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Correspondence Item

Corrections to:* Algebraic Properties of Minimal Degree Spectral Factors†

Corrections aux "Propriétés algébriques de Facteurs Spectraux de Degré Minimal"

Korrekturen zu "Algebraische Eigenschaften von Spektralfaktoren minimalen Grades

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Summary—Corrections and extensions to an earlier paper are presented. The material relates frequency domain, time domain and algebraic statements of a partial ordering of spectral factors.

RECENT correspondence with J. C. Willems has led us to the conclusion that in [1] part of the claims of Theorem 2, and consequential claims of later theorems, may be in error. At least part of the proof of Theorem 2 is certainly in error, but this of course does not necessarily invalidate the claims of the theorem. The purpose of this note is to patch up the situation as far as possible. The following is believed to be a correct result.

Theorem 1

Let $Z(s)$ be a $p\times p$ positive real matrix with $Z(\infty)<\infty$, with poles of all elements in Re $[s]<0$ and with a minimal realization $\{F, G, H, J\}$. Let P_k $(k = 1, 2)$ be two matrices such that

$$M_k = \begin{bmatrix} P_k F+F'P_k & P_k G-H \\ (P_k G-H)' & -(J+J') \end{bmatrix} \leqslant 0. \tag{1}$$

Note that such matrices exist by the Positive Real Lemma as reviewed in [1]. With L_k and W_{0k} any matrices such that

$$M_k = -\begin{bmatrix} L_k \\ W_{0k}' \end{bmatrix} [L_k' \quad W_{0k}], \tag{2}$$

let $W_k(s) = W_{0k}+L_k'(sI-F)^{-1}G$. Let $\mathscr{S}_k$ be the linear system $\dot{x} = Fx+Gu$, $y_k = L_k'x+W_{0k}u$. Then the following three conditions are equivalent

(A) $P_1 \geqslant P_2$.

(B) For any $u(\cdot)$, zero up till $t = 0$, with $x(0) = 0$, corresponding outputs of $\mathscr{S}_1$ and $\mathscr{S}_2$ are related by

$$\int_0^T y_1'(\tau)\, y_1(\tau)\, d\tau \leqslant \int_0^T y_2'(\tau)\, y_2(\tau)\, d\tau. \tag{3}$$

There is equality when $T = \infty$, provided $u(\cdot)$ is in $\mathscr{L}_2[0, \infty)$.

(C) For arbitrary n and s_i, $i = 1, 2, \ldots, n$, in Re $[s]>0$, the matrix with $i-j$ block submatrix

$$\frac{W_2'(s_i^*)\, W_2(s_j) - W_1'(s_i^*)\, W_1(s_j)}{s_i^*+s_j} \tag{4}$$

is non-negative definite.

Proof. (A) $\Leftrightarrow$ (B). We have $y_k(\tau) = W_{0k}\, u(\tau)+L_k'x(\tau)$ and so

$$\begin{aligned} y_k'(\tau)\, y_k(\tau) &= u'(\tau)\, W_{0k}'W_{0k}\, u(\tau)+2x'(\tau)\, L_k W_{0k}\, u(\tau) \\ &\quad +x'(\tau)\, L_k L_k' x(\tau) \\ &= 2u'(\tau)\, Ju(\tau)+2x'(\tau)\, Hu(\tau)-2x'(\tau)\, P_k Gu(\tau) \\ &\quad -2x'(\tau)\, P_k Fx(\tau) \\ &= 2u'(\tau)\, y(\tau)-2x'(\tau)\, P_k \dot{x}(\tau), \end{aligned}$$

where $y = Ju+H'x$. Then

$$\int_0^T y_1'(\tau)\, y_1(\tau) d\tau - \int_0^T y_2'(\tau)\, y_2(\tau)\, d\tau = x'(T)\,(P_2-P_1)\, x(T)$$

For finite T, (A) $\Rightarrow$ (B) and (B) $\Rightarrow$ (A) using complete controllability. For $T = \infty$, $u(\cdot) \in \mathscr{L}_2$ and Re $\lambda_i(F)<0$ yields $x(\infty) = 0$ and the requisite equality in (B).

(B) $\Leftrightarrow$ (C). Notice that if we take $u(\cdot) \in L_2(-\infty, T_1)$, the inequality of (B) can be modified by changing the integration interval to $(-\infty, T]$ for any $T \leqslant T_1$. Moreover, if we take $u(\cdot)$ to be complex, then the integrands should be modified to $y_k'^*(\tau)\, y_k(\tau)$.

