Well-Behaved Itô Equations with Simulations that Always Misbehave

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Abstract—Itô equations, including one arising in the theory of optimal filtering of a random telegraph wave, are examined. Although the solutions of the equations are always bounded, simulations of the equation on a digital computer lead to unbounded solutions.

The purpose of this note is to draw attention to the existence of Itô differential equations whose solutions are well behaved (in fact, bounded with probability one), and with the additional surprising property that digital computer simulations of the equation were not well behaved (in fact, solutions are unbounded). This was observed in the first instance experimentally in simulating a filter for a system generating a random telegraph wave. Other equations were then considered, and a theoretical justification for the phenomenon was sought.

Consider first the following Itô equation for $t \geq 0$ and $x(0)$ such that $|x(0)| < 1$:

$$dx = -\beta x(1 - x) \, dt - \alpha (1 - x^2) \, dt + \beta (1 - x^2) \, dw.$$  
(1)

As usual, $w(t)$ is a Wiener process. Also, $\alpha$ and $\beta$ are real constants with $\beta$ positive. Setting

$$x = \frac{1 + z}{1 - z}$$

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leads via the Ito differential rule [1] to
\[ dz = -2\alpha \, dt + 2\beta \, dw \]
or
\[ s(t) - s(0) = -2\alpha t + 2\beta w(t) \]
so that the solution of (1) is
\[ s(t) = \left( 1 + \frac{z(0)}{1 - z(0)} \right) \exp \left[ -2\alpha t + 2\beta w(t) \right] - 1 \]
From this and the fact that \([z(0)] < 1\), it is easily seen that \([s(t)] < 1\) for all \(t\).

Digital computer simulations of (1) are based on the difference equation
\[ s[k+1] - s[k] = -\rho s[k](1 - s[k]) + \alpha (1 - s[k])^2 \]
where \(s[k], \rho, \alpha\) are the same as described, and the same sort of reasoning provides a heuristic justification of the instability. In simulating both (1) and (4), it was also found that the larger the \(\alpha\), the more pronounced was the effect, in that the jumping out of the region \([-1, 1]\) tended to occur sooner, and the subsequent runaway was more rapid. Earlier breaking out can be attributed to the fact that larger \(\alpha\) means that smaller \([s[k]]\) will cause breakdown, and so the probability of breakdown for any given \(k\) and \(s[k]\) is greater, the greater \(\alpha\) is, and the greater rapidity of buildup can be expected to be correlated with \(s[k]\).

Experimentally, the instability problem can be easily remedied by placing \(s[k]\) on the right-hand side of (3) and the difference equation associated with (3) by \(s[k] = \rho s[k-1] + \alpha (1 - s[k-1])^2\), i.e., \(s[k] = \rho s[k] + \alpha s[k-1] - 1\) for all \(k\) and \(s[k]\) is less than 1. This is the case of the momentum (1) with \([z(0)] > 1\). Equation (3) is still the solution of (1) provided that
\[ \frac{1 + z(0)}{1 - z(0)} \exp \left[ -2\alpha t + 2\beta w(t) \right] + 1 \neq 0 \]
Now, with \([z(0)] > 1\), the right-hand side of the inequality is always positive, and so we can write it as \(\exp(2\beta k')\) with \(k' > 0\) for \([z(0)] < 1\) and \([z(0)] < 1\) for \([z(0)] < 1\). The above inequality then becomes
\[ s(t) < k' + \frac{\alpha}{\beta} \]
Thus, instability will occur whenever a sample function of the Wiener process crosses the line \(y = k' + \alpha/\beta\).

From the “law of the iterated logarithm” [2, p. 560], the sample functions \(w(t)\) will, with probability 1, be bounded by \(w(t)\) if \(t\) sufficiently large and any \(\epsilon > 1\), and secondly, the sample functions will each cross the curve \(w(t)\) infinitely many times. Also, note that \([w(0)] = 0\) for the sample functions of \(w\) are continuous with probability 1. Hence, for \(k' < 0\) and \(\alpha > 0\), it is clear that \([s(t)]\) must intersect the line \(y = k' + \alpha/\beta\) almost always, i.e., the equation (1) is unstable with probability 1 for \([s(0)] < 1\). However, for \([z(0)] < 1\), i.e., for \(k' > 0\), we can say that intersection (and hence instability) will occur with some positive (but strictly less than 1) probability. Further, as \(\alpha\) tends to zero from above, this probability will approach 1 from below, i.e., it will approach the probability of the sample functions crossing the line \(y = k'\). Thus, for \(\alpha = 0\), (1) is unstable with probability 1 for \([s(0)] > 1\).

An equation appearing in the study of filtering of a random telegraph wave [3] is
\[ dz = [-2\alpha z - \rho \alpha(1 - z^2) + \rho \alpha(1 - z^2)\beta] \, dt + \beta(1 - z^2) \, dw \]
where, again, \(w(\cdot)\) is a Wiener process. The variable \(z\) is a conditional probability, and so \(z\) is bounded by \([-1, 1]\) and \(\rho(\cdot)\) is a random process, taking values of \(-1\) and \(+1\) and switching according to a Poisson-type law. The constant \(\alpha\) is positive and governs the switching rate.

Simulations of this equation on a digital computer exhibited the same sort of instability as earlier described, and the same sort of reasoning provides a heuristic justification of the instability. In simulating both (1) and (4), it was also found that the larger the constant \(\beta\), the more pronounced was the effect, in that the jumping out of the region \([-1, 1]\) tended to occur sooner, and the subsequent runaway was more rapid. Earlier breaking out can be attributed to the fact that larger \(\beta\) means that smaller \([s[k]]\) will cause breakdown, and so the probability of breakdown for any given \(k\) and \(s[k]\) is greater, the greater \(\beta\) is, and the greater rapidity of buildup can be described as being associated with \(s[k]\) for \(s[k]\) tends to be associated with the boundary problem (1), which, for \([s(0)] > 1\), causes the buildup.

Experimentally, the instability problem can be easily remedied by placing \(s[k]\) on the right-hand side of (3) and the difference equation associated with (3) by \(s[k] = \rho s[k-1] + \alpha (1 - s[k-1])^2\), i.e., \(s[k] = \rho s[k] + \alpha s[k-1] - 1\) for all \(k\) and \(s[k]\) is less than 1. This is the case of the momentum (1) with \([z(0)] > 1\). Equation (3) is still the solution of (1) provided that
\[ \frac{1 + z(0)}{1 - z(0)} \exp \left[ -2\alpha t + 2\beta w(t) \right] + 1 \neq 0 \]
Now, with \([z(0)] > 1\), the right-hand side of the inequality is always positive, and so we can write it as \(\exp(2\beta k')\) with \(k' > 0\) for \([z(0)] < 1\) and \([z(0)] < 1\) for \([z(0)] < 1\). The above inequality then becomes
\[ s(t) < k' + \frac{\alpha}{\beta} \]
Thus, instability will occur whenever a sample function of the Wiener process crosses the line \(y = k' + \alpha/\beta\).