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Partially Singular Linear-Quadratic Control Problems

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***Abstract*—Necessary and sufficient conditions are given for the nonnegativity of a partially singular quadratic functional associated with a linear system. The conditions parallel known conditions for the totally singular problem, and a known sufficiency condition for the partially singular problem can be derived from them.**

Introduction

Consider the following linear optimal control problem. Minimize

$$J[u(\cdot)] = \int_{t_0}^{t_f} [\tfrac{1}{2}x'Qx + \tfrac{1}{2}u'Ru + u'Cx] + \tfrac{1}{2}x'(t_f)S_f x(t_f) \quad (1)$$

subject to

$$\dot{x} = Ax + Bu \quad x(t_0) = 0 \quad Dx(t_f) = 0. \quad (2)$$

Here, the state vector x is n-dimensional, and the control vector u is m-dimensional. The matrices A, B, C, Q, and R are time varying

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and continuously differentiable with $\ddot{B}$ and $\ddot{C}$ finite; further, (2) is completely controllable over $[t_0,\tau]$ for all $\tau\epsilon(t_0,t_f]$. The matrices S_f and D are constant, with D of s rows and rank s. We take Q, R, and S_f symmetric, and, in order that the minimum not be $-\infty$, $R \geq 0$. Finally we assume $u(\cdot)$ is piecewise continuous on $[t_0,t_f]$. The three cases $R = 0$, $R \neq 0$ but $|R| = 0$, and $R > 0$ are distinguished by the terms totally singular, partially singular, and nonsingular, respectively. Our concern here is with the partially singular problem, and our aim is to present necessary and sufficient conditions for $J[u(\cdot)]$ to be nonnegative for all $u(\cdot)$ (so that the minimum for (1) and (2) becomes zero).

Background

Many insights into the three classes of problems are given (as well as many references to original sources of results concerning the problems) in the survey paper [1] of Jacobson. Among the results, we note that necessary and sufficient conditions are available for both the totally singular and nonsingular problems, and a differential form of the sufficiency conditions extends immediately to the partially singular problem. The necessary and sufficient conditions we present for the partially singular problem parallel those already known for the totally singular and nonsingular problems. The technique establishing these conditions relies on conversion of the partially singular problem to a totally singular problem.

Derivation of Necessity Conditions

Suppose that the control $u(\cdot)$ in (1) and (2) can be written as

$$u(t) = \int_{t_0}^{t} \bar{u}(\tau)\, d\tau \tag{3}$$

for some piecewise continuous $\bar{u}(\cdot)$. Make the following additional definitions:

$$\bar{x} = \begin{bmatrix} x \\ u \end{bmatrix} \quad \bar{A} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad \bar{Q} = \begin{bmatrix} Q & C' \\ C & R \end{bmatrix}$$
$$\cdot\ \bar{S}_f = \begin{bmatrix} S_f & 0 \\ 0 & 0 \end{bmatrix} \quad \bar{D} = [D \quad 0]. \tag{4}$$

Then it is clear that $J[u(\cdot)]$ in (1) is the same as $\bar{J}[\bar{u}(\cdot)]$ given by

$$\bar{J}[\bar{u}(\cdot)] = \int_{t_0}^{t_f} \tfrac{1}{2}\bar{x}'\bar{Q}\bar{x}\, dt + \tfrac{1}{2}\bar{x}'(t_f)\bar{S}_f\bar{x}(t_f) \tag{5}$$

and the problem of minimizing $J(u)$ subject to (2) *with the additional constraint that* (3) *holds* is equivalent to the problem of minimizing $\bar{J}[\bar{u}(\cdot)]$ subject to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \quad \bar{x}(t_0) = 0 \quad \bar{D}\bar{x}(t_f) = 0. \tag{6}$$

Now the various assumptions for the original problem associated with (1) and (2) all imply the corresponding assumptions[1] for the new problem associated with (5) and (6). Admissible controls $\bar{u}(\cdot)$ for the new problem define a subset of the admissible controls for the original problem. Therefore, any necessary condition for the new problem is *a fortiori* a necessary condition for the original problem, and it is therefore irrelevant to the derivation of such a necessary condition that the controls $u(\cdot)$ are restricted by (4). Necessary conditions appear in [1, see theorem 7] and will not be repeated here. Interpreting these conditions to allow for the special form of $\bar{A}$, etc., leads to the following result.

