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Algebraic Properties of Minimal Degree Spectral Factors*†

Propriétés Algébriques de Facteurs Spectraux de Degré Minimal

Algebraische Eigenschaften von Spektralfaktoren minimalen Grades

Алгебраические свойства спектральных членов минимальой степени

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The notion of closeness of a plant to a mimimum phase plant can be explored via time-domain, frequency-domain and algebraic ideas for multivariable and scalar plants to provide interesting links between classical and modern control.

Summary—The paper derives a result connecting frequency-domain and time-domain properties of different spectral factors of the one power spectrum matrix. These results are interpeted from an algebraic point of view, and applied to the linear–quadratic optimal control and filtering problem. Interpretations are given of the phenomenon that many optimal control problems can lead to the same optimal control law but different optimal cost, and likewise many filtering problems can lead to the same optimal filter, but different filter performance.

1. INTRODUCTION

THE genesis of this paper lay in an observation of [1] to the effect that the one optimal Kalman–Bucy filter may be optimum for many different signal processes, and offer different performance, as measured by the error covariance matrix, for these different processes. That signal process for which filter performance is best is a minimum phase process, or the multivariable time-varying generalization of this concept.

This property leads to the conjecture that the less like a minimum phase process the signal process is, the worse will be the performance of the associated filter. Such a loose statement obviously requires more precise formulation before it is investigated; such a formulation is attempted later in the paper, and it is followed by an investigation.

It turns out that the key issue is to define some sort of a partial ordering on the various spectral factors of a prescribed power spectrum matrix, with the "less than" property in some way reflecting the concept of "nearer to minimum phase". This task is tackled in Section 2, where frequency and time-domain interpetations of the property are given.

In Sections 3 and 4, attention is restricted to spectral factors of minimal degree. Then it is shown, with the aid of the Positive Real Lemma [2–5], that the partial ordering of the spectral factors is isomorphic with a partial ordering of constant non-negative definite symmetric matrices appearing in the Positive Real Lemma statement with the minimum phase spectral factor corresponding to a minimum matrix in the set. In Section 5, a "maximum phase" spectral factor is introduced and its properties interpeted with the aid of the Positive Real Lemma.

The material of Sections 3, 4 and 5, by translating the "less than" property into algebraic terms, clears the way for applying the ideas to the linear-quadratic control and filtering problem in Section 6. The filtering situation has been described in rough terms above. The control situation can be regarded as one where many different performance indices can lead to the same optimal control law, but different optimal performance indices; an ordering of the optimal indices is exhibited, which ties in with the ordering of a certain spectral factor intimately associated with the control problem. In general terms, one can conclude that minimum phase plants can be controlled with less cost than non-minimum phase plants. A result of this form has been obtained when the weighting on the control in the performance index is made vanishingly small, see Ref. [6].

Section 7 discusses briefly some time-varying results, and Section 8 contains concluding remarks.

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* Throughout the paper, all transfer function matrices will be assumed to be real rational.

2. SPECTRAL FACTORIZATION, AND AN ORDERING OF SPECTRAL FACTORS

To understand the spectral factorization problem, we recall the notion of a *power spectrum matrix:*

Definition

A $p \times p$ matrix $\Phi(s)$ of real rational functions is a power spectrum matrix* if

(1) $\Phi(\infty)$ is finite.
(2) Each entry of $\Phi(s)$ is analytic for all $s=j\omega$, ω real.
(3) $\Phi(\cdot)$ is parahermitian, i.e. $\Phi'(-s)=\Phi(s)$. Here, the superscript prime denotes matrix transposition.
(4) $\Phi(j\omega)$ is non-negative definite Hermitian, written $\Phi(j\omega)\geq 0$, for all real ω.

A *spectral factorization* of $\Phi(s)$ is then a decomposition of $\Phi(s)$ as

$$\Phi(s)=W'(-s)W(s) \tag{1}$$

where $W(s)$ is a real rational matrix, termed a spectral factor of $\Phi(s)$.

We shall restrict attention henceforth to *stable* spectral factors, i.e. $W(s)$ for which no entry of $W(s)$ has a pole in $Re[s]\geq 0$.

The following facts contained in e.g. [7] will be needed in the sequel.

(1) Among the spectral factors of $\Phi(s)$, there is a special family. Members $\overline{W}(s)$ of this family differ only by premultiplication by a constant orthogonal matrix; members are stable, and possess the following defining property: if $\Phi(s)$ has rank r almost everywhere, $\overline{W}(s)$ is of size $r \times p$, and has rank r throughout $Re[s]>0$. We shall term such a spectral factor *minimum phase.** Note that if $\Phi(s)$ is nonsingular almost everywhere, a minimum phase $\overline{W}(s)$ is square and invertible throughout $Re[s]>0$.
(2) Let $\overline{W}(s)$ be an $r \times p$ minimum phase spectral factor. Then all other stable spectral factors are given by

$$W(s)=V(s)\,\overline{W}(s) \tag{2}$$

where $V(s)$ is such that every entry is analytic in $Re[s]\geq 0$, and

$$V'(-s)V(s)=I_r. \tag{3}$$

The number of rows of $V(s)$ is unspecified, but is obviously bounded below by r.

* The definition is in accord with the description of a scalar transfer function with all zeros and poles in $Re[s]<0$ as minimum phase

The following result characterizes another difference between $W(s)$ and $\overline{W}(s)$.

Theorem 1

Let $\overline{W}(s)$ and $W(s)$ be respectively a stable minimum phase spectral factor, and an arbitrary stable spectral factor of a power spectrum matrix $\Phi(s)$. Then

$$W'(s^*)W(s)\leq \overline{W}'(s^*)\,\overline{W}(s) \tag{4}$$

for all s in $Re[s]\geq 0$, with equality on $s=j\omega$, ω real; equality holds at one point in $Re[s]>0$ if and only if $W(s)$ is also minimum phase, and then equality holds at all points in $Re[s]>0$.

We note that this theorem is similar to a result for scalar spectral factors in [8].

Proof. The inequality (4) is easily seen to be satisfied with equality on $s=j\omega$, by setting $s=j\omega$ in the definition (1). Equality everywhere if $W(s)$ is the minimum phase is also trivially seen. To prove the remainder, it is evidently sufficient to show that with $V(s)$ as in (2) and possessing the properties noted earlier, one has

$$I_r - V'(s^*)V(s)\geq 0 \quad Re[s]\geq 0$$

with equality for one s with $Re[s]>0$ only if $V(s)$ is constant. Constancy of $V(s)$ then implies equality for all points in $Re[s]>0$. Let x be an arbitrary real r-vector, and let $V(s)$ be $n\times r$. Let $y_k(s)$ be the k-th entry of $V(s)x$. Then $y_k(s)$ is analytic in $Re[s]>0$, and by the maximum modulus theorem,

$$y_k(s^*)y_k(s)\leq \max y_k(-_j\omega)y_k(j\omega)$$

with equality for at least one point in $Re[s]>0$ if and only if $y_k(s)$ is constant. Summing over k,

$$x'V'(s^*)V(s)x\leq \underset{\omega}{\text{wax}}\, x'V'(-j\omega)V(j\omega)x=x'x$$

with equality for at least one point and all x if and only if $V(s)$ is constant.

