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# STABILITY THEOREMS FOR THE RELAXATION OF THE STRICTLY POSITIVE REAL CONDITION IN HYPERSTABLE ADAPTIVE SCHEMES

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## ABSTRACT

The hyperstability theorems of Popov have played an important role in establishing the convergence of adaptive schemes, notably adaptive output error identification and adaptive control. The error system of these schemes has the form of a feedback loop with a time-invariant forward path and a passive time-varying feedback path. The strict positive realness of the forward path suffices to establish asymptotic stability of the feedback loop and therefore establishes convergence of the adaptive scheme. In this paper we study conditions which preserve the asymptotic stability but permit relaxation of the strict positive real condition at high frequencies, subject to restrictions on algorithm gain parameters and frequency content of the input signals. These theorems are important for the design of robust adaptive methods.

## INTRODUCTION

The error equations of adaptive control, adaptive output error identification and model reference adaptive schemes have the form of a feedback interconnection of a time-invariant forward path with a passive time-varying feedback path [1,2,23]. The convergence of the adaptive scheme is equivalent to the asymptotic stability of this feedback loop. In turn, this asymptotic stability can be established by appealing to the hyperstability theorems of Popov [3] where the closed loop is stable provided the forward path transfer function is strictly positive real (SPR). This has been the basis of the approach taken by Landau [4,5] and many others. This occurrence of SPR functions in adaptive methods dates back to 1966 [6,7] and several applications are summarized by Ljung in [8].

Our aim here is to develop a new stability theorem for adaptive identification and ultimately adaptive control which allows the relaxation of the SPR requirement on the forward path transfer function in return for a restriction on the frequency content of signals in the loop. The reason for desiring to relax the SPR condition is that the transfer function to which it applies is a priori unknown since it is a function of the actual system parameters and model order selection errors. (The feedback path system is determined by the adaptation algorithm.) Thus in attempting to derive a robust adaptive identification or control procedure it proves necessary to characterize stabilizing forward path systems in such a way as to topologize the set of transfer functions weakly in order that the stability theory be applicable to as wide a class of systems as possible "close" to a nominal SPR forward path.

The inadequacy of the SPR condition alone to establish robustness is demonstrated in [9] where instability is introduced through the inclusion of unmodelled high frequency dynamics in the plant to be controlled. The conditions which we derive for stability allow deviation from SPR at the cost of restricting the frequency content of signals, adapting slowly, and guaranteeing persistence of excitation, which are conditions at variance with those of the stability counterexample of [9]. We work here in continuous-time since then the analysis does not stumble over the notation, and extend the results to discrete-time in the later sections.

The analysis which we present here can be viewed as extending and generalizing the fundamental work by Ioannou and Kokotovic [24] in this area of robustness of adaptive systems. In particular, we apply the tools of input/output stability theory to derive general results applicable to broad classes of input processes and systems, without assumptions of almost periodic inputs, strictly bounded derivatives, finite-dimensionality or strict separation of timescale. Notwithstanding the more general setting and as is to be expected, the stability conditions derived here resemble strongly those of [24], as will be reiterated later.

The major aim of our thrust is to establish a general theory and approach to prove the stability of adaptive control and identification procedures. The results of this paper concentrate on the issues in adaptive identification since this permits the more straightforward derivation of theoretical results and tools. The direct extension to adaptive control is contained in a companion paper [25]. These results may also be applied to more novel forms of adaptive control [26] which actively attempt to satisfy the stability conditions of this paper through the introduction of filtering. Related control ideas have been expressed in [24] and [27].

## PROBLEM STATEMENT

The specific system which we consider is depicted in Figure 1, which represents the error equations as derived in model reference adaptive schemes. With obvious abuse of notation we seek to analyze the exponential asymptotic stability (EAS) of

$$\dot{x}(t) = -\epsilon u(t) Z(s) u'(t) x(t) \tag{1}$$

We know from [10,11,12] that if Z(s) is SPR and u(t) is persistently exciting with $\dot{u}$ bounded, save that u can have a bounded number of step discontinuities in any interval of fixed length, then (1) is EAS. This EAS may be established by Lyapunov analysis [12], Popov's hyperstability theory directly [3] or by the small gain theorem [13]. This situation of (1) with SPR Z(s) arises in the case of adaptive estimation and control without mismatch between the true plant and the model set. The question raised here is whether the EAS of (1) is robust to small departures from SPR of Z(s), since this question naturally has a bearing on the robustness of hyperstable adaptive schemes in general. Persistence of u corresponds to having sufficiently rich input signals to permit consistent identification of plant parameters [28]. We study the case where the deviation of Z(s) from SPR occurs at high frequencies (in the sense that while Z(s) remains strictly stable, the condition $Re Z(j\omega) > 0$ may fail for large $\omega$) and we consider methods for ensuring EAS in spite of this mismatch. The nature of (1) is that if $\epsilon$ is small and u and x are bounded then the derivative of x is small so that, if u is lowpass then u'x will be lowpass. Thus the input to Z will have a frequency spectrum largely concentrated in the region where Re Z is positive. Provided u is maintained persistently exciting as well as lowpass, it is a reasonable conjecture that EAS should follow. For our purposes here the requirements of persistently exciting u(t) are requirements that u(t) be a regulated function of time, i.e. smooth except for possible jumps at well-separated points, and that it satisfy

