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A Simplified Schur-Cohn Test

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Abstract—**A new test is given for deciding whether a prescribed real polynomial of degree n has all its zeros inside the unit circle. The test involves examination of the sign of various linear combinations of the polynomial coefficients, as well as examination to check positive definiteness of a symmetric matrix of dimensions $\frac{1}{2}n \times \frac{1}{2}n$ when n is even, and $\frac{1}{2}(n+1) \times \frac{1}{2}(n+1)$ or $\frac{1}{2}(n-1) \times \frac{1}{2}(n-1)$ when n is odd.**

I. Introduction

A sampled-data system is stable if all the zeros of its characteristic polynomial lie within the unit circle. In this paper, we develop a new procedure for testing whether a prescribed polynomial has this zero distribution property, the chief merit of the test being that of computational simplicity.

For a polynomial of degree $n = 2m$, the test requires the checking for positive definiteness of an $m \times m$ symmetric matrix, and the checking for positivity of various linear combinations of the coefficients. For a polynomial of degree $n = 2m - 1$, the same is true, except that the symmetric matrix is of dimension $m \times m$, or $(m - 1) \times (m - 1)$.

Specific procedures for checking whether a prescribed polynomial possesses all its zeros inside the unit circle (in brief, whether the polynomial is stable) appear to have originated with Schur [1], although Hermite [2] pointed out that his algorithm for checking root distribution in a half plane could be adapted via a bilinear transformation. In effect, Schur offered two tests: one test required the checking for positive definiteness of a Hermitian form, defined using the coefficients of the polynomial under test; the second test required checking, for a specific sign pattern, the signs of a set of determinants, again defined using the coefficients of the polynomial under test. When a polynomial of degree n is under test, the Hermitian form is of order n, and the determinants are of order $2,4,\cdots,2n$.

An alternative derivation of Schur's results was obtained by Cohn [3] by induction on the degree of the polynomial under test. Cohn also extended Schur's results by showing that the rank and signature of Schur's Hermitian form, and the sign pattern of the determinants of Schur's second test, give information about the zero distribution of the polynomial under test, even when its zeros do not all lie in the unit circle.

Fujiwara [4], using ideas originally employed by Hermite [2] in studying conditions for the zeros of a polynomial to be in a half plane, rederived in a simple manner the Schur-Cohn result on the properties of the Hermitian form associated with a prescribed polynomial. He also showed that, when the Hermitian form is not positive definite, its rank and signature give information about the zero distribution of the polynomial.

It is but a small step from Schur's Hermitian form to the associated $n \times n$ Hermitian matrix, the entries of which are integral functions of the underlying polynomial. An explicit formula for the matrix entries has been given in many places, e.g., [5]-[11], with the last two references reproving the stability result by appealing to the method of Lyapunov.

In essence, our new test reduces the computational effort of the Schur-Cohn test by approximately halving the dimension of the matrix whose positive-definite character is to be checked. This is at the expense of the appearance of some easily checked inequalities on linear combinations of the polynomial coefficients.

Other simplifications and modifications of the determinant test have also been made by Jury [12], [14], [15], Jury and Gupta [13], and Jury and Anderson [16]. In [12], it was shown that, when a polynomial of degree n with real coefficients is being tested, each Schur-Cohn determinant of order $2k$ is the product of two determinants of order k, both readily constructible from the polynomial coefficients; it was further shown that a number of the Schur-Cohn conditions become redundant, with the result that approximately n determinants of orders ranging from 1 up to n need to be evaluated to check stability or zero distribution. In essence, this result halved the computational burden associated with the Schur-Cohn determinant test. Further simplifications for restricted degree polynomials appear in [13] and [14].

In [15], it was pointed out that the modified determinantal conditions of [12] guaranteeing stability could be rephrased as a requirement that two matrices be positive innerwise. Further, the signs of the inners of the two matrices give information about the zero distribution in the general case.

Many discrete-time system stability tests have a continuous-time counterpart. Our new test appears to be the analog of the *reduced Hermite criterion* [16]-[18], originally stated by Liénard and Chipart, for checking whether a prescribed polynomial has all its zeros in the left half plane. The reduced Hermite criterion is related to the better known Liénard-Chipart criterion, which approximately halves the testing required of the Hurwitz criterion [18], [19].

Since discrete-time stability results for a polynomial can be obtained by applying a bilinear transformation followed by continuous-time stability results, it might be at first thought that the tests of this paper are nothing more than a combination of a bilinear transformation followed by the reduced Hermite criterion. For several reasons, this is not the case. First, the normal Schur-Cohn matrix associated with a polynomial is not the Hermite matrix associated with a bilinear transformation of that polynomial; the same claim applies to the submatrices of the reduced stability criteria. Second, as close examination of our test shows, we do not compute the bilinear transformation of the polynomial under test, although we do compute approximately half of the coefficients of the transformed polynomial.

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Accordingly, the results of this paper are of interest independent of, for example, the reduced Hermite criterion.

The plan of the paper is as follows. In Section II, we state the main result, separating the statement of it for even and odd degree polynomials. For those interested solely in the mechanics of the test, Section II provides a self-contained treatment. Section III discusses the relation of the test to the Schur–Cohn–Fujiwara test. In indicating the relation, we indicate a simple, and, we believe, new variant on the Schur–Cohn–Fujiwara matrix, which in itself offers a computational simplification when positive definiteness is being checked. We show that the usual matrix of dimension $n \times n$ is orthogonally similar to the direct sum of two matrices of dimensions approximately $n/2 \times n/2$. Positive definiteness of two $n/2 \times n/2$ matrices is of course easier to check than positive definiteness of one $n \times n$ matrix.

Section IV discusses computational aspects. Sections V through VII are devoted to proving the main theorem; in Section V, we explore the significance of the linear inequalities, and we prove the main *result for even degree polynomials in Section VI. The result for odd* degree polynomials is recovered in Section VII by a type of singular perturbation technique, which allows application of the results for even degree polynomials. Section VIII contains conclusions.

It might be argued that proof of the result for even degree polynomials would be sufficient. For, given an odd degree polynomial $f(s)$, one could surely apply the even degree technique to $sf(s)$. Though the reasoning is correct and has been used elsewhere (see, e.g., [20]), one should recognize that the results to be obtained will not be as sharp, or, computationally, as simple.

II. Main Result

In this section, we confine ourselves to stating the main result for the stability of an arbitrary real polynomial $f(s)$ of degree n. In this and subsequent sections, we shall generally need to distinguish between the case of even degree polynomials and odd degree polynomials. In order to illustrate the results, we shall usually study, in a little more detail than for the case of arbitrary n, the result of taking $n = 3$ and $n = 4$.

