p} a_i z^i, \quad a_p \neq 0 \tag{19}$$

and associate with $f(z)$ the $p \times p$ Hermitian matrix $\Gamma = (\gamma_{ij})$ defined by

$$\gamma_{ij} = \sum_{k=1}^{i} (a_{p-i+k} a_{p-j+k}^* - a_{i-k}^* a_{j-k}), \quad i \leqslant j. \tag{20}$$

Then, $f(z) \neq 0$ for $|z| \leqslant 1$ if and only if Γ is negative definite.

Because the negative definiteness of a matrix can be checked by examining the signs of the leading principal minors, and because each entry of Γ is a quadratic function of the coefficients of $f(\cdot)$ and their complex conjugates, it follows that $f(z) \neq 0$ for $|z| \leqslant 1$ if and only if a set of sign definite inequalities are satisfied which are multinomial in the a_j and a_j^*.

Now, apply these ideas to the m-variable polynomial $A_m(z_1, z_2, \cdots, z_m)$ in (17). Thinking of $z_1, z_2, \cdots, z_{m-1}$ as parameters taking values somewhere on the unit circle, $A_m(z_1, z_2, \cdots, z_m)$ can be thought of as a polynomial in z_m, thus

$$A_m(z_1, z_2, \cdots, z_m) = \sum_{i=0}^{p} a_i(z_1, z_2, \cdots, z_{m-1}) z_m^i \tag{21}$$

and it is nonzero inside $|z_m| \leqslant 1$ if and only if the Schur-Cohn matrix is negative definite, i.e., if and only if a number of inequalities which are integral in the a_i are satisfied. Since the a_i are polynomial in $z_1, z_2, \cdots, z_{m-1}$ and since the complex conjugates of the a_i appear in the Schur-Cohn matrix, it follows that these inequalities, p in number, have the form of (18). Realness of the function C on $\cap_{i=1}^{m-1} |z_i| = 1$ follows from the Hermitian nature of Γ, which implies realness of the principal minors.

Notice that the arguments $z_1^*, z_2^*, \cdots, z_{m-1}^*$ in (18) can be replaced if desired by $z_1^{-1}, z_2^{-1}, \cdots, z_{m-1}^{-1}$ since $z_1^* = z_1^{-1}$ for $|z_1| = 1$. Notice also that if the original polynomial $A(z_1, z_2, \cdots, z_n)$ under test is real, each polynomial (18), regarded as a polynomial in $2m - 2$ variables, will have real coefficients; for the realness of $A(\cdot, \cdot, \cdots, \cdot)$ implies realness of $A_m(\cdot, \cdot, \cdots, \cdot)$ which in turn implies that each entry of Γ, and therefore each minor, is a real multivariable polynomial in $z_1, z_1^*, \cdots, z_{m-1}^*$.

Here, our general theory stops. In the next section however, we shall discuss how (18) might be verified when the number of variables n in the original polynomial is not large.

IV. Stability Conditions for Three- and Four-Variable Polynomials

First note that stability testing of a two-variable polynomial $A(z_1, z_2)$ leads via Theorem 2 to a two-variable polynomial—actually $A(z_1, z_2)$—whose properties are of interest on $|z_1| = 1$, $|z_2| \leq 1$, and a one-variable polynomial—actually $A(z_1, 0)$. By the material of Section III, the two-variable polynomial generates a number of polynomials in z_1 and z_1^* and a number of constants, for all of which sign definiteness must be checked. Checking the constants is immediate, and checking the polynomial in z_1 and z_1^* is not much more difficult, with, for example, a Sturm test being readily applicable, see, e.g., [2], [6].

Generally, an n-variable polynomial produces polynomials whose positivity is to be checked which are in up to $2n - 2$ variables, $z_1, z_1^*, z_2, \cdots, z_{n-1}^*$. Specifically, a three-variable polynomial produces constants (where positivity is easily checked), polynomials in z_1 and z_1^* (where positivity is checked via the Sturm test), and polynomials in z_1, z_1^*, z_2, and z_2^*. The checking for positivity of this last set of polynomials can reasonably proceed by plotting values of the polynomials over a two-dimensional grid in a (θ_1, θ_2) plane, taking $z_1 = \exp(j\theta_1)$ and $z_2 = \exp(j\theta_2)$.