$$0 < \alpha I \leq \int_t^{t+T} u(s) u'(s) ds \leq \beta I < \infty \tag{2}$$

for son ositive constants *alpha*, $\beta$ and T and for all t. Our aim here will be to prove that if $\epsilon$ is small, $u(\cdot)$ is lowpass *and* persistently exciting in satisfaction of (2) then Z(s) need not satisfy the SPR condition at high frequencies in order to guarantee EAS of (1).

Suppose that Z(s) is not SPR, but is proper and stable, and suppose it may be written as

$$Z(s) = Z_1(s) + Z_2(s) \tag{3}$$

where $Z_1(s)$ is SPR, $Z_1(0) = Z(0)$. Evidently, $Z_2(s) = sZ_3(s)$ for some stable, strictly proper $Z_3(s)$. Then (1) becomes

$$\dot{x} = -\epsilon u(t) Z_1(s) u'(t) X - \epsilon u(t) Z_3(s) \dot{u}'(t) x$$
$$- \epsilon(t) Z_3(s) u'(t) \dot{x} \tag{4}$$

The approach to be taken in studying the stability of (4) will be to consider separately the equations

$$\dot{x} = -\epsilon u Z_1(s) u' x - \epsilon u Z_3(s) \dot{u}' x \tag{5}$$

and

$$(1 + \epsilon u Z_3(s) u')\dot{x} = -\epsilon M x \tag{6}$$

where $\epsilon M$ is a linear operator with a prescribed degree of stability.

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(Note that (6) corresponds to (4) where $-\epsilon Mx$ is the right hand side of (5).)

### EXPONENTIAL ASYMPTOTIC STABILITY OF (5)

It serves us in this analysis to consider (5) as the cascade of the systems

$$\dot{x} = -\epsilon u Z_1(s) u' x + v \tag{7}$$

and

$$v = \epsilon u Z_3(s) \dot{u}' w \tag{8}$$

with unity negative feedback

$$w = -x \tag{9}$$

We then may approach the stability analysis of (5) using the small gain theorem of [13].

The "ideal" system is

$$\dot{x} = -\epsilon u Z_1(s) u' x \tag{10}$$

A stability analysis of (10) is conducted in [10,11,12] where it is shown that, provided $Z_1(s)$ is SPR, and $u(\cdot)$ is persistently exciting and such that $u, \dot{u}$ are continuous and bounded on $[0,\infty) - C_\Delta$ where $C_\Delta$ is a countable set of points with a bounded number in any interval of prescribed length, and with y(t) representing the state of $Z_1(s)$, there exist positive constants m and $\lambda$ such that

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \leqslant m\, e^{-\lambda t} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \tag{11}$$

or, more precisely, that

$$||\Phi(t,\tau)|| \leqslant m\, e^{-\lambda(t-\tau)} \tag{12}$$

where $\Phi(.,.)$ denotes the transition matrix of (7) from v to $[x'\ y]'$. (The existence of a finite-dimensional state vector y(t) for $Z_1(s)$ is not a restrictive assumption in the context of this paper since an infinite-dimensional SPR transfer function can be approximated arbitrarily closely (in an $L_\infty$ sense on the imaginary axis) by a finite-dimensional one. This arbitrarily small infinite-dimensional error can readily be incorporated into $Z_3(s)$ to produce a correspondingly small change in gain.) The $L_p$ gain of (7) may be obtained by using (12) and Theorem 22 of [13, p113]. Since the impulse response of (7) for $t > \tau$ is $\Phi(t,\tau)$, which satisfies (12), and is zero for $\tau > t$ we have directly

$$\int_\tau^\infty ||\Phi(t,\tau)||dt \leqslant \frac{m}{\lambda} \text{ and } \int_{-\infty}^t ||\Phi(t,\tau)||d\tau \leqslant \frac{m}{\lambda}$$

so that the $L_p$ gain of (7) is overbounded by $m/\lambda$ for any $p \in [1,\infty]$.