Even Degree $f(s)$: Suppose that

$$f(s) = a_{2m}s^{2m} + a_{2m-1}s^{2m-1} + \cdots + a_0, \qquad a_{2m} > 0. \tag{1}$$

Without further comment, the coefficients of $f(\cdot)$ and any other polynomials introduced will be assumed real. With $f(\cdot)$ we associate two $m \times m$ symmetric matrices $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ defined by

$$\alpha_{ij} = \sum_{p=1}^{\min(i,j)} (a_{2m+p-i}a_{2m+p-j} - a_{i-p}a_{j-p}) + \sum_{p=1}^{i} (a_{2m+p-i}a_{j+p-1} - a_{i-p}a_{2m+1-j-p}) \tag{2}$$

and

$$\beta_{ij} = \sum_{p=1}^{\min(i,j)} (a_{2m+p-i}a_{2m+p-j} - a_{i-p}a_{j-p}) - \sum_{p=1}^{i} (a_{2m+p-i}a_{j+p-1} - a_{i-p}a_{2m+1-j-p}). \tag{3}$$

In case $n = 4$, one has, for example, the following:

$$A = \begin{bmatrix} a_4^2 - a_0^2 + a_4a_1 - a_0a_3 & a_4a_3 - a_0a_1 + a_4a_2 - a_0a_2 \\ a_4a_3 - a_0a_1 + a_4a_2 - a_0a_2 & a_4^2 + a_3^2 - a_0^2 - a_1^2 + a_4a_3 + a_3a_2 - a_0a_1 - a_1a_2 \end{bmatrix},$$

$$B = \begin{bmatrix} a_4^2 - a_0^2 - a_4a_1 + a_0a_3 & a_4a_3 - a_0a_1 - a_4a_2 + a_0a_2 \\ a_4a_3 - a_0a_1 - a_4a_2 + a_0a_2 & a_4^2 + a_3^2 - a_0^2 - a_1^2 - a_4a_3 - a_3a_2 + a_0a_1 + a_1a^2 \end{bmatrix}.$$

We also associate with $f(s)$ the polynomial $p(z)$ defined by

$$p(z) = (z-1)^{2m} f\left(\frac{z+1}{z-1}\right) = a_{2m}(z+1)^{2m} + a_{2m-1}(z+1)^{2m-1}(z-1) + \cdots + a_0(z-1)^{2m}. \tag{4}$$

(Evidently, all zeros of $f(s)$ lie in $|s| < 1$ if and only if all zeros of $p(z)$ lie in Re $[z] < 0$.) Suppose that coefficients b_i are defined by

$$p(z) = b_{2m}z^{2m} + b_{2m-1}z^{2m-1} + b_{2m-2}z^{2m-2} + \cdots. \tag{5}$$

Then the b_i are linear combinations of the a_i. In case $n = 4$, we have, for example,

$$\begin{aligned} b_0 &= a_0 - a_1 + a_2 - a_3 + a_4 \\ b_1 &= -4a_0 + 2a_1 - 2a_3 + 4a_4 \\ b_2 &= 6a_0 - 2a_2 + 6a_4 \\ b_3 &= -4a_0 - 2a_1 + 2a_3 + 4a_4 \\ b_4 &= a_0 + a_1 + a_2 + a_3 + a_4. \end{aligned}$$

In general, we have

$$b_i = \sum_{r=0}^{2m} \left[\sum_j (-1)^{r+i-j} a_r \binom{r}{j} \binom{2m-r}{i-j} \right]. \tag{6}$$

[The summation over j is governed by max $(0, 2m - r - i) \le j \le$ min (i,r).]

The stability results are summed up in the following theorem.

Theorem 1$_e$: With definitions as above, necessary conditions for the stability of $f(s)$ are that A and B be positive definite (in brief, $A > 0$ and $B > 0$), and that the b_i be positive. Any of the following is a sufficient condition:

a) $A > 0$; $b_{2i} > 0$, $i = 0,1,\cdots,m$; b) $A > 0$; $b_{2m} > 0$, $b_0 > 0$; $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$; c) $B > 0$; $b_{2i} > 0$, $i = 0,1,\cdots,m$; d) $B > 0$; $b_{2m} > 0$, $b_0 > 0$; $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$; e) $A > 0$; $B > 0$.

Odd Degree $f(s)$: Suppose now that

$$f(s) = a_{2m-1}s^{2m-1} + a_{2m-2}s^{2m-2} + \cdots + a_0, \qquad a_{2m-1} > 0. \tag{7}$$

With $f(\cdot)$ we associate symmetric matrices $A = (\alpha_{ij})$ and $B = (\beta_{ij})$ of dimensions $m \times m$ and $(m-1) \times (m-1)$, respectively, defined by

$$\begin{aligned} \alpha_{ij} = {} & \sum_{p=1}^{\min(i,j)} (a_{2m-1-i+p}a_{2m-1-j+p} - a_{i-p}a_{j-p}) \\ & + \sum_{p=1}^{i} (a_{2m-1-i+p}a_{j+p-1} - a_{i-p}a_{2m-j-p}); \qquad 0 \le i,j \le m-1 \\ = {} & \sum_{p=1}^{\min(i,j)} (a_{2m-1-i+p}a_{2m-1-j+p} - a_{i-p}a_{j-p}); \qquad i = m,\ j < m \text{ and } i < m,\ j = m \\ = {} & \tfrac{1}{2} \sum_{p=1}^{m} (a_{m-1+p}^2 - a_{m-p}^2); \qquad i = j = m \end{aligned} \tag{8}$$

and

$$\beta_{ij} = \sum_{p=1}^{\min(i,j)} (a_{2m-1-i+p}a_{2m-1-j+p} - a_{i-p}a_{j-p}) - \sum_{p=1}^{i} (a_{2m-1-i+p}a_{j+p-1} - a_{i-p}a_{2m-j-p}). \tag{9}$$

In case $n = 3$, one has

$$A = \begin{bmatrix} a_3^2 - a_0^2 + a_1a_3 - a_2a_0 & a_2a_3 - a_1a_0 \\ a_2a_3 - a_1a_0 & \tfrac{1}{2}(a_3^2 + a_2^2 - a_1^2 - a_0^2) \end{bmatrix}$$

$$B = [a_3^2 - a_0^2 - a_1a_3 + a_2a_0].$$

Following the even degree case, we also define the polynomial $p(z)$ with coefficients b_i by

$$p(z) = (z-1)^{2m-1} f\left(\frac{z+1}{z-1}\right) = a_{2m-1}(z+1)^{2m-1} + a_{2m-2}(z+1)^{2m-2}(z-1) + \cdots + a_0(z-1)^{2m-1}$$
$$= b_{2m-1}z^{2m-1} + b_{2m-2}z^{2m-2} + \cdots + b_0. \tag{10}$$

In case $n = 3$, one has, for example,

$$b_0 = -a_0 + a_1 - a_2 + a_3$$
$$b_1 = 3a_0 - a_1 - a_2 + 3a_3$$
$$b_2 = -3a_0 - a_1 + a_2 + 3a_3$$
$$b_3 = a_0 + a_1 + a_2 + a_3.$$

Equation (6) is replaced by

$$b_i = \sum_{r=0}^{2m-1} \left[\sum_j (-1)^{r+i+1-j} a_r \binom{r}{j} \binom{2m-1-r}{i-j}\right]. \tag{11}$$

For odd degree $f(s)$, the stability results are summed up as follows.