In order to illustrate the above ideas, and to motivate a definition of partial ordering on spectral factors given below, we shall briefly consider the case of the 1×1 power spectrum matrix

$$\Phi(s)=\frac{(-s^2+1)(-s^2+9)}{(-s^2+4)(-s^2+16)}$$

for which a minimum phase spectral factor is

$$\overline{W}(s)=\frac{(s+1)(s+3)}{(s+2)(s+4)}.$$

Note that the only other one is the negative of the above. Some other spectral factors are the following where the indexing will be used in remarks below

$$W_1(s)=\begin{bmatrix}\dfrac{s(s+3)}{(s+2)(s+4)}\\[2mm] \dfrac{(s+3)}{(s+2)(s+4)}\end{bmatrix}$$

$$W_2(s)=\begin{bmatrix}\dfrac{s(s-3)}{(s+2)(s+4)}\\[2mm] \dfrac{(s-3)}{(s+2)(s+4)}\end{bmatrix}$$

$$W_3(s)=\frac{(s+1)(s-3)}{(s+2)(s+4)}$$

$$W_4(s)=\frac{(s-1)(s+3)}{(s+2)(s+4)}$$

$$W_5(s)=\frac{(s-1)(s-3)}{(s+2)(s+4)}.$$

It is straightforward to check that

$$\overline{W}(s^*)\,\overline{W}(s)\geq W_3(s^*)W_3(s)\geq W_5(s^*)W_5(s) \quad \text{in } Re[s]\geq 0$$

and

$$\overline{W}(s^*)\,\overline{W}(s)\geq W_4(s^*)W_4(s)\geq W_5(s^*)W_5(s) \quad \text{in } Re[s]\geq 0.$$

One can check also that no comparison of $W_3(s^*)W_3(s)$ and $W_4(s^*)W_4(s)$ can be made over $Re[s]>0$.

The steps $\overline{W}\to W_3\to W_5$ and $\overline{W}\to W_4\to W_5$ involve transfer of zeros of $\overline{W}(s)$ from the left half plane to the mirror image right half plane points. One can check, most easily using this idea, that

$$\text{Arg } \overline{W}(j\omega)<\text{Arg } W_3(j\omega)<\text{Arg } W_5(j\omega)$$

$$\text{Arg } \overline{W}(j\omega)<\text{Arg } W_4(j\omega)<\text{Arg } W_5(j\omega)$$

for all real ω, a fact motivating the name *minimum phase* for $\overline{W}$. It obviously also makes sense to say that W_3 and W_4 are both spectral factors with more phase than $\overline{W}$ and less phase than W_5.

Now consider $W_1(s)$ and $W_2(s)$. One cannot sensibly retain the idea of phase. However, one can check that

$$\overline{W}(s^*)\overline{W}(s)\geq W_1'(s^*)W_1(s)\geq W_4(s^*)W_4(s)$$

and

$$\overline{W}(s^*)\overline{W}(s)\geq W_1'(s^*)W_1(s)\geq W_2'(s^*)W_2(s)$$
$$\geq W_5(s^*)W_5(s)$$

and, in an extended sense, we shall say that W_1 has more phase than $\overline{W}$, and less phase than W_2, W_4 and W_5. The phases of W_1 and W_3 are not comparable.

We carry over these ideas to arbitrary power spectrum matrices as follows.

Definition. Let $\Phi(s)$ be a power spectrum matrix and let $W_1(s)$ and $W_2(s)$ be two stable spectral factors. Then $W_1(s)$ has less phase than $W_2(s)$, written $W_1 \prec W_2$, if

$$W_1'(s^*)W_1(s)\geq W_2'(s^*)W_2(s) \quad Re[s]\geq 0 \tag{6}$$

and the equality fails to hold for at least one s in $Re[s]>0$. We write $W_1\succcurlyeq W_2$ in case (6) holds, with the possibility of equality for all s in $Re[s]\geq 0$. Obviously, $\prec$ defines a partial ordering, and $\overline{W}\preccurlyeq W_1$ where $\overline{W}$ and W_1 are respectively a minimum phase and an arbitrary spectral factor.

Finally, we wish to give a time-domain formulation of the partial ordering notion just defined.

Theorem 2

Let $\Phi(s)$ be a power spectrum matrix and let $W_1(s)$ and $W_2(s)$ be two stable spectral factors. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two systems with transfer function matrices $W_1(s)$ and $W_2(s)$, and denote the outputs of $\mathbf{S}_1$ and $\mathbf{S}_2$ by time-domain quantities $y_1(\cdot)$ and $y_2(\cdot)$. Let $u\,(\cdot)$ be a common input to $\mathbf{S}_1$ and square integrable on $(-\infty, t]$ for all finite t. Then $W_1\prec W_2$ if and only if

$$\int_{-\infty}^{t} y_1'(\tau)y_1(\tau)d\tau\geq\int_{-\infty}^{t} y_2'(\tau)y_2(\tau)d\tau \tag{7}$$

and equality does not hold for all $u(\cdot)$ and t. Further, if $u(\cdot)$ is square integrable on $(-\infty, \infty)$, one has, irrespective of the existence of any ordering on W_1 and W_2,

$$\int_{-\infty}^{+\infty} y_1'(\tau)y_1(\tau)d\tau=\int_{-\infty}^{+\infty} y_2'(\tau)y_2(\tau)d\tau. \tag{8}$$

Before proving the theorem, we remark that (7) and (8) may be interpeted by saying $\mathbf{S}_2$ delays signals more than $\mathbf{S}_1$, [8]. The less like a minimum phase transfer function matrix that a given transfer function matrix is, the more it will delay signals. Also, note that the "only if" part of the theorem is easy to prove if one knows that $W_2=VW_1$ with $V(s)$ such that every entry is analytic in $Re[s]\geq 0$

and $V'(-s)V(s)=I$. One can use arguments as in [8]. However, if $W_1 \prec W_2$, such a $V(s)$ need not exist. In case for example

$$W_1 = \begin{bmatrix} \dfrac{s}{s+2} \\ \\ \dfrac{1}{s+2} \end{bmatrix} \qquad W_2 = \frac{s-1}{s+2}$$

one has $W_1 \prec W_2$, but V clearly cannot exist.

Proof. Suppose first that $W_1 \prec W_2$. Let $\{s_i\}$ be a countably infinite set of points in $Re[s]>0$ chosen so that $\{e^{s_i\tau}\}$ constitutes a complete basis set for functions square integrable on $(-\infty, t]$. Write $u(\tau)=\sum_i e^{s_i\tau}u_i$ where the u_i are constant vectors. Then

$$y_k(\tau)=\sum_i W_k(s_i)e^{s_i\tau}u_i \qquad k=1, 2$$

and

$$\int_{-\infty}^{t} y_k'(\tau)y_k(\tau)d\tau = \sum_i u_i'^* \int_{-\infty}^{t} W_k'(s_i^*)W_k(s_i)e^{(s_i^*+s_i)\tau}d\tau u_i.$$

Then (7) is immediate on using $W_1 \prec W_2$.

