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Well-Behaved Itô Equations with Simulations that Always Misbehave

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***Abstract*—Itô equations, including one arising in the theory of optimal filtering of a random telegraph wave, are examined. Although the solutions of the equations are always bounded, simulations of the equation on a digital computer lead to unbounded solutions.**

The purpose of this note is to draw attention to the existence of Itô differential equations whose solutions are well behaved (in fact, bounded with probability one), and with the additional surprising property that digital computer simulations of the equation are not well behaved (in fact, solutions are unbounded). This was observed in the first instance experimentally in simulating a filter for a system generating a random telegraph wave. Other equations were then considered, and a theoretical justification for the phenomenon was sought.

Consider first the following Itô equation for $t \geq 0$ and $x(0)$ such that $|x(0)| < 1$:

$$dx = -\beta^2 x(1 - x^2)\, dt - \alpha(1 - x^2)\, dt + \beta(1 - x^2)\, dw. \quad (1)$$

As usual, $w(t)$ is a Wiener process. Also, α and β are real constants with β positive. Setting

$$z = \ln \frac{1 + x}{1 - x}$$

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leads via the Itô differential rule [1] to

$$dz = -2\alpha\, dt + 2\beta\, dw$$

or

$$z(t) - z(0) = -2\alpha t + 2\beta w(t)$$

so that the solution of (1) is

$$x(t) = \left\{\frac{1 + x(0)}{1 - x(0)} \exp\left[-2\alpha t + 2\beta w(t)\right] - 1\right\} \cdot \left\{\frac{1 + x(0)}{1 - x(0)} \exp\left[-2\alpha t + 2\beta w(t)\right] + 1\right\}^{-1} \quad (2)$$

From this and the fact that $|x(0)| < 1$, it is easily seen that $|x(t)| < 1$ for all t.

Digital computer simulations of (1) are based on the difference equation

$$x(\overline{k+1}\, T) - x(kT) = \{-\beta^2 x(kT)[1 - x^2(kT)] - \alpha[1 - x^2(kT)]\} T + \{\beta[1 - x^2(kT)]\}\sqrt{T}v(kT) \quad (3)$$

with $v(kT)$ a unit variance zero-mean white Gaussian sequence. Such simulations were found experimentally to be unstable, that is, at some $k = K$ (depending on the particular $v(\cdot)$ realization), $x(kT)$ broke out of $[-1, 1]$, and after $k = K$, $|x(kT)|$ behaved in an unbounded fashion, in the sense that the longer the simulation the greater the value of $\max_k |x(kT)|$.

In retrospect, it is not hard to see heuristically why this should be so. Suppose $|x(kT)| < 1$. Then for each k there is a nonzero probability that $|v(kT)|$ will be sufficiently large in (3) that $|x(k+1)T| > 1$ will result. This accounts for the breaking out of $[-1, 1]$. To account for the subsequent unboundedness, suppose for the moment that $|x(KT)| > 1$, and suppose also that $v(kT) = 0$ for $k \geq K$. Then if $x(KT) > 1$, (3) yields $x(\overline{K+1}\, T) > x(KT)$ and if $x(kT) < -1$ and $\beta^2 x(kT) + \alpha < 0$, (3) yields $x(\overline{K+1}\, T) < x(KT)$. This means that the undriven part of (3) is unstable outside the interval $[-m, 1]$ with $m = \max\{1, \alpha/\beta^2\}$, and it is therefore not surprising that the driven version should also be.