The $L_p$ stability of (5) will follow by the small gain theorem provided the $L_p$ gain of (8) can be overbounded by $\lambda/m$. In order to evaluate the $L_p$ gain of (8) in general one needs to bound

$$||u(t)|| \leqslant m_1 \tag{13}$$

and

$$||\dot{u}(t)|| \leqslant m_2 \tag{14}$$

for all t. (Subsequently, we shall discuss the relaxation of (14) to permit $u(\cdot)$ which are piecewise continuously differentiable but may have step discontinuities.) We then have immediately the following characterization of the asymptotic stability of (5) from application of the small gain theorem, [13].

THEOREM 1

Suppose that: $Z_1(s)$ is SPR, u(t) is persistently exciting *and* satisfies (13) and (14). Further, denote the $L_p$ gain of $Z_3(s)$ for some particular $p \in [1,\infty]$ by $g_3$. Then the solution $x(\cdot)$ of the differential equation (5) will be in $L_p$ if

$$\epsilon \frac{m}{\lambda} m_1 m_2 g_3 < 1 \tag{15}$$

where $\lambda$ and m are defined in (12) and represent the convergence rate of the "ideal" differential equation (10).

This theorem relates the stability of (5) to conditions on the signal u(t) and the $L_p$ gain of $Z_3(s)$, the deviation of Z(s) from SPR. It is worth mentioning here that the $L_\infty$ and $L_2$ gains (induced norms) of $Z_3(s)$ may be found as

$$||Z_3(s)||_\infty = \int_0^\infty |z_3(\tau)|d\tau \tag{16}$$

and

$$||Z_3(s)||_2 = \max_\omega |Z_3(j\omega)| \leqslant ||Z_3(s)||_\infty \tag{17}$$

where $z_3(t)$ is the impulse response of $Z_3(s)$. The condition (15) may then be interpreted as being a constraint on the impulse response or the frequency response of the deviation from SPR, and the p = 2 norm allows less conservative bounds (larger $g_3$) than $p = \infty$.

Given a stability condition for (5) our next requirement is to extend this to an EAS result. To do this we define $x_a(t) = x(t)e^{at}$, $y_a(t) = y(t)e^{at}$, $v_a(t) = v(t)e^{at}$ for $a < \lambda$ and, using (12), write

$$\left|\begin{bmatrix} x(t)e^{at} \\ y(t)e^{at} \end{bmatrix}\right| \leqslant \int_0^t e^{a(t-\tau)} m e^{-\lambda(t-\tau)} v_a(\tau) d\tau + k(t) \tag{18}$$

where k(t) is due to initial conditions and decays exponentially to zero. This may be rewritten in terms of $x_a$, $y_a$ and $v_a$ to yield

$$\left|\begin{bmatrix} x_a(\cdot) \\ y_a(\cdot) \end{bmatrix}_t\right|_p \leqslant \frac{m}{\lambda - a} ||[v_a(\cdot)]_t||_p + k(t) \tag{19}$$

for all $p \in [1,\infty]$ and all t > 0. Here the subscript t denotes the restriction to the interval [0,t) and $||\cdot||_p$ the $L_p$ norm.

We now consider the analog of (8), (9)

$$v_a(t) = -\epsilon e^{at} u(t) Z_3(s) \dot{u}(t) e^{-at} x_a(t) \tag{20}$$

and denote

$$\int_0^\infty |z_i(t)| e^{at} dt = g_i(a) \quad i = 1,3 \tag{21}$$

$$\max_\omega |Z_i(j\omega - a)| = \bar{g}_i(a) \quad i = 1,3 \tag{22}$$

We have, after application of the small gain theorem,

THEOREM 2

Suppose that the conditions of Theorem 1 hold, constant a is chosen such that $0 < a < \lambda$ and $Z_3(s-a)$ has no poles in $Re(s) \geqslant 0$. Then x(t), the solution of (5) satisfies

$$e^{at} x(t) \in L_2 \text{ if } \epsilon m_1 m_2 \bar{g}_3(a) m(\lambda - a)^{-1} < 1 \tag{23}$$

and

$$e^{at} x(t) \in L_\infty \text{ if } \epsilon m_1 m_2 g_3(a) m(\lambda - a)^{-1} < 1 \tag{24}$$

Further, condition (23) implies that $e^{at}x(t) \in L_\infty$ and $e^{at}x(t) \to 0$ as $t \to \infty$.

Proof: The earlier parts stem from application of the small gain theorem of [13]. The final statement follows from the implication that $e^{at}x(t) \in L_2$ forces $e^{at}\dot{x}(t) \in L_2$ since x satisfies (5). Thus $d/dt\,(e^{at}x(t)) \in L_2$ and hence $e^{at}x(t) \in L_\infty$ and $|e^{at}x(t)| \to 0$ as $t \to \infty$, see [13, p.237].