Conversely, suppose that (7) holds for all square integrable $u(\cdot)$. Selecting $u(\tau)=e^{s_i\tau}u_i$ for $Re[\cdot]>0$ and u_i arbitrary leads, by reversal of the above argument, to the conclusion that $u_i'^*W_1'(s_i^*)W_1(s_i)u_i \geq u_i'^*W_2'(s_i^*)W_2(s_i)u_i$. In view of the arbitrary nature of u_i and s_i, it follows that $W_1 \prec W_2$.

We now establish (8). If $u(\cdot)$ is square integrable on $(-\infty, \infty)$ so are the $y_k(\cdot)$. [Recall that $W_k(s)$ is stable and rational, and $\Phi(\infty)<\infty$ implies $W_k(\infty)<\infty$]. Because the $y_k(\cdot)$ are square integrable one has by Parseval's Theorem

$$\int_{-\infty}^{+\infty} y_k'(\tau)y_k(\tau)d\tau$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty} U'(-j\omega)W_k'(-j\omega)W_k(j\omega)U(j\omega)d\omega$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty} U'(-j\omega)\Phi(j\omega)U(j\omega)d\omega \qquad k=1, 2.$$

We remark that if in the statement of the above theorem, the condition $W_1 \prec W_2$ is changed to $W_1 \preccurlyeq W_2$, then the possibility arises that equality in (7) could hold for all $u(\cdot)$ and t. If in fact $W_1'(s^*)W_1(s)=W_2'(s^*)W_2(s)$ for all s in $Re[s]>0$, then equality always holds in (7).

3. POSITIVE REAL MATRICES, AND A CLASS OF SPECTRAL FACTORS

A concept related to that of the power spectrum matrix is that of a *positive real matrix* [2–5].

Definition

A $p \times p$ matrix $Z(s)$ of real rational functions is positive real if

(1) Entries of $Z(s)$ are analytic in $Re[s]>0$

(2) Poles of entries of $Z(s)$ on $s=j\omega$ (ω real) are simple, and the associated residue matrix is non-negative hermitian

(3) $Z'(j\omega)+Z(j\omega)\geq 0$ for all real ω, with $j\omega$ not a pole of any element of $Z(s)$.

If $Z(s)$ is positive real, and has $Z(\infty)<\infty$, then

$$\Phi(s)=Z(s)+Z'(-s) \tag{9}$$

is a power spectrum matrix. Note that even if one or more entries of $Z(\cdot)$ possesses a pole on $s=j\omega$, ω real, this is cancelled in the addition of $Z(s)$ and $Z'(-s)$, so that $\Phi(\cdot)$, at least after cancellations, has no entries with $j\omega$-axis poles.

Conversely, if $\Phi(s)$ is a power spectrum matrix, a positive real $Z(s)$ satisfying (9) can be constructed. One procedure for doing this to carry out a partial fraction expansion of each entry of $\Phi(s)$. Summands with poles in $Re[s]<0$ are then added together to give an entry of $Z(s)$. It should be noted that the remaining summands form the corresponding entry of $Z'(-s)$.

In view of the relation between positive real matrices and power spectrum matrices, it makes sense to talk of spectral factors $W(s)$ associated with a positive real matrix $Z(s)$, that is, spectral factors of the associated power spectrum matrix. Such spectral factors satisfy

$$Z(s)+Z'(-s)=W'(-s)W(s). \tag{10}$$

If $Z(s)$ is real rational with $Z(\infty)<\infty$, it possesses completely controllable realizations $\{F, G, H, J\}$, the realization being a quadruple of real constant matrices such that

$$Z(s)=J+H'(sI-F)^{-1}G \tag{11}$$

with the pair $[F, G]$ completely controllable.

One has the following important characterization of positive real matrices, see [5, 9]:

Positive real lemma. Let $Z(s)$ be a $p\times p$ real rational matrix with $Z(\infty)<\infty$, and let $\{F, G, H, J\}$ be a completely controllable realization. Then $Z(s)$ is positive real if and only if there exists a triple of real matrices P, L and W_0 with $P=P'\geq 0$ such that

$$\begin{aligned} PF+F'P&=-LL' \\ PG&=H-LW_0 \\ W_0'W_0&=J+J'. \end{aligned} \tag{12}$$

Moreover, if $[F, H]$ is completely observable, P is nonsingular.

Procedures for finding solutions of (12) are discussed *in* e.g. [5]. *Each pair* L *and* W_0 *defines a* stable spectral factor $W(s)$ satisfying (10) by

$$W(s)=W_0+L'(sI-F)^{-1}G \tag{13}$$

and, actually, *all* minimal degree spectral factors* with entries analytic in $Re[s]>0$ are definable this way [10]. For a great many applications, minimal degree spectral factors are desired, and accordingly, there is great interest in studying the restricted class of spectral factors defined by (12) and (13).

In case F has an imaginary eigenvalue, the formula (13) would appear to yield poles on $s=j\omega$ (ω real) of entries of $W(s)$. However, there is a pole-zero cancellation. Equivalently, the associated modes of F are not observed by L.

Each spectral factor $W(s)$ of the form (13) defines a non-negative definite symmetric P by the first of (12). Conversely, suppose a non-negative definite matrix P is known which satisfies (12), without L and W_0 being known. (Actually, the *existence* of L and W_0 is of course known.) Then we can derive a family of $W(s)$ associated with $Z(s)$ as follows. Form the matrix

$$M=\begin{bmatrix} PF+F'P & PG-H \\ (PG-H)' & -(J+J') \end{bmatrix}. \tag{14}$$

If L and W_0 were known, we would have

$$M=-\begin{bmatrix} L \\ W_0' \end{bmatrix}[L' \quad W_0]. \tag{15}$$

Since they are not known, pairs L and W_0 can be found by factoring M as $M=-NN'$ and partitioning N. Such L and W_0 then define a spectral factor $W(s)$ by (13).

The important point is that although there are an infinity of such $W(s)$, the relation (15) forces $W'(s^*)W(s)$ to be independent of the particular L and W_0 chosen as shown below. In other words, the matrix P is associated with an infinite family of different spectral factors $W(s)$, but $W'(s^*)W(s)$ is the same for all members of the family. To establish this property, observe that

$$\begin{aligned} W'(s^*)W(s)&=[W_0'+G'(s^*I-F')^{-1}L][W_0+L'(sI-F)^{-1}G] \\ &=W_0'W_0+G'(s^*I-F')^{-1}LW_0 \\ &\quad + W_0'L'(sI-F)^{-1}G+G'(s^*I-F')^{-1}\times LL'(sI-F)^{-1}G=-[G'(s^*I-F)^{-1}I]M\begin{bmatrix} (sI-F)^{-1}G \\ I \end{bmatrix}. \end{aligned}$$

The last equality follows by (14) and (15). It shows that $W'(s^*)W(s)$ depends only on F, G, H, J and P, and not on the particular L and W_0 appearing in $W(s)$.

4. ORDERING OF SPECTRAL FACTOR PHASE AND THE P MATRIX

The matrices P in the solution triples $\{P, L, W_0\}$ of (12) can be given the natural partial ordering of symmetric matrices. Our main result shows that P and the associated spectral factor family have the same ordering.