Conceivably, this idea could be extended to a four-variable polynomial $A(z_1, z_2, z_3, z_4)$, which would demand the checking for positivity of polynomials across a $(\theta_1, \theta_2, \theta_3)$ space.

V. Example

To obtain an example which requires but little computation, one needs an exceptionally simple polynomial. We shall choose

$$A(z_1, z_2, z_3) = z_1^2 z_2 + z_1 - z_2 + z_3 + 4. \tag{22}$$

To demonstrate stability we need to check that

$$A(z_1, z_2, z_3) \neq 0, \quad |z_1| = |z_2| = 1, |z_3| \leq 1 \tag{23}$$

$$A(z_1, z_2, 0) = z_1^2 z_2 + z_1 - z_2 + 4 \neq 0, \qquad |z_1| = 1, \quad |z_2| \leq 1 \tag{24}$$

$$A(z_1, 0, 0) = z_1 + 4 \neq 0, \quad |z_1| \leq 1. \tag{25}$$

Equation (25) is immediate, and (24) almost immediate, by direct verification. [In considering (24), observe that $|z_1^2 z_2 + z_1 - z_2| \leq |z_1|^2 |z_2| + |z_1| + |z_2| \leq 3$.] For (23), applying the Schur-Cohn procedure to $A(z_1, z_2, z_3)$ regarded as a polynomial in z_3, or proceeding immediately using the linear nature of $A(z_1, z_2, z_3)$ in z_3, we discover that (23) is equivalent to

$$|z_1^2 z_2 + z_1 - z_2 + 4| > 1, \quad |z_1| = 1, \; |z_2| = 1. \tag{26}$$

We could plot out values of the left side of (26) for various z_1 and z_2. However, in this case, (26) may be directly verified by *ad hoc* means. Since $|z_1^2 z_2 + z_1 - z_2| \leq 3$ as noted earlier, it follows that (26) fails if and only if for some z_1 and z_2 on $|z_1| = 1$, $|z_2| = 1$ we have

$$z_1^2 z_2 + z_1 - z_2 = -3.$$

To see this is impossible, notice that this relation implies

$$z_2 = \frac{3 + z_1}{1 - z_1^2}.$$

Now, $|z_2| = 1$ implies $|1 - z_1^2| = |3 + z_1|$; for z_1 on $|z_1| = 1$, one has $0 \leq |1 - z_1^2| \leq 2$ and $2 \leq |3 + z_1| \leq 4$. So we require $2 = |3 + z_1| = |1 - z_1^2|$, which cannot be satisfied by z_1 with $|z_1| = 1$, as is easily seen. Therefore, the polynomial $A(z_1, z_2, z_3)$ is nonzero inside $\cap_{i=1}^3 |z_i| \leq 1$.

VI. Two Extensions

In [4], conditions of the form

$$A(z_1, z_2, \cdots, z_n) \neq 0, \quad \left\{\bigcap_{i=1}^{r} |z_i| \leq 1\right\} \cap \bigcap_{i=r+1}^{s} |z_i| \geq 1 \cap \left\{\bigcap_{i=s+1}^{n} |z_i| = 1\right\} \tag{27}$$

are introduced in stability studies. In case $A(\cdot, \cdot, \cdots, \cdot)$ is polynomial in the z_i, let us note how the earlier ideas can be employed to simplify the checking of stability. First observe that setting $z_i' = z_i^{-1}$ for $i = r + 1$ through s makes the region appearing in (27) more amenable; second, by multiplication by appropriate powers of the z_i' for $i = r + 1$ through s, $A(z_1, z_2, \cdots, z_{r+1}'^{-1}, \cdots, z_s'^{-1}, z_{s+1}, \cdots, z_n)$ becomes a polynomial $B(\cdot, \cdot, \cdots, \cdot)$ in $z_1, \cdots, z_r, z_{r+1}', \cdots, z_s', z_{s+1}, \cdots, z_n$ which is nonzero in $\{\cap_{i=1}^r |z_i| \leq 1\} \cap \{\cap_{i=r+1}^s |z_i'| \leq 1\} \cap \{\cap_{i=s+1}^n |z_i| = 1\}$, precisely when (27) holds. The testing of this condition can then proceed as indicated earlier in this paper, with the restriction $\cap_{i=s+1}^n |z_i| = 1$ making no essential difference to the results.