Theorem 1_o: With definitions as above, necessary conditions for $f(s)$ to be stable are that $A > 0$, $B > 0$, and $b_i > 0$, $i = 0,1,\cdots,2m-1$. Any of the following is a sufficient condition:

a) $A > 0$; $b_{2m-1} > 0$; $b_{2i} > 0$, $i = 0,1,\cdots,m-1$; b) $A > 0$; $b_0 > 0$; $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$; c) $B > 0$; $b_{2m-1} > 0$; $b_{2i} > 0$, $i = 0,1,\cdots,m-1$; d) $B > 0$; $b_0 > 0$; $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$; e) $A > 0$; $B > 0$.

It may be noted that conditions c) and d) can be utilized advantageously for the even case. In this case the even degree conditions can be obtained as a limiting case from the odd degree conditions.

III. Computational Aspects

In this section, we take up two points, viz., calculation of the b^i and comparison of our stability test with one based on the use of a bilinear transformation and application of, say, the reduced Hermite criterion.

The formulas (6) and (11) are of course trivial to derive. Their presentation in matrix form is, however, interesting; it was first done by Mansour [21]. Other material illustrating the matrix connection is contained in [22]–[24], with the latter reference containing the simplest yet combinational rule for generating the matrix relating the vector of a_i to the vector of b_i. As is almost self-evident and as discussed in detail in [24], computation of only half the b_i is a lesser task than computation of all of them.

Next, we shall consider testing the fifth degree polynomial

$$f(s) = a_5s^5 + a_4s^4 + \cdots + a_0$$

for which the associated polynomial $p(s)$ is

$$p(s) = b_5s^5 + b_4s^4 + \cdots + b_0.$$

Via the reduced Hermite test, $f(s)$ has all its zeros inside the unit circle if and only if: 1) either b_0, b_2, b_4, and b_5 are positive or b_0, b_1, b_3, and b_5 are positive; and 2)

$$\begin{bmatrix} b_4b_3 - b_5b_1 & -b_5b_0 + b_4b_1 \\ -b_5b_0 + b_4b_1 & b_2b_1 - b_3b_0 \end{bmatrix}$$

is positive definite.

Via the method of this paper, we conclude that $f(s)$ has all its zeros inside the unit circle if and only if: 1) either b_0, b_2, b_4, and b_5 are positive or b_0, b_1, b_3, and b_5 are positive; and 2)

$$\begin{bmatrix} a_5^2 - a_0^2 - a_5a_1 + a_0a_4 & a_5a_4 - a_0a_1 - a_5a_2 + a_0a_3 \\ a_5a_4 - a_0a_1 - a_5a_2 + a_0a_3 & a_4^2 - a_1^2 - a_4a_2 + a_1a_3 + a_5^2 - a_0^2 - a_5a_3 + a_0a_2 \end{bmatrix}$$

is positive definite.

The principal differences between the two approaches are as follows.

1) All the b_i are needed for the first approach, and only half are needed for the second approach.

2) The b_i must be calculated prior to checking positive definiteness in the first approach, but not for the second, allowing the possibility of simultaneous calculation in the second approach.

Even this is not the full story. For instance, if the a_i are literals rather than numbers, the second method has a clear advantage; on the other hand, if the a_i are numbers, computation of all the b_i gives a set of necessary conditions for stability, which, if not satisfied, would mean that the matrix positive definiteness would not be checked.

Finally, we comment that, just as the Hermite principal minors can be related to the Hurwitz determinants, so, by minor modification of ideas in [15], the principal minors of the matrices A and B of our test can be related to the set of inners determinants associated with the checking of stability.

IV. Connection with the Schur–Cohn–Fujiwara Criterion

In this section, our purpose is threefold. First, we shall review the relatively well-known Schur–Cohn criterion for the stability of a polynomial, as well as a less well-known extension of it, due to Fujiwara, which gives information about the zero distribution of a polynomial which is not stable. Second, we shall indicate the relationship of the matrices A and B introduced in the previous section to the Schur–Cohn matrix. Finally, and this step proves trivial, we shall establish part of the claims of Theorem 1, viz., that the conditions $A > 0$ and $B > 0$ are necessary and sufficient for stability.

The Schur–Cohn–Fujiwara Criterion: Consider the real polynomial

$$f(s) = \sum_{i=0}^{n} a_is^i, \qquad a_n > 0 \tag{12}$$

and associate with it the $n \times n$ symmetric matrix $C = (\gamma_{ij})$ defined by

$$\gamma_{ij} = \sum_{p=1}^{\min(i,j)} (a_{n-i+p}a_{n-j+p} - a_{i-p}a_{j-p}). \tag{13}$$

For $n = 3$, one has, for example,

$$C = \begin{bmatrix} a_3^2 - a_0^2 & a_2a_3 - a_1a_0 & a_1a_3 - a_2a_0 \\ a_2a_3 - a_1a_0 & a_3^2 + a_2^2 - a_0^2 - a_1^2 & a_2a_3 - a_1a_0 \\ a_1a_3 - a_2a_0 & a_2a_3 - a_1a_0 & a_3^2 - a_0^2 \end{bmatrix}$$

and, for $n = 4$, one has

$$C = \begin{bmatrix} a_4^2 - a_0^2 & a_3a_4 - a_1a_0 & a_2a_4 - a_2a_0 & a_1a_4 - a_3a_0 \\ a_3a_4 - a_1a_0 & a_4^2 + a_3^2 - a_1^2 - a_0^2 & a_3a_4 + a_2a_3 - a_2a_1 - a_1a_0 & a_2a_4 - a_2a_0 \\ a_2a_4 - a_2a_0 & a_3a_4 + a_2a_3 - a_2a_1 - a_1a_0 & a_4^2 + a_3^2 - a_1^2 - a_0^2 & a_3a_4 - a_1a_0 \\ a_1a_4 - a_3a_0 & a_2a_4 - a_2a_0 & a_3a_4 - a_1a_0 & a_4^2 - a_0^2 \end{bmatrix}.$$

These two examples illustrate an easily established property of arbitrary dimension C: Not only is there symmetry about the main diagonal, but also symmetry about the other diagonal.

Let us adopt the notation $n_+(X)$, $n_-(X)$, and $n_0(X)$ to denote the number of positive, negative, and zero eigenvalues of a symmetric matrix X. The result of Schur and Cohn, as extended by Fujiwara, is as follows [4].