Theorem 3

Let $Z(s)$ be a $p\times p$ positive real matrix with $Z(\infty)<\infty$, with poles of all elements in $Re[s]<0$ and possessing a completely controllable minimal realization $\{F, G, H, J\}$ with $Re\ \lambda_i(F)<0$. Let $P_k=P_k'\geq 0$ for $k=1, 2$ be two solutions of the positive real lemma equations and let $W_k(s)$, $k=1, 2$ be representatives from each of the associated spectral factor families. Then

$$P_1\leq P_2 \quad \text{if and only if} \quad W_1\leqslant W_2$$

* A minimal degree spectral factor is one with McMillan degree equal to the minimum McMillan degree of all possible spectral factors of a prescribed spectrum matrix $\Phi(s)$. As shown in [4], the degree of such spectral factors is one half the degree of $\Phi(s)$. Note also that the degree of a transfer function matrix $W(s)$ with $W(\infty)$ finite is the dimension of a minimal state-space realization of $W(s)$. Thus the dimension of F in (13) is an upper bound on the degree of $W(s)$, and is the degree of $W(s)$ if $[F, L]$ is completely observable.

and $P_1=P_2$ if and only fi $W_1'(s^*)W_1(s)=W_2'(s^*)\times W_2(s)$ for all s in $Re[s]\geq 0$.

Proof. We will show first that $W_1\lesseqq W_2$ implies $P_1\leq P_2$. Let x_0 be an arbitrary vector of dimension equal to that of P_k, and let S_k, $k=1$, 2 be systems with transfer function matrices $W_k(s)$ and state-space equations

$$\dot{x}=Fx+Gu$$
$$y_k=L_k'x+W_{0k}u.$$

[Here, L_k and W_{0k} are such that $W_k(s)=W_{0k}+L_k'(sI-F)^{-1}G$ for $k=1, 2$.] Let $u(\cdot)$ be an input to each S_k which is zero up till some time $t_0<0$, zero after $t=0$, and which takes $x(t_0)=0$ to $x(0)=x_0$. Existence is guaranteed by complete controllability. By Theorem 2,

$$\int_{-\infty}^{0} y_1'(\tau)y_1(\tau)d\tau\geq\int_{-\infty}^{0} y_2'(\tau)y_2(\tau)d\tau$$

and

$$\int_{-\infty}^{+\infty} y_1'(\tau)y_1(\tau)d\tau=\int_{-\infty}^{+\infty} y_2'(\tau)y_2(\tau)d\tau.$$

Hence

$$\int_{0}^{\infty} y_1'(\tau)y_1(\tau)d\tau\leq\int_{0}^{\infty} y_2'(\tau)y_2'(\tau)d\tau$$

or

$$x_0'\int_0^\infty e^{F'\tau}L_1L_1'e^{F\tau}d\tau x_0 \quad \leq x_0'\int_0^\infty e^{F'\tau}L_2L_2'e^{F\tau}d\tau x_0$$

or [see (12)]

$$x_0'P_1x_0\leq x_0'P_2x_0.$$

A minor variation on the above establishes that if $W_1'(s^*)W_1(s)=W_2'^*(s)W_2(s)$, then $P_1=P_2$.

Now suppose that $P_1\leq P_2$. Let L_k and W_{0k} ($k=1$, 2) together with P_k satisfy (12). Observe that

$$\begin{aligned}Z'(s^*)+Z(s)&=J+J'+G'(s^*I-F')^{-1}(P_kG+L_kW_{0k})+(P_kG+L_kW_{0k})'(sI-F)^{-1}G\\&=J+J'+W_{0k}L_k'(sI-F)^{-1}G\\&\quad+G'(s^*I-F)^{-1}L_kW_{0k}\\&\quad+G'(s^*I-F')^{-1}[P_k(sI-F)+(s^*I-F')P_k](sI-F)^{-1}G\\&=J+J'+W_{0k}'L_k'(sI-F)^{-1}G\\&\quad+G'(s^*I-F')^{-1}L_kW_{0k}\\&\quad+G'(s^*I-F')^{-1}L_kL_k'(sI-F)^{-1}G\\&\quad+(s+s^*)G'(s^*I-F')^{-1}P_k(sI-F)^{-1}G\\&=W_k'(s^*)W_k(s)+(s+s^*)G'(s^*I-F')^{-1}P_k(sI-F)^{-1}G.\end{aligned}$$

Since $P_1\leq P_2$ implies

$$(s+s^*)G'(s^*I-F')^{-1}P_1(sI-F)^{-1}G\leq(s+s^*)G'(s^*I-F')^{-1}P_2(sI-F)^{-1}G$$

for all s in $Re[s]\geq 0$, we have

$$W_1'(s^*)W_1(s)\geq W_2'(s^*)W_2(s)$$

for all s in $Re[s]\geq 0$, as required. Further, $P_1=P_2$ implies $W_1'(s^*)W_1(s)=W_2'(s^*)W_2(s)$.

We remark that the theorem can be extended easily to the case when poles of entries of $Z(s)$ can occur on $Re[s]=0$. The easiest procedure is to decompose $Z(s)$ as $Z(s)=Z_1(s)+Z_2(s)$, where $Z_1(s)$ has entries free of poles on $Re[s]>0$, and $Z_2(s)$ is lossless. Then, see [4, 5], spectral factors associated with $Z_1(s)$ are the same as those associated with $Z_2(s)$, while any ordering on the P matrices associated with $Z(s)$ induces a like ordering on those associated with $Z_1(s)$.

Partial results along the lines of Theorem 3 can be found in [10]. There, it is shown for example, in case $J+J'$ is nonsingular, that $\bar{P}\leq P$ where $\bar{P}$ is associated with the minimum phase and P is associated with an arbitrary spectral factor. This result has also been derived in [11]. Theorem 3 shows immediately that $\bar{P}\leq P$ without the restriction that $J+J'$ is nonsingular.

5. MAXIMUM PHASE SPECTRAL FACTORS

By analogy with the ordering property involving the minimum phase spectral factor of a prescribed power spectrum matrix $\Phi(s)$, see (4), one might be led to seek a maximum phase spectral factor $\tilde{W}(s)$ with the property that

$$W'(s^*)W(s)\geq\tilde{W}'(s^*)\tilde{W}(s) \qquad (16)$$

for all s in $Re[s]\geq 0$ and for all other spectral factors $W(s)$ of $\Phi(s)$. Such a search is futile, because, given a putative $\tilde{W}(s)$, one has

$$\tilde{W}'(s^*)\tilde{W}(s)\geq\left[\tilde{W}'(s^*)\frac{s^*-a}{s^*+a}\right]\left[\frac{s-a}{s+a}\tilde{W}(s)\right]$$

for all s in $Re[s]\geq 0$ and for arbitrary $a>0$. This calculation is straightforward, and it is easily seen that equality cannot hold.

Nevertheless, in a restricted sense there does not exist a maximum phase spectral factor: as we shall show, $\tilde{W}(s)$ exists satisfying (16) if both $W(s)$ and $\tilde{W}(s)$ are restricted to being minimal degree spectral factors. Further, we can show that $P \leq \tilde{P}$ where P and $\tilde{P}$ are the matrices associated with $W(s)$ and $\tilde{W}(s)$ in the manner described in the previous section.

The notion of a maximum phase spectral factor does not appear to have arisen in classical treatments of spectral factorization. Its occurrence in the context of minimal degree spectral factors is slightly covered in [5, 10, 11].