### EXPONENTIAL ASYMPTOTIC STABILITY OF (1)

We have shown that (1) is equivalent to (4) and, further, that the related equation (5) is EAS subject to the conditions of Theorem 2. We now make the connection between the EAS of (5) and that of (1) by examining (6), i.e.

$$(1 + \epsilon u Z_3(s) u')\dot{x} = -\epsilon M x$$

where $\epsilon M$ is a linear operator with a prescribed degree of stability. Specifically,

$$M = u Z_1(s) u' + u Z_3(s) \dot{u}' \tag{25}$$

and writing

$$N = u Z_3 u' \tag{26}$$

we define the operator K by the relation

$$K = (I + \epsilon N)^{-1} - I \tag{27a}$$

$$= -\epsilon N + \epsilon^2 N^2 - \epsilon^3 N^3 + \cdots$$

provided

$$\epsilon ||N|| < 1. \tag{27b}$$

We then have

$$||K|| \leqslant \epsilon ||N|| + \epsilon^2 ||N||^2 + \epsilon^3 ||N||^3 + \cdots$$

$$= \frac{\epsilon ||N||}{1 - \epsilon ||N||} \tag{28}$$

Using this result we may rewrite (6) as

$$\dot{x} = -\epsilon M x - K \epsilon M x$$

which, in turn, may be described as the feedback interconnection

$$\dot{x} = -\epsilon Mx + v \quad (29)$$

$$v = \epsilon KMw$$

$$w = -x$$

By Theorem 2, we can guarantee that the first equation of (29) has an $L_p$ gain of $m'/\lambda'$ (where $\lambda' < \lambda$ is the exponent governing the rate of decay of a transition matrix which can be associated with the homogeneous version of the equations augmented by the state of $Z_3(s)$). By applying the small gain theorem, we see that stability follows if

$$\frac{\epsilon m'}{\lambda'} \frac{\epsilon ||N||}{1-\epsilon ||N||} ||M|| < 1 \quad (30)$$

Exponential asymptotic stability also carries through in a similar manner to before by considering $x_b(t) = e^{bt}x(t)$ for $0 < b < \lambda'$, $Z_3(s-b)$ etc. In particular we have

THEOREM 3

Define

$$g_i(b) = \int_0^\infty |z_i(t)| e^{bt} dt \quad i = 1,3 \quad (31)$$

$$\tilde{g}_i(b) = \max_\omega |Z_i(j\omega - b)| \quad (32)$$

Suppose that $Z_1(s)$ is SPR and there is a $b > 0$ such that $Z_1(s-b)$ and $Z_3(s-b)$ have no poles in $Re(s) > 0$. Suppose that for all $t \in [0,\infty)$, u(t) satisfies

$$0 < \alpha I \leqslant \int_t^{t+T} u(s)u'(s)ds \quad \text{for some fixed finite } T \quad (33)$$

$$||u(t)|| \leqslant m_1 \quad (34)$$

$$||\dot{u}(t)|| \leqslant m_2 \quad (35)$$

Then (1) will be exponentially asymptotically stable if the following modifications of (24), (27b) and (30) hold

$$(i) \quad \epsilon m_1 m_2 g_3(b) \frac{m}{\lambda - b} < 1 \quad (36)$$

$$(ii) \quad \epsilon m_1^2 g_3(b) < 1 \quad (37)$$

$$(iii) \quad \epsilon^2 \frac{m'}{\lambda' - b} \frac{m_1^2 g_3(b)}{1 - \epsilon m_1^2 g_3(b)} [m_1^2 g_1(b) + m_1 m_2 g_3(b)] < 1 \quad (38)$$

where m, $\lambda$ are associated with the exponential convergence rate of the ideal algorithm (10), m', $\lambda'$ are associated with the exponential convergence rate of (5), and b is chosen $0 < b < \lambda'$. Alternatively, $g_i(b)$ can be replaced by $\tilde{g}_i(b)$.

The general form of the inequalities of Theorem 3 is strikingly similar to that of the inequalities of Chapter 2 in [24] and indeed some of the terms are identical. This is strong evidence of the ability of these results to specialize well to situations of weakly observable singularly perturbed parasitic $Z_3(s)$. The underlying conceptual formulation of the results of [24] and those here is essentially the same, although the general setting and techniques differ substantially.

Having presented sufficient conditions for the EAS of (1) it is pertinent to enquire into the requirements for the satisfaction of these conditions in terms of the signal and transfer function properties. In particular, it is of interest to determine whether there are any free variables to ensure stability for given $Z_1$ and $Z_3$ by selecting $\epsilon$, $m_1$ or $m_2$.

## *ATTAINMENT OF CRITERIA FOR EAS*

We shall analyze separately the three inequalities of Theorem 3 with an eye towards interpreting them as operating or design constraints for the EAS of a particular given system. The approach will be to regard $Z_1(s)$ and $Z_3(s)$ as fixed and to examine the effect of varying $\epsilon$, $m_1$ and $m_2$.