Theorem 2: With $f(s)$ and C as defined above, the number of zeros s_i of $f(s)$ for which $|s_i| < 1$ and for which s_i^{-1} is not a zero are $n_+(C)$; the number of zeros s_i for which $|s_i| > 1$ and for which s_i^{-1} is not a zero are $n_-(C)$; the number of zeros s_i for which either $|s_i| = 1$ or for which s_i^{-1} is also a zero are $n_0(C)$.

Evidently, $f(s)$ is stable if and only if $C > 0$.

Relation of the Matrices C, A, and B: Consider first the case of even $n = 2m$. We shall show how A and B can be found via an orthogonal transformation of C. Examination of the definitions of α_{ij}, β_{ij}, and γ_{ij} shows easily that

$$\alpha_{ij} = \gamma_{ij} + \gamma_{i,2m+1-j} \qquad \beta_{ij} = \gamma_{ij} - \gamma_{i,2m+1-j} \tag{14}$$

so that the individual entries of A and B are easily related to individual entries of C. Also, however, it is not hard to verify, using the double symmetry of C, that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} I & I \\ -I & I \end{bmatrix} J_e C J_e \begin{bmatrix} I & -I \\ I & I \end{bmatrix} \tag{15}$$

where

$$J_e = \left[\begin{array}{c|ccccc} I_m & & & 0_m & & \\ \hline & 0 & \cdot & \cdot & \cdot\ 0 & 1 \\ & 0 & \cdot & \cdot & \cdot\ 1 & 0 \\ & \cdot & & & \cdot & \cdot \\ 0_m & \cdot & & \cdot & & \cdot \\ & \cdot & \cdot & & \cdot & \cdot \\ & 1 & \cdot & \cdot & \cdot\ 0 & 0 \end{array}\right]. \tag{16}$$

(This result is almost intuitive; the effect of postmultiplication by J_e in (15) is to reverse the order of the last m columns of C, while the postmultiplication by

$$\frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$$

then serves to form the sum and difference in the formulas (14) for α_{ij} and β_{ij}.)

In the case of odd $n = 2m - 1$, examination of the definitions of α_{ij}, β_{ij}, and γ_{ij} shows that

$$\begin{aligned} \alpha_{ij} &= \gamma_{ij} + \gamma_{i,2m-j}, && i,j \le m-1 \\ \alpha_{ij} &= \gamma_{ij}, && i = m, j < m \text{ or } i < m, j = m \\ \alpha_{mm} &= \tfrac{1}{2}\gamma_{mm} && \end{aligned} \tag{17}$$

while

$$\beta_{ij} = \gamma_{ij} - \gamma_{i,2m-j}. \tag{18}$$

It is not hard to verify that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \tfrac{1}{2}\left[\begin{array}{c|c|c} I_{m-1} & 0 & I_{m-1} \\ \hline 0 & 1 & 0 \\ \hline -I_{m-1} & 0 & I_{m-1} \end{array}\right] J_0 C J_0 \left[\begin{array}{c|c|c} I_{m-1} & 0 & -I_{m-1} \\ \hline 0 & 1 & 0 \\ \hline I_{m-1} & 0 & I_{m-1} \end{array}\right] \tag{19}$$

where

$$J_0 = \left[\begin{array}{c|ccccc} I_m & & & 0_{m\times m-1} & & \\ \hline & 0 & \cdot & \cdot & \cdot\ 1 & 0 \\ & 0 & \cdot & \cdot & \cdot\ 1 & 0 \\ 0_{m-1\times m} & \cdot & & & \cdot & \cdot \\ & \cdot & & \cdot & & \cdot \\ & \cdot & \cdot & & \cdot & \cdot \\ & 1 & \cdot & \cdot & \cdot\ 0 & 0 \end{array}\right]. \tag{20}$$

With $\dotplus$ denoting direct sum, it is clear for even and odd n from (15) and (19) that $n_+(A \dotplus B) = n_+(C)$, $n_-(A \dotplus B) = n_-(C)$, and $n_0(A \dotplus B) = n_0(C)$. Almost certainly, the computational burden in computing, for example, $n_+(A \dotplus B)$ from A and B will be less than the computational burden in computing $n_+(C)$ from C. Therefore, one could recommend the matrix $A \dotplus B$ as being more convenient to use, at least in Theorem 2, than the matrix C.

Implications of Theorem 1: If $f(s)$ is stable, Theorem 2 implies $C > 0$. Hence $A > 0$ and $B > 0$, as noted in the necessity claims of Theorem 1. Conversely, if $A > 0$ and $B > 0$, then $C > 0$, and, by Theorem 2, $f(s)$ is stable. This establishes the fifth sufficiency claim in Theorem 1.

V. Significance of the Linear Inequalities

In this section, we shall illustrate the necessity of the linear inequalities of Theorem 1. Then we shall explore a consequence of the restricted set of linear inequalities in the sufficiency part of Theorem 1.

Necessity of the Linear Inequalities: If $f(s)$ is stable, all zeros of $p(z)$ lie in Re $[z] < 0$. Then, as is well known, the coefficients b_i of $p(z)$ are all of the same sign. Hence, if we can show that any one of the b_i is positive, it follows that they are all positive. Now the fact that $a_n > 0$ implies that $f(s) > 0$ for large real s. Since $f(s)$ is stable, it must have constant sign for $s \in [1, \infty)$. Hence $f(1) > 0$. However, as is easily seen, $f(1) = \Sigma a_i = b_n$. This completes the proof of the necessity part of Theorem 1.

We remark that the condition $f(1) > 0$ for stability has been noted before; so has the condition $(-1)^n f(-1) > 0$, or $b_0 > 0$. See, e.g., [15].

Consequences of the Sufficiency Condition Linear Inequalities: For the remainder of this section, we shall suppose that $n = 2m$. Let us write

$$p(z) = q(z) + zr(z) \tag{21}$$

where $q(z)$ and $r(z)$ are both even, i.e., $q(z) = \sum_{i=0}^{m} b_{2i}z^{2i}$ and $r(z) = \sum_{i=1}^{m} b_{2i-1}z^{2i-2}$. The two types of linear inequalities we need to cope with are given in conditions a) and c) and conditions b) and d), respectively, of Theorem 1_e. The first type clearly implies

$$q(z) > 0, \qquad \forall \text{ real } z; \tag{22}$$

and the second type clearly implies

$$r(z) > 0, \qquad \forall \text{ real } z \tag{23}$$

and

$$q(z) > 0, \qquad \text{for sufficiently small and sufficiently large real } z. \tag{24}$$