To prove the claims, let $Z(s)$ be the positive real matrix associated with $\Phi(s)$, and let $Z(s)$ possess a minimal realization $\{F, G, H, J\}$. Then $Z'(s)$ is positive real, as is well known, and has a minimal realization $\{F', H, G, J'\}$. Now it is easily checked that if $\{Q, M, W_0\}$ is a solution triple of the positive real lemma equations for $Z'(s)$, then $\{P=Q^{-1}, L=PM, W_0\}$ is a solution triple of the positive real lemma equations for $Z(s)$. [Conversely, each solution $\{P, L, W_0\}$ associated with the $Z(s)$ equations determines a solution $\{Q=P^{-1}, M=-QL, W_0\}$ of the $Z'(s)$ equations]. Now let $\bar{Q}$ be associated with the minimum phase spectral factor family of $Z'(s)+Z(-s)=\Phi'(s)$, so that $\bar{Q} \leq Q$ for any other solution Q of the $Z'(s)$ equations. Associate with $\bar{Q}$ two matrices $\bar{M}$ and W_0 which define a particular member of the minimum phase spectral factor family. Without loss of generality and for covenience, assume W_0 is symmetric. Setting $\tilde{P}=\bar{Q}^{-1}$, it follows from $\bar{Q} \leq Q$ that $\tilde{P} \geq P$. The spectral factor of $Z(s)$ associated with $\{\tilde{P}, -\tilde{P}M, W_0\}$, viz.

$$\tilde{W}(s)=W_0-\bar{M}'\tilde{P}(sI-F)^{-1}G \qquad (17)$$

is then a maximum phase spectral factor among the set of minimal degree spectral factors, by Theorem 3.

Further insight into the maximum phase property is obtainable. When the spectral factors $W(s)$ are scalar, the minimum phase spectral factor, as is known, has all its zeros and poles in $Re[s] \leq 0$ and $Re[s]<0$ respectively. The maximum phase spectral factor is obtained by moving each zero, but not the poles, into the right half plane mirror image point with respect to the $j\omega$-axis. More generally, as we argue below in precise terms, whereas an $r \times n$ minimum phase spectral factor $\bar{W}(s)$ has constant rank in $Re[s]>0$, or perhaps $Re[s] \geq 0$, a maximum phase spectral factor $\tilde{W}(s)$ has constant rank in $Re[s]<0$, except at those points which are poles of entries of $\tilde{W}(s)$. To see this, we proceed as follows. With quantities as defined above,

$$\begin{aligned}
&[I-\bar{M}'(sI+F'+\tilde{P}\bar{M}\bar{M}')^{-1}\tilde{P}\bar{M}][W_0 - \bar{M}'(sI+F')^{-1}H] \\
&= W_0-\bar{M}'(sI+F'+\tilde{P}\bar{M}\bar{M}')^{-1}\tilde{P}\bar{M}W_0 - \bar{M}'(sI+F')^{-1}H \\
&\quad + \bar{M}'(sI+F'+\tilde{P}\bar{M}\bar{M}')^{-1}[(sI+F' + \tilde{P}\bar{M}\bar{M}')-(sI+F')](sI+F')^{-1}H \\
&= W_0-\bar{M}'(sI+F'+\tilde{P}\bar{M}\bar{M}')^{-1}(\tilde{P}\bar{M}W_0+H) \\
&= W_0-\bar{M}'\tilde{P}(sI+\tilde{P}^{-1}F'\tilde{P}+\bar{M}\bar{M}'\tilde{P})^{-1}(\bar{M}W_0 + \tilde{P}^{-1}H) \\
&= W_0-\bar{M}'\tilde{P}(sI-F)^{-1}G. \qquad (18)
\end{aligned}$$

The first three equalities follow by direct manipulation, and the last by using the positive real lemma equations. Let $\bar{V}(s)$ be the minimum phase spectral factor of $Z'(s)+Z(-s)=\Phi'(s)$ associated with the solution triple $\{\bar{Q}, \bar{M}, W_0\}$ of the positive real lemma equations. Then

$$\bar{V}(s)=W_0+\bar{M}'(sI-F')^{-1}H$$

so that (18) may be written

$$[I-\bar{M}'(sI+F'+\tilde{P}\bar{M}\bar{M})^{-1}\tilde{P}M]\,\bar{V}(-s)=\tilde{W}(s).$$

Now it is not difficult to show that

$$\det[I-\bar{M}'(sI+F'+\tilde{P}\bar{M}\bar{M})^{-1}\tilde{P}M] = \frac{\det(sI+F')}{\det(sI-F)}.$$

Using the constancy of rank $\bar{V}(s)$ in $Re[s]>0$ (or $Re[s] \geq 0$, as the case may be), one then sees that $\tilde{W}(s)$ has constant rank in $Re[s]<0$ (or $Re[s] \leq 0$), save at those points for which $\tilde{W}(s)$ has a pole. This establishes the claim made above.

Note also that (18) generalizes the idea that a maximum phase spectral factor has zeros which are imaginary axis reflections of the zeros of the minimum phase spectral factor: each zero of det $\tilde{W}(s)$, when $\tilde{W}(s)$ is nonsingular, is a zero of det $\bar{V}(-s)$, or the negative of a zero of det $\bar{V}(s)$.

6. APPLICATION TO OPTIMAL CONTROL AND FILTERING

Consider the problem of minimizing the performance index

$$V(x_0, u(\cdot))=\int_0^\infty [u'u+x'L_kL_k'x]dt \qquad (19)$$

associated with the completely controllable system

$$\dot{x}=Fx+Gu \quad x(0)=x_0 \qquad (20)$$

and subject to the constraint $\lim_{T\to\infty} x(T)=0$. Different L_k lead normally to different optimal control laws $u=-K_k'x$ and different performance indices $x_0'P_kx_0$. However, in case two L_k, $k=1, 2$ are such that

$$W_k'(-s)W_k(s) = G'(-sI-F')^{-1}L_kL_k'(sI-F)^{-1}G \quad (21)$$

is the same for $k=1, 2$, the resulting optimal control laws are the same, though the performance indices are different.*

To establish the first claim, one can use the well known fact, see e.g. [12], that

$$[I+G'(-sI-F)^{-1}K_k][I+K_k'(sI-F)^{-1}G] = I+G'(-sI-F')^{-1}L_kL_k'(sI-F)^{-1}G \quad (22)$$

and $Re\lambda_i(F-GK_k')<0$. Since the right side is independent of k, by assumption,

$$[I+G'(-sI-F')^{-1}K_1][I+K_1'(sI-F)^{-1}G] = [I+G'(-sI-F')^{-1}K_2][I+K_2'(sI-F)^{-1}G].$$

Post multiply by $[I+K_2'(sI-F)^{-1}G]^{-1}$ and premultiply by $[I+G'(-sI-F')^{-1}K_1]^{-1}$. There results

$$I+(K_1-K_2)'[sI-(F-GK_2')]^{-1}G = I+G'[-sI-(F-GK_1')']^{-1}(K_2-K_1).$$

Poles of all entries of $(K_1-K_2)'[sI-(F-GK_2')]^{-1}G$ lie in $Re[s]<0$ and of all entries of $G'[-sI-(F-GK_1')']^{-1}(K_2-K_1)$ in $Re[s]>0$. Hence these quantities are zero. Complete controllability then yields $K_1=K_2$, as claimed.