The second extension is to the checking of conditions of the type

$$B(s_1, s_2, \cdots, s_n) \neq 0, \quad \bigcap_{i=1}^{n} \operatorname{Re} s_i \geq 0 \tag{28}$$

for multivariable polynomials $B(\cdot, \cdot, \cdots, \cdot)$. The case $n = 2$ has been considered in [7], but also see [2] for a restatement of the results of [7] which avoids difficulties in [7] over closed and open half-planes. By setting

$$s_i = \frac{1 - z_i}{1 + z_i} \tag{29}$$

and clearing fractions, one can reduce (28) to a condition of the type (1). Then one can simplify (1) in the manner shown earlier. From the computational point of view, instead of working with the conditions noted in Theorem 2, see (7), (14)-(16), one can work with the s-plane equivalent of these conditions. Thus (28) is equivalent to the following:

$$B(s_1, s_2, \cdots, s_n) \neq 0, \qquad \left\{\bigcap_{i=1}^{n-1} \operatorname{Re} s_i = 0\right\} \cap \{\operatorname{Re} s_n \geqslant 0\} \tag{30}$$

$$B(s_1, s_2, \cdots, s_{n-1}, 1) \neq 0, \qquad \left\{\bigcap_{j=1}^{n-2} \operatorname{Re} s_i = 0\right\} \cap \{\operatorname{Re} s_{n-1} \geqslant 0\} \tag{31}$$

$$B(s_1, s_2, \cdots, s_{n-2}, 1, 1) \neq 0, \qquad \left\{\bigcap_{i=1}^{n-3} \operatorname{Re} s_i = 0\right\} \cap \{\operatorname{Re} s_{n-1} \geqslant 0\} \tag{32}$$

$$\vdots$$

$$B(s_1, 1, \cdots, 1, 1) \neq 0, \qquad \operatorname{Re} s_i \geqslant 0. \tag{33}$$

Each of these conditions can be reformulated as a set of positivity conditions over smaller regions. Thus (31) can be replaced by a set of positivity conditions of the form

$$C(s_1, s_1^*, \cdots, s_{n-2}^*, s_{n-2}^*) > 0, \qquad \bigcap_{i=1}^{n-2} \operatorname{Re} s_i = 0 \tag{34}$$

where C takes on only real values in the region of interest. The polynomials C follow by applying the Hermite criterion, see, e.g., [5], to the polynomial $B(s_1, s_2, \cdots, s_{n-1}, 1)$ viewed as a polynomial in s_{n-1} with s_1 through s_{n-2} as parameters. (The Hermite criterion plays the same role here as played by the Schur–Cohn criterion in Section III.)

Note that a minor variation on the above conditions can be obtained by using the bilinear transformation $s_i = \alpha_i(1 - z_i)(1 + z_i)^{-1}$ for arbitrary positive α_i. The various 1's appearing as arguments in (31) through (33) are not sacrosanct.

By analogy with [7], it appears that these ideas may have relevance in discussing realizability properties of impedances of networks comprising inductors, capacitors, transformers, and transmission lines, where the transmission lines may be of incommensurate lengths.

VII. Concluding Remarks

The following key points were raised in this paper. First, the checking of the stability condition for a prescribed multivariable polynomial can be replaced by the checking of simpler stability conditions for a set of multivariable polynomials with different numbers of variables. The simplification lies in the fact that all but one variable in each of this set of polynomials lies on the unit circle, rather than either inside or on the unit circle. Second, the checking of the simpler stability condition for each of these polynomials can proceed by checking the positivity (for variable values on the unit circle only) of a number of polynomials, which are readily computable (and this ready computability is an important consequence of the use of the Schur–Cohn criterion). Finally, in case the original polynomial whose stability is to be checked is three-variable, or possibly four-variable, the actual computations involved in checking positivity appear feasible. Secondary points raised in the paper include the extension of the stability results to cover regions other than the unit circle interiors.

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