Let us first examine consequences of (22). It is clear that

$$y_1(z) = -\frac{zr(z)}{q(z)} \tag{25}$$

is continuous for all real z, is zero for $z = 0$, and tends to zero as $z \to \infty$. Further, z_i is a positive real zero of $p(z)$ if and only if $q(z_i) + z_i r(z_i) = 0$ or $y_1(z_i) = 1$, and $-z_i$ is a negative real zero of $p(z)$ if and only if $y_1(z_i) = -1$. [Use the evenness of $q(\cdot)$ and $r(\cdot)$.] Suppose that one lists in increasing size the positive solutions z_i of the equations $y_1(z) = \pm 1$. Denote a solution of $y_1(z) = +1$ by z_i^+, and one of $y_1(z) = -1$ by z_i^-. There will be up to $n = 2m$ such solutions, and the continuity of $y_1(z)$ ensures that they must occur in even blocks, in the sense that the list has the following form:

$$\underbrace{z_1^+, z_2^+, \cdots, z_{2r_1}^+}_{\text{even}},\ \underbrace{z_{2r_1+1}^-, z_{2r_1+2}^-, \cdots, z_{2r_2}^-}_{\text{even}},\ \underbrace{z_{2r_2+1}^+, \cdots, z_{2r_3}^+}_{\text{even}}, \cdots$$

or

$$z_1^-, z_2^-, \cdots, z_{2r_1}^-, z_{2r_1+1}^+, z_{2r_1+2}^+, \cdots, z_{2r_2}^+, z_{2r_2+1}^-, \cdots, z_{2r_3}^-, \cdots.$$

(Fig. 1 may help to make the idea clear.)

It follows that the positive real zeros of $f(z)$ may be arranged in pairs $[z_{2i}^+, z_{2i+1}^+]$ such that there is no negative real zero $-z_k^-$ with $z_{2i}^+ \le z_k^- \le z_{2i+1}^+$. Similarly, the negative real zeros of $f(z)$ may be arranged in pairs $[-z_{2j}^-, -z_{2j+1}^-]$ with no positive real zero z_k^+ with $z_{2j}^- \le z_k^+ \le z_{2j+1}^-$.

Translating this second statement into a property of $f(s)$ leads to the following result.

Lemma 1: Let $f(s)$ be the real polynomial of (1) with degree $n = 2m$. Suppose quantities b_i are defined as in Section II and that $b_{2i} > 0$ for $i = 0, 1, \cdots, m$. Then the real zeros of $f(s)$ lying in $|s| < 1$ may be arranged in pairs (s_i, s_{i+1}) such that there is no real zero s_j of $f(s)$ with $|s_j| > 1$ and with $s_i \le s_j^{-1} \le s_{i+1}$.

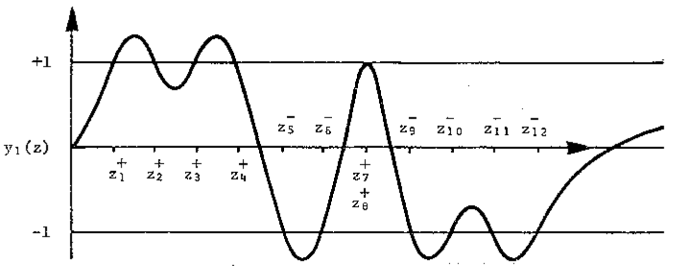

Fig. 1. Graph of $y_1(z)$. An even number of intersections with $y_1 = +1$ are followed by an even number of intersections with $y_1 = -1$.

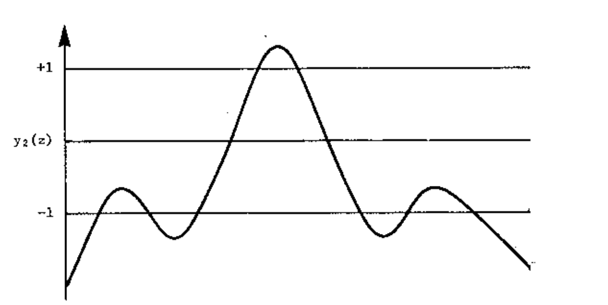

Fig. 2. Graph of $y_2(z)$. As z increases, there is first an odd number of intersections with $y_2 = -1$, then an even number with $y_2 = +1$, and finally there is an odd number with $y_2 = -1$.

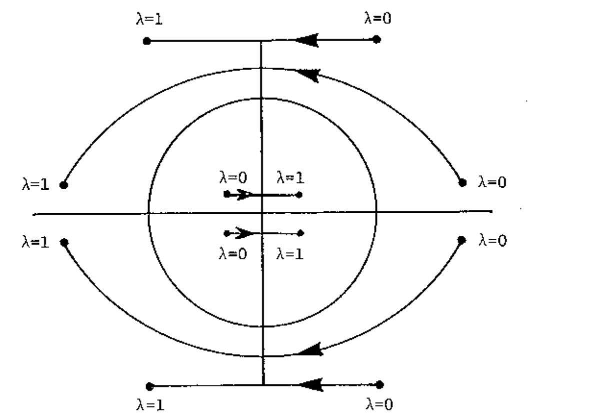

Fig. 3. Paths avoiding the unit circle linking points with their negatives.

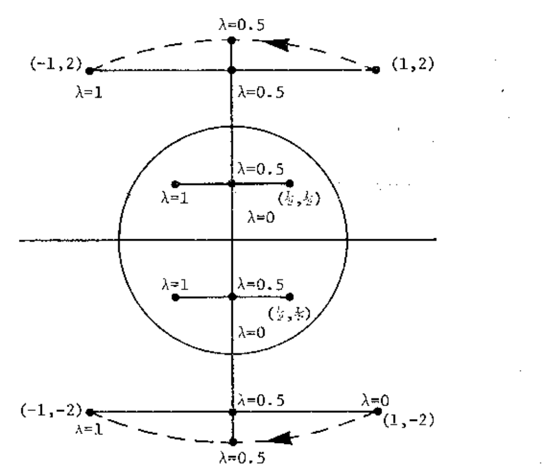

Fig. 4. The undistorted paths (solid line) at $\lambda = 0.5$ contain reciprocal points. The distorted paths (broken line) contain no reciprocal points.

We remark that it does not prove necessary in the sequel to discuss the pairing of zeros outside $|s| = 1$. In any case, this is complicated by the fact that the interval $[0, \infty)$ in the z-plane maps into two *disjoint* intervals $(-\infty, -1]$ and $[1, \infty)$ on the s-plane; a positive real z-plane pair can be split up in this mapping.