The main aim is now to examine the relation between the different costs associated with those control problems with the same control law. The key result is as follows.

Theorem 4

For the optimal control problem as posed above, suppose that

$$W_k(s)=L_k'(sI-F)^{-1}G \qquad k=1, 2 \quad (23)$$

and that $W_1'(-s)W_1(s)=W_2'(-s)W_2(s)$. Then $W_1 \leqslant W_2$ if and only if $P_1\leq P_2$, where the optimal performance index is $x_0'P_kx_0$, $k=1, 2$. Further, $P_1=P_2$ if and only if $W_1'(s^*)W_1(s)=W_2'(s^*)W_2(s)$ for all s in $Re[s]>0$.

Before proving the theorem, we comment on the significance of this result. For vector L_k and G, it says that the more right half plane zeros there are in the transfer function $L_k(sI-F)^{-1}G$, the greater is the cost of control; the situation is therefore similar to that in classical control. More generally though, the result says that outputs which make the plant minimum phase are easier to control than outputs which make the plant nonminimum phase, with outputs making the plant maximum phase the worst of all. Results along these lines for the case when the control weighting in (19) becomes vanishingly small appear in [6].

Proof. The matrices P_k are non-negative definite solutions of

$$P_kF+F'P_k-P_kGG'P_k+L_kL_k'=0$$

and the associated optimum control law $u=-K'x$ is given by

$$P_kG=K. \quad (24)$$

These equations lead to

$$P_k(F-\tfrac{1}{2}GK')+(F-\tfrac{1}{2}GK')'P_k=-L_kL_k'. \quad (25)$$

Now (24) and (25) together indicate via the Positive Real Lemma that the matrix $K'(sI-F-\tfrac{1}{2}GK')^{-1}G$ is positive real, with $X_k(s)=L_k'(sI-F-\tfrac{1}{2}GK')^{-1}G$ as associated spectral factors for $k=1, 2$. Hence

$$P_1\leq P_2 \quad \text{if and only if} \quad X_1 \leqslant X_2.$$

Now

$$\begin{aligned} W_k(s) &= L_k'(sI-F)^{-1}G \\ &= L_k'(sI-F-\tfrac{1}{2}GK')^{-1}G[I+\tfrac{1}{2}K'(sI-F)^{-1}G] \\ &= X_k(s)Y(s) \end{aligned}$$

for some nonsingular $Y(s)$. Immediately, $X_1 \leqslant X_2$ is equivalent to $W_1 \leqslant W_2$. This proves the main part of the theorem, and the remainder is trival to establish.

Notice the theorem also shows that, associated with a certain optimal control law, there will be a worst and a best loss function and associated optimal indices, defined by the maximum and minimum phase spectral factors.

* The question becomes trivial in case $W'_1(-s)W_2(s) = W'_2(-s)W_2(s)$ should imply $L_1L'_1=L_2L'_2$. This would be so if for example G was square and nonsingular. However, in general it will not be true and should not be expected if the control dimension is less than the state dimension.

The situation for filtering is also easily covered. One considers the signal generating system

$$\dot{x}=Fx+G_k u$$
$$y=H'x$$

where $u(\cdot)$ is a white noise process. The measurement process is

$$z=H'x+v$$

with v a white noise process, independent of u. The Kalman filter is defined via F, H and a certain gain matrix, and one can show, dualizing the control argument in the obvious way, that the gain matrix will be the same for any two G_k such that

$$H'(sI-F)^{-1}G_1G_1'(-sI-F')^{-1}H$$
$$=H'(sI-F)^{-1}G_2G_2'(-sI-F')^{-1}H.$$

The Kalman filter performance, as measured by the error covariance matrix P_k, depends on the particular G_k. The main result is that with $W_k(s)$ the transfer function matrix of the signal generating system, i.e.

$$W_k(s)=H'(sI-F)^{-1}G_k$$

and with

$$W_k(s)W_k'(-s)=\Phi(s)$$

independent of k, then $P_1 \leq P_2$ if and only if

$$W_1(s)W_1'(s^*)\geq W_2(s)W_2'(s^*)$$

or

$$W_1(s^*)W_1'(s)\geq W_2(s^*)W_2'(s)$$

which, in our earlier notation, is $W_1' \leqslant W_2'$. In this instance, minimum phase plants give the lowest error covariance, maximum phase plants the highest, and other plants an in-between covariance. The first part of this result was also found, by an entirely different and somewhat complicated procedure, in [1].

Let Π_1 and Π_2 denote the state covariance matrices of the signal generating systems. Thus

$$\Pi_k F' + F\Pi_k = -G_k G_k'.$$

Then it is easily seen that an ordering $P_1 \leq P_2$ implies and is implied by $\Pi_1 \leq \Pi_2$, so that, for example, minimum phase plants have the least state covariance matrix. This follows in fact either from $W_1' \leqslant W_2'$, or the fact that $P_k - \Pi_k$ is independent of k. This last interesting property of the Kalman filter appears to have first been pointed out in [14].

7. TIME-VARYING RESULTS

The point of this section is to argue, in a nondetailed fashion, that many of the preceding results can be extended to time-varying systems.

To fix ideas, consider the linear–quadratic control problem of minimizing

$$V(x_0, t_0, u(\cdot))$$
$$=\int_{t_0}^{\infty}[u'u+x'L_k(t)L_k'(t)x]dt \quad (26)$$

for the system

$$\dot{x}=F(t)x+G(t)u \quad x(t_0)=x_0. \quad (27)$$

Assume that $F(\cdot)$, $G(\cdot)$ and $L_k(\cdot)$ satisfy conditions which will guarantee existence of a control law $u(t)=-K_k'(t)x(t)$ making the closed-loop system asymptotically stable, see e.g. [13]. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two systems defined by (27) and the equation

$$y_k(t)=L_k'(t)x(t).$$

Suppose that $u(t)=0$ for $t<t_0$ and $x(t_0)=0$. If $L_k(\cdot)$ is such that for $k=1, 2$ one has

$$\int_{t_0}^{\infty} y_1'(t)y_1(t)dt=\int_{t_0}^{\infty} y_2'(t)y_2(t)dt$$

for all $u(\cdot)$, one would expect the optimal control laws for the problem posed above to be the same. If also it were true that for all $u(\cdot)$ and t,

$$\int_{t_0}^{t} y_1'(\tau)y_1(\tau)d\tau \geq \int_{t_0}^{t} y_2'(\tau)y_2(\tau)d\tau$$

one would expect $P_1 \leq P_2$, where $x_0'P_kx_0$ is the optimal performance index for the two optimal control problems.

Some of these ideas have been developed for the filtering problem in [1]. The minimum phase property actually corresponds to a property termed causal invertibility.

8. CONCLUSIONS

The preceding material has indicated a partial ordering on the set of spectral factors of a prescribed power spectrum matrix. For spectral factors of arbitrary degree, we have described this partial ordering in frequency-domain and time-domain terms, and we have given an algebraic interpetation for minimal degree spectral factors.

This algebraic interpetation has then been applied to the linear–quadratic control and filtering problems, and the conclusion has been reached that the nearer a plant is to being minimum phase, the less will be the cost of control, or the more accurate the filtering will be.