Theorem 1: With the aforementioned notation and assumptions, define matrices $\bar{W}_{11}$, $\bar{W}_{12}'$, $\bar{W}_{22}$, $\bar{\Phi}_{11}$, $\bar{\Phi}_{12}$, and Z via

[1] The only nonimmediate correspondence is that of the two controllability assumptions, but the correspondence does follow with several lines of algebra.

$$\begin{aligned} -\dot{\bar{W}}_{11} &= Q + A'\bar{W}_{11} + \bar{W}_{11}A & \bar{W}_{11}(t_f) &= S_f \\ -\dot{\bar{W}}_{12}' &= A'\bar{W}_{12} + \bar{W}_{11}B + C' & \bar{W}_{12}(t_f) &= 0 \\ -\dot{\bar{W}}_{22} &= R + B'\bar{W}_{12} + \bar{W}_{12}'B & \bar{W}_{22}(t_f) &= 0 \end{aligned} \tag{7}$$

$$\begin{aligned} -\dot{\bar{\Phi}}_{11} &= \bar{\Phi}_{11}A & \bar{\Phi}_{11}(t_f) &= I \\ -\dot{\bar{\Phi}}_{12} &= \bar{\Phi}_{11}B & \bar{\Phi}_{12}(t_f) &= 0 \end{aligned} \tag{8}$$

and

$$Z = \begin{bmatrix} D_1^{-1}D_2 \\ I \end{bmatrix}. \tag{9}$$

Then a necessary condition for the nonnegativity of $J[u(\cdot)]$ is that there exist a real symmetric $(m+n)\times(m+n)$ $\hat{\bar{P}}(t)$ for each $t \in (t_0,t_f)$ such that, with obvious partitioning of $\hat{\bar{P}}(\cdot)$

$$\hat{\bar{P}} \text{ is monotone increasing in } t \tag{10}$$

$$Z'\hat{\bar{P}}_{11}(t_f^-)Z \leq 0 \tag{11}$$

$$\bar{W}_{12} + \bar{\Phi}_{11}'\hat{\bar{P}}_{11}\bar{\Phi}_{12} + \bar{\Phi}_{11}'\hat{\bar{P}}_{12} = 0$$
$$\bar{W}_{22} + \bar{\Phi}_{12}'\hat{\bar{P}}_{11}\bar{\Phi}_{12} + \hat{\bar{P}}_{12}'\bar{\Phi}_{12} + \bar{\Phi}_{12}'\hat{\bar{P}}_{12} + \hat{\bar{P}}_{22} = 0. \tag{12}$$

We comment that, as for the totally singular case, the interval (t_0,t_f) can be replaced by $(t_0,t_f]$ when the final-time constraint in (2) is absent. The variation in argument is trivial.

Derivation of Sufficiency Conditions

The sufficiency conditions differ only marginally from the necessity conditions, and are given in Theorem 2. Note that Theorem 2 cannot be obtained by applying known sufficiency results for the nonnegativity of $\bar{J}[\bar{u}(\cdot)]$. While necessary conditions for nonnegativity of $\bar{J}[\bar{u}(\cdot)]$ are, *a fortiori*, necessary conditions for nonnegativity of $J[u(\cdot)]$, sufficient conditions for nonnegativity of $\bar{J}[\bar{u}(\cdot)]$ are not sufficient conditions for nonnegativity of $J(u)$ because not all admissible controls $u(\cdot)$ for the original problem can be associated with admissible controls $\bar{u}(\cdot)$ via (4). Rather the reverse is true; sufficient conditions for nonnegativity of $J[u(\cdot)]$ are, *a fortiori*, sufficient conditions for the nonnegativity of $\bar{J}[\bar{u}(\cdot)]$.

Theorem 2: With the above notation and assumptions, including the definitions of the matrices $\bar{W}_{ij}$, $\bar{\Phi}_{ij}$, and Z in Theorem 1, a sufficient condition for the nonnegativity of $J[u(\cdot)]$ is that there exist a real symmetric $(m+n)\times(m+n)$ $\hat{\bar{P}}(t)$ for each $t \in [t_0,t_f]$ such that (10), (11), and (12) hold, with $\hat{\bar{P}}_{11}(t_f^-)$ in (11) replaced by $\hat{\bar{P}}_{11}(t_f)$.

Note that the gap between the necessity and sufficiency results involves only the question of the interval over which $\hat{\bar{P}}(t)$ has certain properties; in one case the interval is (t_0,t_f). in the other, $[t_0,t_f]$. This is as for the totally singular case, see [1].