Now let us examine the consequences of (23) and (24). We form the quantity

$$y_2(z) = -\frac{q(z)}{zr(z)}, \tag{26}$$

which is evidently continuous for all $z \neq 0$, and approaches $-\infty$ when $z > 0$, $z \to 0$, and when $z \to \infty$. Clearly z_i is a positive real zero of $p(z)$ if and only if $y_2(z_i) = 1$, and $-z_i$ is a negative real zero if and only if $y_2(z_i) = -1$. Note that $z_i = 0$ cannot be a zero of $p(z)$ since $q(0) > 0$. As suggested by Fig. 2, when we list the solutions $z_i^{\pm}$ of $y_2(z) = \pm 1$ in increasing order, we obtain a list of the following type:

$$\underbrace{z_1^-, z_2^-, \cdots, z_{2r_1-1}^-}_{\text{odd}},\ \underbrace{z_{2r_1}^+, \cdots, z_{2r_2-1}^+}_{\text{even}},\ \underbrace{z_{2r_2}^-, \cdots, z_{2r_3-1}^-}_{\text{even}},$$

$$\cdots,\ \underbrace{z_{2r_{n-2}}^+, \cdots, z_{2r_{n-1}-1}^+}_{\text{even}},\ \underbrace{z_{2r_{n-1}}^-, \cdots, z_{2r_n}^-}_{\text{odd}}.$$

Let us now translate this into a property of $f(s)$. The negative real zeros $-z_1^-, -z_2^-, \cdots$ of $p(z)$ define zeros of $f(s)$ inside $(-1, 1)$. Since there are evidently an even number of negative real zeros of $p(z)$, there are an even number of real zeros of $f(s)$ in $[-1, 1]$. The above remarks then lead to the following.

Lemma 2: Let $f(s)$ be the real polynomial of (1) with degree $n = 2m$. Suppose quantities b_i are defined as in Section II with $b_0 > 0$, $b_{2m} > 0$, and $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$. List the real zeros of $f(s)$ in the interval $[-1, 1]$ as $s_1, \cdots, s_{2r}$ where $s_i \leq s_{i+1}$. Then there is no real zero s_j of $f(s)$ in $|s| > 1$ such that $s_j^{-1} \leq s_1$ or $s_2 \leq s_j^{-1} \leq s_3$, or $s_4 \leq s_j^{-1} \leq s_5, \cdots$, or $s_{2r-2} \leq s_j^{-1} \leq s_{2r-1}$, or $s_{2r} \leq s_j^{-1}$.

We remark that Lemmas 1 and 2 become trivial in the case when all the b_i are positive, for then $p(z)$ can have no nonnegative real zero at all, and all real zeros of $f(s)$ accordingly lie in $|s| < 1$.

VI. Proof of Main Result, Even Degree Polynomials

In this section, we establish the sufficiency part of Theorem 1_e.

The proof of Theorem 1_e hinges on the following Lemma.

Lemma 3: Let $f(s)$ of degree $n = 2m$, $A = (\alpha_{ij})$, $B = (\beta_{ij})$, and b_i be defined as in Section II, see (1)-(3) and (6). Suppose that either $b_{2i} > 0$, $i = 0,1,\cdots,m$, or that $b_{2m} > 0$, $b_0 > 0$, and $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$. Suppose further that, if s_i is a zero of $f(s)$, s_i^{-1} is not a zero, and that $f(s)$ has no zeros s_i with $|s_i| = 1$. Then there exists a continuously varying family of real polynomials $g_\lambda(s)$, indexed by λ, with $g_0(s) = f(s)$, $g_1(s) = f(-s)$, and $n_+(A_\lambda \dotplus B_\lambda)$ and $n_-(A_\lambda \dotplus B_\lambda)$ constant for $0 \leq \lambda \leq 1$. [Here, A_λ and B_λ are the matrices associated with $f(s)$.]

Proof: First, observe that associated with each zero of $f(s)$, inside or outside the unit circle, and real or complex, there is a similarly located zero of $f(-s)$, real or complex as the case may be. We construct a set of paths linking each zero of $f(s)$ to the corresponding zero of $f(-s)$ in accordance with the following rules.

Rule 1: If points along the paths are parametrized continuously by the index λ, with $\lambda = 0$ corresponding to the initial point, and $\lambda = 1$ to the final point, then points on the various paths with the same λ fall into complex conjugate pairs or are real.

Rule 2: In the case of complex zeros outside the unit circle, the paths stay outside the unit circle, and, in the case of complex zeros inside the unit circle, the paths stay inside the unit circle. See Fig. 3 for illustration.

Rule 3: For any given λ, the corresponding points on the various paths do not form reciprocal pairs. This is easy to ensure when the paths are not confined to the real axis. See Fig. 4 for illustration.

Rule 4a: Suppose $b_{2i} > 0$ for $i = 0,1,\cdots,m$. Real zeros within the unit circle are grouped in the pairs described in Lemma 1; the zeros of one pair are moved toward each other until they coalesce, and then they are moved off the real axis. Rule 3 is not violated, since it is guaranteed by Lemma 1. Real zeros in the range $[1, \infty)$, of which there are an even number (because $f(1) > 0$ and $f(s) > 0$ for large s), may then be coalesced in pairs without contravening Rule 3, since there are no real zeros within $(-1, 1)$ at this point. Then they are moved off the real axis. Remaining off the real axis, the paths are traced out in accordance with Rules 2 and 3 to the negative of the point at which the paths left the real axis. Then the mirror image in the vertical axis of the first part of the paths is followed. See Fig. 5. There is an obvious extension to the case when there are real zeros in $(-\infty, 1]$.

Rule 4b: Suppose $b_0 > 0$, $b_{2m} > 0$, and $b_{2i+1} > 0$, $i = 0,1,\cdots,m-1$. The situation is like that of Rule 4a. Zeros inside the unit circle, other than the most negative and most positive, are first made complex. Then zeros outside the unit circle are made complex.

Finally, the most negative and most positive zero inside the unit circle are made complex. The procedure is then worked in reverse. Lemma 2 guarantees that Rule 3 is not violated.

With the paths defined and with points on the paths parametrized continuously by λ, $g_\lambda(s)$ is defined as the polynomial with zeros consisting of the path points with parameter λ. Clearly, $g_0(s) = f(s)$ and $g_1(s) = f(-s)$. The rules ensure that the number of zeros of $g_\lambda(s)$ within the unit circle are constant for all λ, as is the number outside. Further, for no λ are there zeros on $|s| = 1$ or reciprocal zeros. The Lemma then follows by Theorem 2 and the formula (15).

We remark that, if all the b_i are known to be positive, the above proof is greatly simplified. This is because the polynomial $f(s)$ has no real zeros outside $|s| = 1$, as remarked earlier; Rules 4a and 4b then become much less complicated.

There is an immediate strengthening of Lemma 3, as follows.

Lemma 4: Let the assumptions of Lemma 3 hold. Then $n_+(A_\lambda)$, $n_+(B_\lambda)$, $n_-(A_\lambda)$, and $n_-(B_\lambda)$ are constant for $0 \leq \lambda \leq 1$.