Though results are available for time-varying systems, they are almost certainly of less interest, and their delineation in detailed terms at this point would seem largely irrelevant. An interesting open problem however would be to establish results for smoothing akin to those for filtering. This seems difficult, and it is not clear that results can even be established. On the other hand, results for discrete time systems like those of this paper can be found, most simply it would seem via the bilinear transformation technique of [15] relating continuous and discrete time linear–quadratic problems.

REFERENCES

[1] B. D. O. ANDERSON and T. KAILATH: The choice of signal-process models in Kalman–Bucy filtering, *J. Math. Analysis Applic.* **35**, 659–668 (1971).

[2] R. E. KALMAN: On a New Characterization of Linear Passive Systems, Proceedings of the First Allerton Conference on Circuit and System Theory, University of Illinois, pp. 456–470 (1963).

[3] V. M. POPOV: Hyperstability and optimality of automatic systems with several control functions, *Rev. Roum. Sci. Tech.—Electrotechn. et Energ.* **9**, 629–690 (1964).

[4] B. D. O. ANDERSON: A system theory criterion for positive real matrices, *SIAM J. Control* **5**, 171–182 (1967).

[5] B. D. O. ANDERSON and S. VONGPANITLERD: *Network Analysis and Synthesis—A Modern Systems Approach.* Prentice Hall, New Jersey (1972).

[6] H. KWAKERNAAK and R. SIVAN, The maximally achievable accuracy of linear optimal regulators and linear optimal filters, *IEEE Trans. Aut. Control* **AC-11**, 79–86 (1972).

[7] D. C. YOULA: On the factorization of rational matrices, *IRE Trans. Inform. Theory* **IT-7**, 172–189 (1961).

[8] E. A. ROBINSON: External representation of stationary stochastic processes, *Arkiv für Mathematik* **4**, 379–384 (1962).

[9] B. D. O. ANDERSON and J. B. MOORE: Algebraic structure of generalized positive real matrices, *SIAM J. Control* **6**, 615–624 (1968).

[10] B. D. O. ANDERSON: The inverse problem of stationary covariance generation, *J. Statist. Phys.* **1**, 133–147 (1969).

[11] J. C. WILLEMS: Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Aut. Control* **AC-16**, 621–634 (1971).

[12] B. D. O. ANDERSON and J. B. MOORE, *Linear Optimal Control.* Prentice-Hall, New Jersey (1971).

[13] R. E. KALMAN: Contributions to the Theory of Optimal Control, Boletin de la Sosiedad Matematica Mexicana, pp. 102–119 (1960).

[14] B. D. O. ANDERSON and J. B. MOORE, State Estimation Via the Whitening Filter, Proceedings of the Joint Automatic Control Conference, Ann Arbor, Michigan, pp. 123–129 (1968).

[15] K. L. HITZ and B. D. O. ANDERSON: An iterative method of computing the limiting solution of the matrix Riccati differential equation, *Proc. IEE* **119**, 1402–1406 (1972).

Résumé—L'exposé dérive un résultat reliant les propriétés du domaine de fréquence et du domaine de temps de différents facteurs spectraux de la seule matrice de spectre de puissance. Ces résultats sont interprêtés du point de vue algébrique, et appliqués au problème de filtrage et de contrôle optimal quadratique linéaire. Des interprétations sont données du phénomène selon lequel de nonbreux problèmes de contrôle optimal peuvent mener à la même loi de contrôle optimal mais à un coût optimal différent, et de la même façon de nombreux problèmes de filtration peuvent mener au même filtre optimal, mais à une performance différente du filtre.

Zusammenfassung—Abgeleitet werden Beziehungen, die die Eigenschaften verschiedener Spektralfaktoren der gleichen Leistungsspektrum-Matrix im Frequenzbereich und Zeitbereich verbinden. Diese Resultate werden von einem algebraischen Gesichtspunkt aus interpretiert und auf das linear-quadratische Problem optimaler Steuerung und Filterung angewandt. Gegeben werden Interpretationen des Phänomens, daß viele Probleme der optimalen Steuerung zu dem gleichen Gesetz der optimalen Steuerung, aber verschiedenen optimalen Kosten führen können und daß in gleicher Weise viele Filterprobleme zu dem gleichen Optimalfilter aber verschiedenem Filterverhalten führen können.

Резюме—В статье приводится вывод результата исследований связи свойств различных спектральных элементов спектральной матрицы первой степени во временной и частотной областях. Полученные результаты интерпретируются с алгебраической точки зрения и применены к линейно-квадратичному оптимальному управлению и проблеме фильтрации. Даны объяснения того, что многие проблемы оптимального управления могут привести к одинаковому закону оптимального управления но с различной стоимостью оптимальности. Аналогично многие проблемы фильтрации могут привести к одному оптимальному фильтру, но с различной работой фильтра.

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Correspondence Item

Corrections to:* Algebraic Properties of Minimal Degree Spectral Factors†

Corrections aux "Propriétés algébriques de Facteurs Spectraux de Degré Minimal"

Korrekturen zu "Algebraische Eigenschaften von Spektralfaktoren minimalen Grades

BRIAN D. O. ANDERSON‡

Summary—Corrections and extensions to an earlier paper are presented. The material relates frequency domain, time domain and algebraic statements of a partial ordering of spectral factors.

RECENT correspondence with J. C. Willems has led us to the conclusion that in [1] part of the claims of Theorem 2, and consequential claims of later theorems, may be in error. At least part of the proof of Theorem 2 is certainly in error, but this of course does not necessarily invalidate the claims of the theorem. The purpose of this note is to patch up the situation as far as possible. The following is believed to be a correct result.

Theorem 1

Let $Z(s)$ be a $p\times p$ positive real matrix with $Z(\infty)<\infty$, with poles of all elements in Re $[s]<0$ and with a minimal realization $\{F, G, H, J\}$. Let P_k $(k = 1, 2)$ be two matrices such that

$$M_k = \begin{bmatrix} P_k F+F'P_k & P_k G-H \\ (P_k G-H)' & -(J+J') \end{bmatrix} \leqslant 0. \tag{1}$$

Note that such matrices exist by the Positive Real Lemma as reviewed in [1]. With L_k and W_{0k} any matrices such that

$$M_k = -\begin{bmatrix} L_k \\ W_{0k}' \end{bmatrix} [L_k' \quad W_{0k}], \tag{2}$$

let $W_k(s) = W_{0k}+L_k'(sI-F)^{-1}G$. Let $\mathscr{S}_k$ be the linear system $\dot{x} = Fx+Gu$, $y_k = L_k'x+W_{0k}u$. Then the following three conditions are equivalent

(A) $P_1\geqslant P_2$.

(B) For any $u(\cdot)$, zero up till $t = 0$, with $x(0) = 0$, corresponding outputs of $\mathscr{S}_1$ and $\mathscr{S}_2$ are related by

$$\int_0^T y_1'(\tau)\, y_1(\tau)\, d\tau \leqslant \int_0^T y_2'(\tau)\, y_2(\tau)\, d\tau. \tag{3}$$

There is equality when $T=\infty$, provided $u(\cdot)$ is in $\mathscr{L}_2[0,\infty)$.