The instability may also be viewed by considering the solution of (1) with $|x(0)| > 1$. Equation (2) is still the solution of (1) provided that

$$\frac{1 + x(0)}{1 - x(0)} \exp\left[-2\alpha t + 2\beta w(t)\right] + 1 \neq 0$$

or

$$\exp\left[-2\alpha t + 2\beta w(t)\right] \neq \frac{x(0) - 1}{x(0) + 1}.$$

Now, with $|x(0)| > 1$, the right-hand side of the inequality is always positive, and so we can write it as $\exp(2\beta k')$ with $k' > 0$ for $x(0) < -1$ and $k' < 0$ for $x(0) > 1$. The above inequality then becomes

$$w(t) \neq k' + \frac{\alpha}{\beta}t.$$

Thus, instability will occur whenever a sample function of the Wiener process crosses the line $y = k' + (\alpha/\beta)t$.

From the "law of the iterated logarithm" [2, p. 560], the sample functions $w(t)$ will, with probability 1, be bounded by $c\sqrt{2t\ln \ln t}$ for t sufficiently large and any $c > 1$, and secondly, the sample functions will each cross the curve $c\sqrt{2t\ln \ln t}$, $0 < c < 1$ infinitely many times. Also, note that $w(0) = 0$ and the sample functions of w are continuous with probability 1. Hence, for $k' < 0$ and $\alpha > 0$, it is clear that $w(t)$ must intersect the line $y = k' + (\alpha/\beta)t$ almost always, i.e., the equation (1) is unstable with probability 1 for $x(0) > 1$. However, for $x(0) < 1$, i.e., for $k' > 0$, we can say that intersection (and hence instability) will occur with some positive (but strictly less than 1) probability. Further, as α tends to zero from above, this probability will approach 1 from below, i.e., it will approach the probability of the sample functions crossing the line $y = k'$. Thus, for $\alpha = 0$, (1) is unstable with probability 1 for $|x(0)| > 1$.

An equation appearing in the study of filtering of a random telegraph wave [3] is

$$dx = [-2\nu x - \beta^2 x(1 - x^2) + \beta^2(1 - x^2)y]\, dt + \beta(1 - x^2)\, dw \quad (4)$$

where, again, $w(\cdot)$ is a Wiener process. The variable x is a conditional probability, and so lies in $[-1, 1]$, and $y(\cdot)$ is a random process, taking values of -1 and $+1$ and switching according to a Poisson-type law. The constant ν is positive and governs the switching rate.

Simulations of this equation on a digital computer exhibited the same sort of instability as earlier described, and the same sort of reasoning provides a heuristic justification of the instability. In simulating both (1) and (4), it was also found that the larger the constant β, the more pronounced was the effect, in that the jumping out of the region $[-1, 1]$ tended to occur sooner, and the subsequent runaway was more rapid. Earlier breaking out can be attributed to the fact that larger β means that smaller $|v(kT)|$ will cause breakout (and so the probability of breakout for any given k and $x(kT)$ is greater, the greater β is), and the greater rapidity of buildup can be ascribed to the $\beta^2 x(kT)[1 - x^2(kT)]$ term in (3), which, for $|x(kT)| > 1$, causes the buildup.

Experimentally, the instability problem can be easily remedied by placing $x(kT)$ on the right-hand side of (3) [and the difference equation associated with (4)] by $z(kT) = \text{sat}[x(kT)]$, i.e., $z(kT) = x(kT)$ for $|x(kT)| \leq 1$, $z(kT) = 1$ for $x(kT) \geq 1$, and $z(kT) = -1$ for $x(kT) \leq -1$.

Another approach to the problem of instability is the following. Since the breaking out of the interval $[-1, 1]$ is solely due to large values of $|v(kT)|$ in (3), it is reasonable to expect that, if this quantity is replaced by a saturation function with some scaling, and α, β, and $x(0)$ are given constants with $|x(0)| < 1$, then there will exist some value of T, say T^*, such that if $0 < T < T^*$, then $|x(kT)| < 1$ for all k. Detailed calculations establishing this are straightforward; the result follows from considering the inequality

$$|x - \beta^2 x(1 - x^2)T - \alpha(1 - x^2)T + \beta(1 - x^2)\sqrt{T}v| < 1$$

with $|x| < 1$.

References

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