Proof: Because $f(s)$ has no zeros s_i such that s_i^{-1} is also a zero and no zero s_i with $|s_i| = 1$, $A_0 \dotplus B_0$ is nonsingular, being orthogonally similar, as explained in Section IV, to the Schur–Cohn matrix C_0. Hence $A_\lambda \dotplus B_\lambda$ is nonsingular for all $\lambda \in [0, 1]$, and so A_λ and B_λ are individually nonsingular for $\lambda \in [0, 1]$. The eigenvalues of A_λ are continuously dependent on the entries of A_λ, and therefore on λ. Therefore, if $n_+(A_\lambda)$ changed at any point λ_1, an eigenvalue of A_{λ_1} would be zero; this contradicts the nonsingularity of A_λ. (This result can be found in [3, p. 129] for the case when A_λ has entries which are polynomial in λ; continuity is, however, all that is needed for the proof of [3] to be valid.)

We can now turn to the proof of the main result, at least if we exclude for the moment polynomials $f(s)$ possessing reciprocal zeros, or zeros of magnitude unity. The general idea of the proof is as follows. Suppose, for the sake of argument, that $A > 0$ and $b_{2i} > 0$, $i = 0,1,\cdots,m$. Identifying A with $A_\lambda|\lambda = 0$, we can conclude that $A_\lambda|\lambda = 1$ is positive definite. From this, we shall find that $B > 0$. Then simultaneous positive definiteness of A and B implies positive definiteness of C, and thus stability.

The matrix A_1 is obtained by applying the definition of (2) to the polynomial $f(-s)$. Since the coefficients of $f(-s)$ are, in descending order, $a_{2m}, -a_{2m-1}, a_{2m-2}, \cdots, a_0$, we have

$$\begin{aligned}(A_1)_{ij} &= \sum_{p=1}^{\min(i,j)} [(-1)^{2m+p-i}(-1)^{2m+p-j}a_{2m+p-i}a_{2m+p-j} \\ &\quad - (-1)^{i-p}(-1)^{j-p}a_{i-p}a_{j-p}] \\ &\quad + \sum_{p=1}^{i} [(-1)^{2m+p-i}(-1)^{j+p-1}a_{2m+p-i}a_{j+p-1} \\ &\quad - (-1)^{i-p}(-1)^{2m+1-p-j}a_{i-p}a_{2m+1-p-j}] \\ &= (-1)^{i+j}\beta_{ij}.\end{aligned}$$

Accordingly,

$$A_1 = \operatorname{diag}[1, -1, 1, -1, \cdots]B \operatorname{diag}[1, -1, 1, -1, \cdots]. \tag{27}$$

By Lemma 4, if either condition a) or condition b) of Theorem 1_e is fulfilled, we have $A_1 > 0$. Then $B > 0$ by (27), and stability follows.

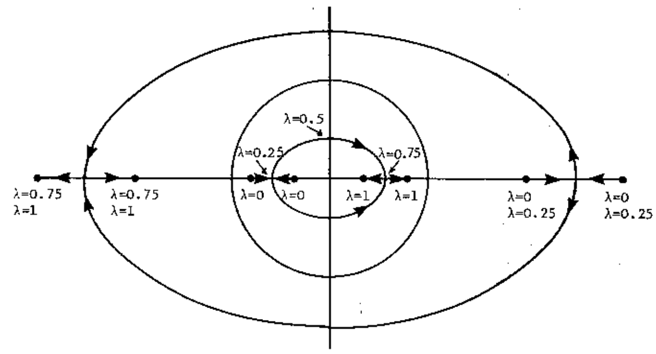

Fig. 5. As λ varies from 0 to 0.25, all real zeros inside the unit circle are made complex. As λ varies from 0.25 to 0.75, the zeros outside the unit circle are moved from starting point to finishing point.

The case when either condition c) or condition d) of Theorem 1_e is fulfilled is trivially different.

It only remains to deal with the case of a polynomial $f(s)$ possessing either reciprocal zeros or zeros on $|s| = 1$. Suppose, for example, that condition a) of Theorem 1_e holds. We shall obtain a contradiction.

Perturb the coefficients of $f(s)$ slightly to force the presence of a zero outside $|s| = 1$. Condition a) inequalities of the theorem will not be disturbed by a small enough perturbation. Our results apply to the perturbed polynomial. Therefore, $B > 0$ for the perturbed polynomial, as well as $A > 0$. Then $C > 0$, and so the perturbed polynomial is stable, which is a contradiction.

VII. Proof of Main Result, Odd Degree Polynomials

In this section, we shall establish the sufficiency part of Theorem 1_o. The general idea will be to deduce results concerning the odd degree $f(s)$ with the aid of results for the even degree polynomial $\bar{f}^\epsilon(s)$ defined by

$$\bar{f}^\epsilon(s) = (s + 1 - \epsilon)f(s). \tag{28}$$

Of course, $\bar{A}^\epsilon$, $\bar{B}^\epsilon$, etc., are the matrices associated with $\bar{f}^\epsilon$, as in Section II. We shall assume that $f(s)$ does not possess reciprocal zeros or zeros on $|s| = 1$. The technique at the end of the last section will cover this case.

Relation of $\bar{A}^0$ and $\bar{B}^0$ to A and B: From (28), it follows that, with $f(s) = \sum_{i=0}^{2m-1} a_i s^i$,

$$\bar{a}_{2m}^0 = a_{2m-1} \qquad \bar{a}_i^0 = a_i + a_{i-1}, \quad 0 < i < 2m, \qquad \bar{a}_0^0 = a_0.$$

It follows, using (13), that

$$\bar{\gamma}_{ij}^0 = \gamma_{ij} + \gamma_{i-1,j} + \gamma_{i,j-1} + \gamma_{i-1,j-1} \tag{29}$$

with $\gamma_{kl} = 0$ if k or l is outside the range $1,2,\cdots,2m-1$. Then we have, via (14) and (17),

$$\begin{aligned}\bar{\alpha}_{ij}^0 &= \alpha_{ij} + \alpha_{i-1,j} + \alpha_{i,j-1} + \alpha_{i-1,j-1}, & 1 \leq i,j \leq m-1 \\ \bar{\alpha}_{mj}^0 &= 2\alpha_{m,j} + 2\alpha_{m,j-1} + \alpha_{m-1,j} + \alpha_{m-1,j-1}, & j < m \\ \bar{\alpha}_{mm}^0 &= 4\alpha_{m,m} + 2\alpha_{m,m-1} + 2\alpha_{m-1,m} + \alpha_{m-1,m-1}\end{aligned} \tag{30}$$

with, naturally, $\alpha_{kl} = 0$ if k or l is outside the range $1,2,\cdots,m$. Equation (30) may be written as

$$\bar{A}^0 = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & 1 & & & & & 0 \\ 0 & 1 & 1 & & & & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & 1 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 1 & & & & \cdot \\ \cdot & & 1 & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & 1 & 0 \\ \cdot & & & & & 1 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & 0 & 2 \end{bmatrix}. \tag{31}$$

This shows that $n_+(\bar{A}^0) = n_+(A)$ and $n_-(\bar{A}^0) = n_-(A)$. Also, $n_0(\bar{A}^0) = n_0(A) = 0$. (Because $f(\cdot)$ has no reciprocal or unity modulus roots, A and B are nonsingular.)