Proof: The proof parallels that used for the totally singular case, [1, see theorem 4]. We have

$$J[u(\cdot)] = \int_{t_0}^{t_f} [\tfrac{1}{2}x'Qx + \tfrac{1}{2}u'Ru + u'Cx + x'\hat{\bar{P}}_{11}(Ax + Bu - \dot{x})]\, dt + \tfrac{1}{2}x'(t_f)S_f x(t_f). \tag{13}$$

The matrix $\bar{P}_{11}$ is defined as follows. Let $\bar{W}$ be the matrix with submatrices $\bar{W}_{ij}$ already defined, and $\bar{\Phi}$ the transition matrix associated with $\bar{A}$, possessing the submatrices $\bar{\Phi}_{11}$ and $\bar{\Phi}_{12}$ defined earlier. Then $\bar{P}_{11}$ is the appropriate submatrix of

$$\bar{P} = \bar{W} + \bar{\Phi}'\hat{\bar{P}}\bar{\Phi}. \tag{14}$$

Now because $\hat{\bar{P}}$ is monotone increasing, each entry must be of bounded variation, and because $\bar{W}$ and $\bar{\Phi}$ are differentiable, $\bar{P}_{11}$ must have entries of bounded variation. This allows reformulation of $J[u(\cdot)]$ after integration by parts as

$$J[u(\cdot)] = \int_{t_0}^{t_f} \tfrac{1}{2}[x'\ u'] \cdot \begin{bmatrix} (Q + \bar{P}_{11}A + A'\bar{P}_{11})\,dt + d\bar{P}_{11} & (P_{11}B + C')\,dt \\ (B'\bar{P}_{11} + C)\,dt & R\,dt \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \tfrac{1}{2}x'(t_f)[S_f - \bar{P}_{11}(t_f)]x(t_f). \quad (15)$$

(Note that $u(\cdot)$ is piecewise continuous. Therefore, so is $x(\cdot)$, making (15) well defined.)

Nonnegativity of the term outside the integral follows as in [1]. Nonnegativity of the term inside the integral follows from the monotonicity of $\hat{\bar{P}}_1$ and the following easily derived relation:

$$\begin{bmatrix} (Q + \bar{P}_{11}A + A'\bar{P}_{11})\,dt + d\bar{P}_{11} & (\bar{P}_{11}B + C')\,dt \\ (B'\bar{P}_{11} + C)\,dt & R\,dt \end{bmatrix} = \begin{bmatrix} \Phi_{11}' & 0 \\ \Phi_{12}' & I \end{bmatrix} \begin{bmatrix} d\hat{\bar{P}}_{11} & d\hat{\bar{P}}_{12} \\ d\hat{\bar{P}}_{12}' & d\hat{\bar{P}}_{22} \end{bmatrix} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & I \end{bmatrix}. \quad (16)$$

Differential Form of Sufficiency Conditions

If the matrix $\hat{\bar{P}}$ of the sufficiency conditions of Theorem 2 is differentiable, it is evident from (16) that monotonicity of $\hat{\bar{P}}$ implies

$$\begin{bmatrix} \dot{\bar{P}}_{11} + Q + \bar{P}_{11}A + A'\bar{P}_{11} & \bar{P}_{11}B + C' \\ B'\bar{P}_{11} + C & R \end{bmatrix} \geq 0. \quad (17)$$

A minor variation of the proof of Theorem 2 then yields the sufficiency theorem reviewed in [1], see also [4], to the effect that $J[u(\cdot)]$ is nonnegative if for some $\bar{P}_{11}(t)$ defined for $t \in [t_0,t_f]$, (17) holds as well as

$$Z'[S_f - \bar{P}_{11}(t_f)]Z \geq 0. \quad (18)$$

Recovery of Conditions for Totally Singular Problem

Suppose that $R(t) = 0$ for all t, so that we have a special case of the preceding results. Taking the necessity conditions, it follows using (16) on (t_0,t_f) that $\bar{P}_{11}B + C' = 0$ almost everywhere on (t_0,t_f). Arguments as in [3] establish equality everywhere, and the standard necessity conditions (see [1, theorem 7]) are recovered. In a similar way, the almost identical sufficiency conditions can be obtained.

Conclusion

In this paper, we have filled a gap in the various sets of necessary and sufficient conditions for singular linear-quadratic control problems. It is possible that our alternative derivation could be obtained via the limiting approach of [3], and this would then allow numerical computation of, for example, the matrix $\bar{P}_{11}$. Nevertheless, the method given in this paper would seem more efficient if the conditions alone are required, without numerical values of all the quantities involved.

We have also recently sighted unpublished work of B. Molinari in which a totally different approach is used to prove the same main result as this paper. A key step is to demonstrate that the performance index (1), with nonzero $x(t_0)$, has an infimum over all admissible controls which must be quadratic in $x(t_0)$. This demonstration works for the nonsingular, partially singular, and totally singular cases.

Yet another approach could conceivably be adopted using Robbin's ideas [5], which amount to replacement of a partially singular problem by a combination of nonsingular and totally singular problems. However, the separation is not complete, and it would seem that a demonstration via this means could be very hard.

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