(C) For arbitrary n and s_i, $i = 1, 2, \ldots, n$, in Re $[s]>0$, the matrix with $i-j$ block submatrix

$$\frac{W_2'(s_i^*)\, W_2(s_j)-W_1'(s_i^*)\, W_1(s_j)}{s_i^*+s_j} \tag{4}$$

is non-negative definite.

Proof. (A) $\Leftrightarrow$ (B). We have $y_k(\tau) = W_{0k}\, u(\tau)+L_k'x(\tau)$ and so

$$\begin{aligned} y_k'(\tau)\, y_k(\tau) &= u'(\tau)\, W_{0k}'W_{0k}\, u(\tau)+2x'(\tau)\, L_k W_{0k}\, u(\tau) \\ &\quad +x'(\tau)\, L_k L_k' x(\tau) \\ &= 2u'(\tau)\, Ju(\tau)+2x'(\tau)\, Hu(\tau)-2x'(\tau)\, P_k Gu(\tau) \\ &\quad -2x'(\tau)\, P_k Fx(\tau) \\ &= 2u'(\tau)\, y(\tau)-2x'(\tau)\, P_k \dot{x}(\tau), \end{aligned}$$

where $y = Ju+H'x$. Then

$$\int_0^T y_1'(\tau)\, y_1(\tau) d\tau - \int_0^T y_2'(\tau)\, y_2(\tau)\, d\tau = x'(T)\,(P_2-P_1)\, x(T)$$

For finite T, (A) $\Rightarrow$ (B) and (B) $\Rightarrow$ (A) using complete controllability. For $T=\infty$, $u(\cdot)\in\mathscr{L}_2$ and Re $\lambda_i(F)<0$ yields $x(\infty) = 0$ and the requisite equality in (B).

(B) $\Leftrightarrow$ (C). Notice that if we take $u(\cdot)\in L_2(-\infty, T_1)$, the inequality of (B) can be modified by changing the integration interval to $(-\infty, T]$ for any $T\leqslant T_1$. Moreover, if we take $u(\cdot)$ to be complex, then the integrands should be modified to $y_k'^*(\tau)\, y_k(\tau)$.

Now take

$$u(\tau) = \sum_{i=1}^{n} \exp(s_i\,\tau)\, u_i, \quad \tau\in(-\infty,\infty)$$

for arbitrary complex u_i. Then

$$y_k(\tau) = \sum_i W_k(s_i) \exp(s_i\,\tau)\, u_i$$

and

$$\int_{-\infty}^0 y_k'^*(\tau)\, y_k(\tau)\, d\tau = \sum_{i,j} u_i^* \frac{W_k'(s_i^*)\, W_k(s_j)}{s_i^*+s_j}\, u_j.$$

The inequality of (B) implies the non-negativity claimed in (C).

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The inequality of (B) follows from (C) by using the fact that the functions $\exp(s_i \tau)$, $\text{Re}[s_i] > 0$ span $\mathscr{L}_2[-\infty, T]$ for all finite T, and by reversing the argument immediately above. To prove equality when $T = \infty$, $u(\cdot)$ is in $\mathscr{L}_2[0, \infty)$ and the lower limit of integration is 0, the earlier argument relating (A) and (B) applies.

Remarks

1. Though the above proof is complete in itself, it is probably interesting to obtain a direct connection between (A) and (C). Some minor algebra will show that

$$Z'(s_i^*)+Z(s_j) = W_k'(s_i^*)\, W_k(s_j) + (s_i^*+s_j)\, G'(s_i^* I-F')^{-1} P_k(s_j I-F)^{-1} G$$

or that

$$\frac{W_2'(s_i^*)\, W_2(s_j)-W_1'(s_i^*)\, W_1(s_j)}{s_i^*+s_j} = G'(s_i^* I-F')^{-1}(P_1-P_2)(s_j I-F)^{-1} G$$

This identity can be used to show the equivalence of (A) and (C).

2. The connection between (B) and (C) does not depend on the rationality or otherwise of the $W_k(s)$. The equality part of (B) does require that

$$W_k'(-s)\, W_k(s) = Z(s)+Z'(-s),$$

an equality which is implicit in the rational case in our definitions of $W_k(s)$.

The frequency domain-ordering used in [1] was

(D) $$W_2'(s_i^*)\, W_2(s_i)-W_1'(s_i^*)\, W_1(s_i) \geqslant 0$$

for all s_i in $\text{Re}[s_i] > 0$, and in [1] we claimed the equivalence of (A), (B) and (D).

Now (C) $\Rightarrow$ (D), as is seen immediately by taking $n = 1$. We are also able to state a set of circumstances under which (D) $\Rightarrow$ (A), (B) and (C).

Theorem II

Suppose $W_2(s)$ has a right inverse. Then condition (D) $\Rightarrow$ (A), (B) and (C).

Proof. Set $S(s) = W_1(s)\, W_2^{-1}(s)$. If $S(s)$ is not square, add rows or columns as required to make it square. Then (D) $\Rightarrow$

$$I-S'(s_i^*)\, S(s_i) \geqslant 0 \tag{5}$$

for all s_i in $\text{Re}[s] > 0$. At this point, there are two ways to proceed; either we can appeal to a theorem of Pick [2] to conclude that the matrix with $i-j$ block submatrix

$$\frac{I-S'(s_i^*)\, S(s_j)}{s_i^*+s_j} \tag{6}$$

is positive definite, or we can use the fact that $S(s)$ is a scattering matrix [3] in view of (5). Then for a system with input $v(\cdot)$, output $w(\cdot)$ and transfer function matrix $S(s)$ one has for all T

$$\int_{-\infty}^{T} (v'v-w'w)\, dt \geqslant 0,$$

assuming the integrals are well defined. Taking

$$w(t) = \sum \exp(s_i t)\, w_i, \quad T = 0,$$

yields non-negativity of the matrix with $i-j$ block submatrix (6). From (6) and the definition of $S(s)$, non-negativity of the matrix with $i-j$ block (4) follows.

Remarks

1. It remains an open question as to whether condition (D) *always* implies (C). In view of work of Willems related to a similar problem [4], it would be fair to conjecture that (D) does not always imply (C), causing Theorem 2 of [1] to be probably incorrect.

2. Evidently, condition (C) is a more fundamental frequency domain condition than is condition (D). One could presume it would be relevant to the problem considered by Willems in [4], where a frequency domain condition analogous to (D) is noted as being inadequate.

3. A variant on the proof of Theorem 2 will show that if there exists a rational $V(s)$ with $W_1 = VW_2$ and $I-V'(s^*)\, V(s) \geqslant 0$ in $\text{Re}[s] > 0$, then condition (D) implies (C).

References

[1] B. D. O. ANDERSON: Algebraic properties of minimal degree spectral factors. *Automatica* **9**, 491–500 (1973).

[2] G. PICK: Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden. *Mathematische Annalen* **77**, 7–23 (1916).

[3] B. D. O. ANDERSON and S. VONGPANITLERD: *Network Analysis and Synthesis—A Modern Systems Theory Approach.* Prentice Hall, New Jersey (1973).

[4] J. C. WILLEMS: On the existence of a nonpositive solution to the Riccati equation. *IEEE Trans. Aut. Control*, **AC-19**, No. 5 (1974).