In a similar manner, we can show that

$$\bar{B}^0 = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 1 & 1 & & & & 0 \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & 1 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} B \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & & & & \cdot \\ \cdot & 1 & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & & & 1 & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 1 & 1 \end{bmatrix}, \tag{32}$$

and, again, $n_+(\bar{B}^0) = n_+(B)$ and $n_-(\bar{B}^0) = n_-(B)$. However, we have $n_0(\bar{B}_0) = 1$, although $n_0(B) = 0$.

Relation of $\bar{A}^\epsilon$ and $\bar{B}^\epsilon$ to A and B: Because $\bar{A}^0$ is nonsingular, it is clear that, for suitably small $\epsilon > 0$, $n_+(\bar{A}^\epsilon) = n_+(\bar{A}^0) = n_+(A)$ and $n_-(\bar{A}^\epsilon) = n_-(A)$. However, appeal to the Fujiwara extension of the Schur–Cohn result shows that $n_+(\bar{A}^\epsilon \dotplus \bar{B}^\epsilon) = 1 + n_+(\bar{A}^0 + B^0)$, because $\bar{f}^\epsilon(s)$ possesses one more zero inside $|s| < 1$ than $\bar{f}^0(s)$. Hence $n_+(\bar{B}^\epsilon) = 1 + n_+(\bar{B}^0)$. Also, because $\bar{B}^\epsilon$ must be nonsingular, one finds that $n_-(\bar{B}^\epsilon) = n_-(\bar{B}^0)$. In particular, we have the following important result.

Lemma 5: Let $f(s)$ be a polynomial of degree $n = 2m - 1$, and let A and B be defined as in Section II. Let $\bar{f}_\epsilon(s) = (s + 1 - \epsilon)f(s)$ and let $\bar{A}^\epsilon$ and $\bar{B}^\epsilon$ be defined as described in Section II for even degree polynomials. Then, if $f(s)$ has no reciprocal or unity modulus zeros, and if $\epsilon > 0$ is suitably small: 1) $A > 0$ implies and is implied by $\bar{A}^\epsilon > 0$; and 2) $B > 0$ implies and is implied by $\bar{B}^\epsilon > 0$.

Relation of the Coefficients $\bar{b}_i^\epsilon$ to the Coefficients b_i: Now we shall see how inequalities on the b_i map into inequalities on the $\bar{b}_i^\epsilon$. We define

$$\bar{p}^\epsilon(z) = (z - 1)^{2m}\bar{f}^\epsilon\left(\frac{z+1}{z-1}\right). \tag{33}$$

Then it is easy to check that

$$\bar{p}^\epsilon(z) = \left(z + \frac{\epsilon}{2}\right)p(z), \tag{34}$$

so that

$$\bar{b}_{2m}{}^\epsilon = b_{2m-1}$$

$$\bar{b}_i{}^\epsilon = b_{i-1} + \frac{\epsilon}{2}\, b_i, \qquad 1 \leq i \leq 2m - 1$$

$$\bar{b}_0{}^\epsilon = \frac{\epsilon}{2}\, b_0. \tag{35}$$

The following lemma is then immediate.

Lemma 6: Let $f(s)$ be a polynomial of degree $2m - 1$, and let the coefficients b_i be defined as in Section II. Let $\bar{f}^\epsilon(s) = (s + 1 - \epsilon)f(s)$, and let coefficients $\bar{b}_i^\epsilon$ be defined from $\bar{f}^\epsilon(s)$, as described in Section II for even degree polynomials. Then if $\epsilon > 0$ is suitably small: 1) $b_{2m-1} > 0$ and $b_{2i} > 0$ for $i = 0,1,\cdots,m-1$ implies and is implied by $\bar{b}_{2m}{}^\epsilon > 0$, $\bar{b}_0{}^\epsilon > 0$, and $\bar{b}_{2i+1}{}^\epsilon > 0$ for $i = 0,1,\cdots,m-1$; and 2) $b_0 > 0$ and $b_{2i+1} > 0$ for $i = 0,1,\cdots,m-1$ implies and is implied by $\bar{b}_{2i}{}^\epsilon > 0$ for $i = 0,1,\cdots,m$.

Lemmas 5 and 6, together with Theorem 1_e, provide the desired proof of the sufficiency conditions of Theorem 1_o. Suppose, for example, that $A > 0$, $b_{2m-1} > 0$, and $b_{2i} > 0$ for $i = 0,1,\cdots,m-1$. Then Lemmas 5 and 6 imply that, for suitably small $\epsilon > 0$, $\bar{A}^\epsilon > 0$, $\bar{b}_{2m}{}^\epsilon > 0$, $\bar{b}_0{}^\epsilon > 0$, and $\bar{b}_{2i+1}{}^\epsilon > 0$. The second set of sufficiency conditions of Theorem 1_e now implies that $\bar{f}^\epsilon(\cdot)$ is stable; then $\bar{B}^\epsilon > 0$. From Lemma 5 again, $B > 0$. Finally, $A > 0$ and $B > 0$ jointly yield stability of $f(s)$. Alternatively, stability of $f(s)$ is immediate from stability of $\bar{f}^\epsilon(s)$.

VIII. Conclusions

Throughout the paper, we have commented on the computational aspects of the new test. In this section, we shall sum up the principal factors related to computation, and offer one or two further remarks.

For a polynomial of degree $n = 2m$ or $2m - 1$, the Schur–Cohn–Fujiwara test for stability requires the checking for positive definiteness of an $n \times n$ matrix. The entries of the matrix are simple integral functions of the polynomial coefficients. We showed, in Section II, that examination of the $n \times n$ matrix could be replaced by examination of two $m \times m$ matrices, for $n = 2m$, and an $(m - 1) \times (m - 1)$ and an $m \times m$ matrix in the case $n = 2m - 1$. These matrices have entries which again are simple integral functions of the coefficients and are, in fact, easily determinable from the Schur–Cohn–Fujiwara matrix.

The bulk of the paper was devoted to showing that, when $n = 2m$, only one of the $m \times m$ matrices need be checked for positive definiteness, and, when $n = 2m - 1$, only one of the $(m - 1) \times (m - 1)$ and the $m \times m$ matrix need be checked, provided some inequalities involving linear combinations of the polynomial coefficients are checked. In case $n = 2m$, these inequalities are $m + 1$ or $m + 2$ in number, and in the case $n = 2m - 1$, they are $m + 1$ in number.

Finally, we remark that there still remains the open question of relating these ideas to the Markov parameter approach of [20].

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