We begin by considering the factors affecting $m/\lambda$ since this is the only unquantified dependence left in the inequalities (36)-(38). For this analysis we need to consider the effects on the transition matrix of (12). This problem has been considered in [14] for $Z_1(s) = I$ and in [15] for more general $Z_1(s)$. The results of both these works are that for $\epsilon$ small and $Tm_1^2/\alpha$ fixed in (33) the gain $m/\lambda$ depends linearly on $(m_1^2\epsilon)^{-1}$. A typical measure of smallness for $\epsilon$ is that $\epsilon T$ be strictly less than one, which is consistent with the notion that the averaging properties of the differential equation are the dominant feature of its behavior.

The satisfaction of (37) may thus be achieved by taking $\epsilon$ sufficiently small and satisfaction of (36) can now be seen as admitting $m_2/m_1$ as a free parameter. That is, the ratio of the maximum input derivative to the maximum input magnitude is the design variable. Clearly, this quantity is a measure of the frequency content of u(t). However, all of the preceding theory is valid only under the assumption of persistence of excitation, i.e. satisfaction of (33), and we must be careful not to invalidate (33) through the low frequency restriction on u(t).

The interaction of $m_2/m_1$ and (33) can in part be studied using the results and methods of [14] and their extension in [15]. In particular, for some systems the physical origins of $u(\cdot)$ are such that it is possible to maintain the exponential degree of stability for (6) by taking a given u(t) and then making adjustments, perhaps to external excitations, the effect of which is to stretch the time axis, replacing u(t) by $u(t/l)$ for $l > 1$. This has the effect of maintaining $m_1$ and causing $m_2, \alpha, T$ to become $m_2/l$, $\alpha l$, $lT$. Provided $\epsilon T$ and $\epsilon lT$ are both small the result is that the degree of stability, $\lambda$, of (10) is unchanged but the left hand side of (36) has been reduced by a factor of $l^{-1}$. In many situations, however, such as ARMA model equation error identification the signal u is a vector with entries comprising designed input signals and measured output signals of a real plant. If this plant has zero (or relatively very low) d.c. response then clearly there is no guarantee that as the frequency content of the input is reduced the whole of u(t) exhibits the time-stretching phenomenon. The value of $\alpha$ will not remain fixed. For these systems we need to restrict $Z_3$ compared with $Z_1$, i.e. keep $g_3(b)$ small, in order to guarantee stability, while for other plants we may get away with reducing $m_2$.

Having $\epsilon$ small is necessary for the satisfaction of (37) and (38), which are not violated by a reduction of $m_2$. Indeed, it appears that the best interpretation of these latter requirements is that they restrict $\epsilon$ to be sufficiently small.

There are then tradeoffs perceptible in these conditions for EAS, where the convergence rate of the ideal algorithm is balanced against gain of the deviation from the SPR property, the speed of adaptation and the signal frequency content.

The conclusion here, argued in outline below, is that the requirements for EAS of (1) can frequently be met by keeping $\epsilon$ small and restricting the frequency content of u(t). We shall comment again on this in the concluding section.

## *DISCRETE-TIME EXTENSIONS*

As remarked at the outset, the equation (1) arises in adaptive systems as the differential equation describing the evolution of the error. Stability of the equation is identified with convergence of the adaptive scheme. Since many adaptive procedures are computer based, one is led to the question of whether the results extend to discrete-time.

The discrete-time equivalent to (1), for finite-dimensional Z(z) with state-variable realization {A,b,c,d}, is given by

$$e_{k+1} = Ae_k + bw_k \quad (39)$$

$$v_k = c'e_k + dw_k \quad (40)$$

$$w_k = u_k'x_k - \alpha\epsilon u_k'\Gamma u_k v_k \quad (41)$$

$$x_{k+1} = x_k - \epsilon\Gamma u_k v_k \quad (42)$$

where $\Gamma$ is a positive definite gain matrix which we scale here by the parameter $\epsilon$, Z(z) is a discrete transfer function and $\alpha$ is a constant greater than 0.5. The exponential asymptotic stability of this difference equation for discrete SPR Z(z) and persistently exciting $\{u_k\}$ has been shown, at least for finite-dimensional Z(z), in [16,17]. A result for infinite-dimensional Z(z) appears in [15]. The question to be asked here is what is the effect on stability of Z(z) not being SPR, and how may this be alleviated by restricting $\epsilon\Gamma$ and/or the frequency content of $\{u_k\}$?