Now take

$$u(\tau) = \sum_{i=1}^{n} \exp(s_i \tau)\, u_i, \quad \tau \in (-\infty, \infty)$$

for arbitrary complex u_i. Then

$$y_k(\tau) = \sum_i W_k(s_i) \exp(s_i \tau)\, u_i$$

and

$$\int_{-\infty}^{0} y_k'^*(\tau)\, y_k(\tau)\, d\tau = \sum_{i,j} u_i^* \frac{W_k'(s_i^*)\, W_k(s_j)}{s_i^*+s_j} u_j.$$

The inequality of (B) implies the non-negativity claimed in (C).

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The inequality of (B) follows from (C) by using the fact that the functions $\exp(s_i \tau)$, $\text{Re}[s_i] > 0$ span $\mathscr{L}_2[-\infty, T]$ for all finite T, and by reversing the argument immediately above. To prove equality when $T = \infty$, $u(\cdot)$ is in $\mathscr{L}_2[0, \infty)$ and the lower limit of integration is 0, the earlier argument relating (A) and (B) applies.

Remarks

1. Though the above proof is complete in itself, it is probably interesting to obtain a direct connection between (A) and (C). Some minor algebra will show that

$$Z'(s_i^*)+Z(s_j) = W_k'(s_i^*)\, W_k(s_j) + (s_i^*+s_j)\, G'(s_i^* I-F')^{-1} P_k(s_j I-F)^{-1} G$$

or that

$$\frac{W_2'(s_i^*)\, W_2(s_j)-W_1'(s_i^*)\, W_1(s_j)}{s_i^*+s_j} = G'(s_i^* I-F')^{-1}(P_1-P_2)(s_j I-F)^{-1} G$$

This identity can be used to show the equivalence of (A) and (C).

2. The connection between (B) and (C) does not depend on the rationality or otherwise of the $W_k(s)$. The equality part of (B) does require that

$$W_k'(-s)\, W_k(s) = Z(s)+Z'(-s),$$

an equality which is implicit in the rational case in our definitions of $W_k(s)$.

The frequency domain-ordering used in [1] was

$$\text{(D)} \qquad W_2'(s_i^*)\, W_2(s_j)-W_1'(s_i^*)\, W_1(s_j) \geq 0$$

for all s_i in $\text{Re}[s_i] > 0$, and in [1] we claimed the equivalence of (A), (B) and (D).

Now (C) $\Rightarrow$ (D), as is seen immediately by taking $n = 1$. We are also able to state a set of circumstances under which (D) $\Rightarrow$ (A), (B) and (C).

Theorem II

Suppose $W_2(s)$ has a right inverse. Then condition (D) $\Rightarrow$ (A), (B) and (C).

Proof. Set $S(s) = W_1(s)\, W_2^{-1}(s)$. If $S(s)$ is not square, add rows or columns as required to make it square. Then (D) $\Rightarrow$

$$I-S'(s_i^*)\, S(s_j) \geq 0 \qquad (5)$$

for all s_i in $\text{Re}[s] > 0$. At this point, there are two ways to proceed; either we can appeal to a theorem of Pick [2] to conclude that the matrix with $i-j$ block submatrix

$$\frac{I-S'(s_i^*)\, S(s_j)}{s_i^*+s_j} \qquad (6)$$

is positive definite, or we can use the fact that $S(s)$ is a scattering matrix [3] in view of (5). Then for a system with input $v(\cdot)$, output $w(\cdot)$ and transfer function matrix $S(s)$ one has for all T

$$\int_{-\infty}^{T} (v'v - w'w)\, dt \geq 0,$$

assuming the integrals are well defined. Taking

$$w(t) = \sum \exp(s_i t)\, w_i, \quad T = 0,$$

yields non-negativity of the matrix with $i-j$ block submatrix (6). From (6) and the definition of $S(s)$, non-negativity of the matrix with $i-j$ block (4) follows.

Remarks

1. It remains an open question as to whether condition (D) *always* implies (C). In view of work of Willems related to a similar problem [4], it would be fair to conjecture that (D) does not always imply (C), causing Theorem 2 of [1] to be probably incorrect.

2. Evidently, condition (C) is a more fundamental frequency domain condition than is condition (D). One could presume it would be relevant to the problem considered by Willems in [4], where a frequency domain condition analogous to (D) is noted as being inadequate.

3. A variant on the proof of Theorem 2 will show that if there exists a rational $V(s)$ with $W_1 = VW_2$ and $I-V'(s^*)\, V(s) \geq 0$ in $\text{Re}[s] > 0$, then condition (D) implies (C).

References

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