Firstly, we write (42) in a more convenient form. Premultiplying (42) by $u_k'$ and subtracting it from (41) yields

$$w_k = u_k'x_{k+1} + (1-\alpha)\epsilon u_k'\Gamma u_k v_k \quad (43)$$

Also we may write from (39) and (40) that

$$v_k = Z(z)w_k \quad (44)$$

so that (42), (43) and (44) allow description of the algorithm as shown in Figure 2 which, in turn, may be redrawn as Figure 3.

The feedback path of Figure 3 may readily be shown to be passive for all $\alpha > 0.5$ and it is then a direct consequence of the Popov hyperstability theorem that the closed loop is EAS if Z(z) is SPR. We shall consider departures of Z(z) from SPR characterized by $Z_3(z)$ in the following decomposition of Z(z)

$$Z(z) = Z_1(z) + Z_2(z) = Z_1(z) + (1-z^{-1})Z_3(z) \qquad (45)$$

where $Z_3(z)$, and hence Z(z), is stable and $Z_1(z)$ is SPR. Block diagram manipulations allow us to redraw Figure 3 as Figure 4 where the diagram is seen to involve the feedback interconnection of an EAS upper part (in dotted lines) with a feedback block $\epsilon\Gamma u_k(1-z^{-1})Z_3(z)u_k'$. We may now appeal to the small gain theorem as before to ensure EAS.

Consider the system described by (39) and (40) together with

$$w_k = u_k' x_k + \alpha n_k \qquad (46)$$

$$n_k = \epsilon\Gamma u_k v_k + p_k \qquad (47)$$

$$x_{k+1} = x_k + n_k \qquad (48)$$

$$h_k = x_{k+1} + (\alpha-1)n_k \qquad (49)$$

As before, the gain of the EAS forward path operator is $m/\lambda$ provided the transition matrix from $p_k$ to $h_k$ is bounded by

$$||\Phi(k,j)|| \le m\lambda^{k-j} \qquad (50)$$

for $\lambda < 1$. This may be proven by considering the impulse response and using the results of [16,17] provided $\{u_k\}$ is persistently exciting, i.e. there exist positive constants $\alpha$, $\beta$, $T$ such that for all t

$$0 < \alpha I \le \sum_{t}^{t+T} u_k u_k' \le \beta I < \infty \qquad (51)$$

To guarantee EAS of the closed loop system in Figure 4, it becomes necessary to consider the gain of the operator

$$\epsilon\Gamma u_k(1-z^{-1})Z_3(z)u_k' = \epsilon\Gamma u_k Z_3(z)[u_k' - u_{k-1}']$$

$$+\epsilon\Gamma u_k Z_3(z)u_{k-1}'(1-z^{-1}) \qquad (52)$$

Under the conditions

$$||u_k|| \le m_1 < \infty \quad \text{for all } k \qquad (53)$$

$$||u_k - u_{k-1}|| \le m_2 < \infty \quad \text{for all } k \qquad (54)$$

and denoting the $1_p$ gain of $Z_3(z)$ by $m_3$, the gain of the first term on the right hand side of (52) may be bounded by

$$|\epsilon\Gamma u_k Z_3(z)[u_k' - u_{k-1}']| \le \epsilon||\Gamma|| m_1 m_2 m_3 \qquad (55)$$

The second term is, however, more difficult to bound tightly.

Consider the signals $h_k$ and $q_k$ where

$$q_k = \epsilon\Gamma u_k Z_3(z)u_{k-1}'(h_k - h_{k-1}) \qquad (56)$$

and the signals $n_k$ and $x_k$ from Figure 4. We have from (48) and (49) that

$$h_k - h_{k-1} = \alpha n_k - (\alpha-1)n_{k-1} \qquad (57)$$

and, further, that

$$n_k = \epsilon\Gamma u_k Z(z)u_k' h_k \qquad (58)$$

so we may write (57) as

$$h_k - h_{k-1} = \epsilon[\alpha-(1-\alpha)z^{-1}]\Gamma u_k Z(z)u_k' h_k \qquad (59)$$

which, in turn, may be substituted into (55) to bound the gain of the operator from $h_k$ to $q_k$ by $\epsilon^2||\Gamma||^2(\alpha+|1-\alpha|)m_1^4 g_3 g_0$ where $g_3$ and $g_0$ are the $1_p$ gains of the operators $Z_3(z)$ and Z(z) respectively.

The upshot of this anlaysis is that the continuous-time results of Theorem 3 carry over to discrete-time with little modification. In particular, the EAS of (39)-(42) with Z(z) stable and given by (45) may be assured by choosing the adaptation gain sufficiently small and selecting an input sequence $\{u_k\}$ which is both persistently exciting and lowpass. We shall next sketch the qualitative extension of these results to situations where $\{u_k\}$ (or u(t)) is not directly manipulable by the designer.

## COPING WITH NONDIFFERENTIABLE REGRESSION VECTOR

It will be recalled that in computing the $L_p$ gain of the operator defined in (8), mapping $w(\cdot)$ into $v(\cdot)$, we assumed that |u| and $||\dot{u}||$ were bounded. We now show how to compute a bound for the case when $||\dot{u}||$ is bounded on all but a set $\Delta$ of points $\tau_1, \tau_2, \cdots$, where the $\tau_i$ have no finite limit point. We further assume that at the points $\tau_i$, $u(\cdot)$ may have a jump of magnitude no greater than $\nu$. Assuming then that (13) holds on $[0,\infty)$, (14) now holds on $[0,\infty) - \Delta$, it follows that

$$||v(t)|| \le \epsilon m_1 m_2 \int_0^\infty |Z_3(t)|d\tau ||[w(\cdot)]_t||_\infty$$

$$+ \epsilon m_1 \nu \sum_{\tau_i < t} |z_3(t-\tau_i)| |w(\tau_i)| \qquad (60)$$

The effect is to increase $m_2 g_3$ to $m_2 g_3 + \nu \max_t \sum_{\tau_i < t} |Z_3(t-\tau_i)|$.

Evidently, step changes in $u(\cdot)$ need not destroy EAS; they do make it harder to secure, and they must be restricted in their magnitude and frequency of occurrences. This condition is appealing, in that it is consistent with earlier results, and is of welcome practical utility.

## EXTENSION TO OUTPUT ERROR AND ADAPTIVE CONTROL

The details of the application of the ideas here to adaptive control and adaptive output error identification will be dealt with elsewhere. However, we will identify the problems here, indicate the approach to their resolution and interpret the conditions for EAS in terms of system knowledge.

The distinguishing feature of output error identification and adaptive control methods from the equation error schemes implicitly studied in the earlier sections is that in these former schemes the vector process u is a function of the parameter error x, which therefore makes the differential equations inherently nonlinear. Thus for equation error, $u(\cdot)$ comprises filtered values of the subject plant input signals and output signals and so the algorithm input u depends in no way upon the parameter error, while in contrast, output error methods use a u vector which is composed of filtered values of plant input signals and *adjustable* model outputs, leading to a dependence of u upon x. Similarly, in adaptive control the plant input values are functions of x, again leading to the nonlinear occurrence of x in the analysis. The extension of the results of this paper to include adaptive control and output error methods must therefore reflect the nonlinear system underlying the situation.

The key idea to handling the EAS of this nonlinear set of equations is to consider the behavior of the "ideal" system, which is shown in [16] to be EAS under persistence of excitation of reference trajectory. Then, given this behavior of the ideal system, one introduces additive disturbance components due to nonideal transfer functions. To achieve this additive correction to the ideal equations it becomes necessary to linearize x(t) about the solution of the ideal system. The major deviation from the analysis for equation error is that initial condition terms necessarily arise. Thus it transpires that for these nonlinear adaptive schemes one has an extra component to consider in ensuring EAS, i.e. the initial state of $Z_3(s)$ must lie within a prescribed ball. This condition also arises in the analysis of [24]. The radius of this ball may be traded off against the magnitudes of the other system restrictions such as frequency content, $\epsilon$, $m_3$, etc. As remarked earlier, in adaptive control the problem of ensuring that u is persistently exciting and has bounded derivative is more difficult than in the equation error case as is reflected by the introduction of additional restrictions.

## COMPLEMENTS AND CONCLUSIONS

We have analyzed the problem of ensuring EAS of certain error models arising in adaptive systems when the SPR condition has been violated at high frequencies. Specifically, EAS is still achievable subject to restricting the inputs u to being low frequency and persistently exciting and restricting the algorithm gain $\epsilon$ to being small. The conditions described in Theorem 3 demonstrate that there are several design tradeoffs to ensure EAS. Clearly the magnitude of deviation from SPR, $g_3$, may be weighed against the need to limit the derivative magnitude of the input etc. In particular, it is evident that as $g_3$ tends to zero we begin to recover the simple requirement for EAS that u be persistently exciting. There have been several other recent analyses of robust adaptive control [18-20]. These are based on an analysis of the robustness of the "tuned" system when disturbances generated by the model/plant mismatch impinge upon it. These results have been derived using methods similar to those of this paper, i.e. small gain theorems and input/output properties, but without reference to a frequency region in which there occurs deviation from SPR. Clearly, for small $Z_2(s)$ in (3) and $Z_1(s)$ SPR, one still has Z(s) SPR, and to a certain extent many of the earlier results reflected this general robustness of SPR systems. However, the results of this paper clearly admit the consideration of non-SPR Z(s) but place other restrictions on design signals.

Several extensions and specializations of the theory here are possible. Since we are concerned with convolution operators acting on error signals it is possible to consider procedures for bounding more strongly the gain of some operators such as N in (26) by specifically frequency restricting u with respect to the actual frequency response of $Z_3$ or $Z_2$. These tighter bounds requiring closer knowledge of $Z_2$ are investigated in [21]. Further, by explicitly studying the double convolution integrals involved in the analysis of (5), it is conceptually possible (especially when u is periodic and admits a Fourier series description) to specify more accurately the constraints on $Z_1(s)$ and $Z_3(s)$. Such a situation pertains in instances such as those considered in [24] where the nonoverlapping time-scales would greatly restrict the gain of these operators. The stability requirements on $\dot{u}$ could conceivably also be reduced by considering the logarithmic variation methods of [22] rather than appealing strictly to the $L_\infty$ gain of the derivative of u, although it may be difficult to interface this theory with the underlying persistence of excitation requirements. Finally, as we have indicated one may further relax the restrictions on u for EAS to include u's with step changes.

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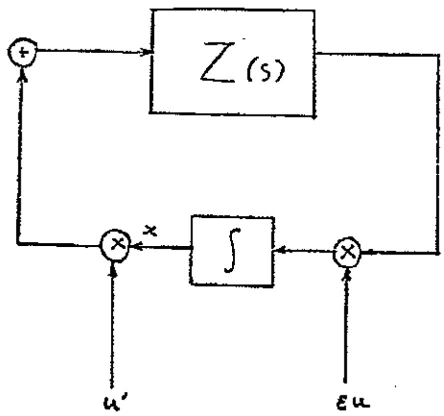


Figure 1: Feedback System

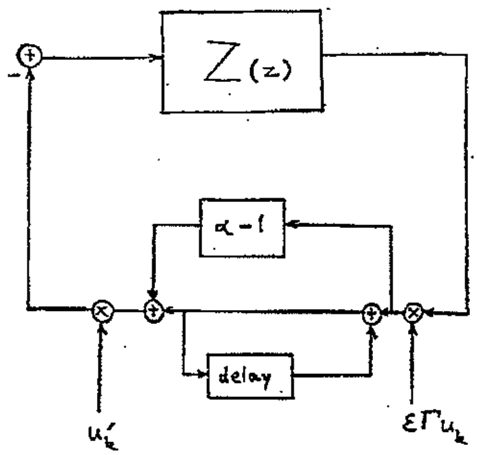


Figure 3: Redrawing of Figure 2 as Passive Feedback System

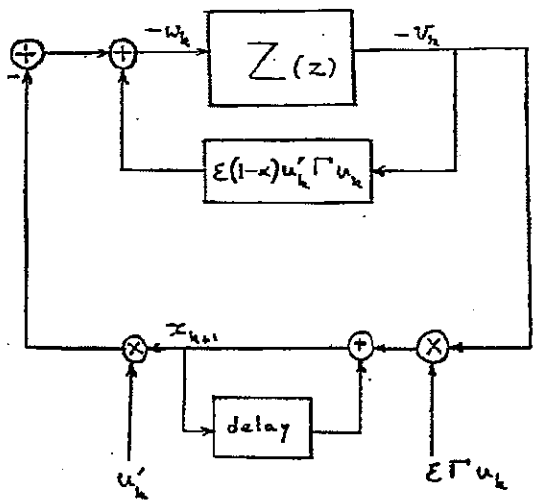


Figure 2: Discrete-time Feedback Interconnection

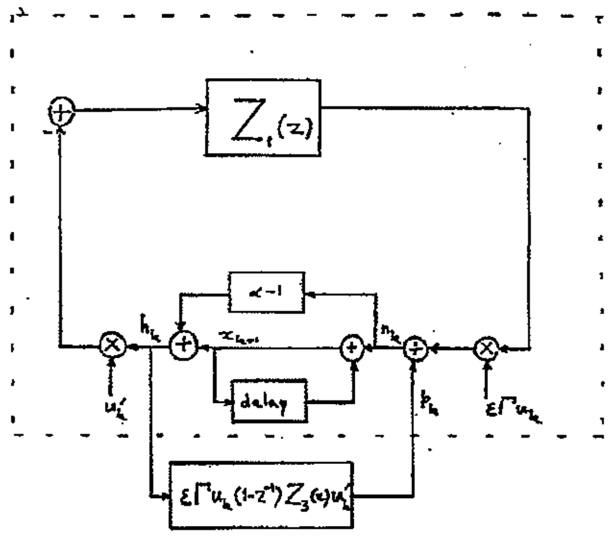


Figure 4: Description of Nonideality as Feedback Interconnection

"We're right in the middle of a *?#!%$& reptile zoo! And somebody's giving booze to these *?#!%@¢ things! It won't be long before they tear us to shreds." H.S.T.