

Model-Independent Trajectory Tracking of Euler-Lagrange Agents on Directed Networks

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Abstract—The problem of trajectory tracking of a moving leader for a directed network where each fully-actuated agent has Euler-Lagrange self-dynamics is studied in this paper using a distributed, model-independent control law. We show that if the directed graph contains a directed spanning tree, with the leader as the root node, then a model-independent algorithm semi-globally achieves the trajectory tracking objective exponentially fast. By model-independent we mean that each agent can execute the algorithm with no knowledge of the agent self-dynamics, though reasonably, certain bounds are known. For stability, a pair of control gains for each agent are required to satisfy lower bounding inequalities and so design of the algorithm is centralised and requires some limited knowledge of global information. Numerical simulations are provided to illustrate the algorithm’s effectiveness.

I. INTRODUCTION

Multiagent systems research is an area which has grown in significance in the last two decades. A primary focus is to control and coordinate a group of cooperating agents to achieve a common objective. In certain applications this offers advantages over a single complex agent. There are a broad range of applications because the term “agent” can refer to many different controllable subsystems. See [1] for a survey of recent topics on cooperative multiagent systems.

Tracking of a leader’s trajectory by a network of follower agents is a commonly studied problem in multiagent coordination. Given a common state variable(s) for every agent, a distributed control law is designed such that the followers track the trajectory of the leader’s time-varying state variable(s). By this definition, the follower agents also achieve synchronisation. Distributed controllers are often desired for multiagent coordination; by distributed we mean that each agent can execute its individual control law without requiring information about the network as a whole, or without a central command [2]. The connectivity constraints of a network are linked to the agent self-dynamics when studying control laws which guarantee achievement of the coordination objective; reducing the topology constraints allows greater flexibility during the design phase. Consensus and various synchronisation problems for agents with simple linear and nonlinear self-dynamics are covered in [2], [3].

The nonlinear Euler-Lagrange equations of motion can be used to model the dynamics of a large class of mechanical, electrical and electromechanical systems. As such, multiagent coordination problems where each agent has self-dynamics described by Euler-Lagrange equations are well motivated. Some existing literature has studied control laws which require exact knowledge of the agent dynamical model [4], [5]. Other results use a linear parametrisation for adaptive algorithms [6], [7]. This allows for uncertain agent parameters but requires knowledge of the exact equation structure. Adaptive tracking algorithms are studied in [8], [9]. An adaptive algorithm for rendezvous to a stationary leader with collision avoidance capabilities is proposed in [10]. Containment control, a variation of trajectory tracking, is studied in [11], [12], [13].

In comparison, there have been relatively few works studying *model-independent* algorithms, i.e. algorithms for obtaining robust controllers. The pioneering work in [14] considered leaderless position consensus and assumed neighbouring agent interactions modelled by an undirected, connected graph. In [15], flocking is achieved assuming an undirected graph. Tracking of a leader with nonconstant velocity is studied in [16] but the subgraph of followers is undirected and two-hop information is required. In general, directed graphs representing unilateral information flow are more desirable than undirected graphs (i.e. bilateral information flow). The passivity analysis in [17] showed synchronisation of the velocities (but not the positions) on strongly connected graphs. Rendezvous to a stationary leader is studied on a directed spanning tree in [18].

Further study of model-independent algorithms is desirable for several reasons. In the context of adaptive algorithms, given a unique Euler-Lagrange equation, determining the minimum number of parameters required is difficult in general [6]. By their definition, the same form of a model-independent controller can be applied to various agents with minor alterations. Thus one can view a model-independent algorithm as a robust controller; stability is guaranteed given limited knowledge of upper bounds on parameters of the multiagent system. This is reminiscent of robust control. By their nature, adaptive controllers typically yield asymptotic stability. In [18], and as we show in this paper, model-independent controllers have exponential stability properties which offer better rejection to noise and disturbances.

The key contribution of this paper is to show that a network of heterogeneous Euler-Lagrange follower agents can synchronise and track the trajectory of a dynamic leader with nonconstant velocity if the interaction topology contains

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a directed spanning tree. The algorithm studied in this paper requires that a pair of scalar control gains, used by every follower agent, satisfy a set of lower bounding inequalities. Some limited knowledge of the bounds on the agent dynamic parameters, limited knowledge of the network topology and a set of all possible initial conditions (which may be arbitrarily large) is required to solve the inequalities. The last requirement means the algorithm is semi-globally stable. In other words, a larger set of initial conditions simply requires recomputation of the control gains and stability is assured.

The paper is organised according to the following structure. In Section II we provide background information which will be used in analysing the proposed algorithm. At the same time a formal definition of the synchronised tracking objective is given. The algorithm and stability proof are detailed in Section III. Simulations are presented in Section IV and the paper is concluded in Section V.

II. BACKGROUND AND PROBLEM STATEMENT

A. Mathematical Notation and Matrix Theory

In this section we provide definitions of notation and several lemmas and theorems and corollaries for later use. We use \otimes to denote the Kronecker product, see [19] for the properties of Kronecker products. The $p \times p$ identity matrix is \mathbf{I}_p and the column vector of all ones as $\mathbf{1}$. The l_1 -norm and Euclidean norm of a vector are denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. The spectral norm of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the matrix norm induced by the Euclidean vector norm and denoted as $\|\mathbf{A}\|_2$. See [19] for the properties of the spectral norm, which will be frequently used in this paper. The signum function is denoted as $\text{sgn}(\cdot)$. For an arbitrary vector \mathbf{x} , the function $\text{sgn}(\mathbf{x})$ is defined element-wise.

A matrix \mathbf{A} which is positive definite (respectively non-negative definite) is denoted by $\mathbf{A} > 0$ (respectively $\mathbf{A} \geq 0$). For two symmetric matrices \mathbf{A}, \mathbf{B} , the expression $\mathbf{A} > \mathbf{B}$ is equivalent to $\mathbf{A} - \mathbf{B} > 0$. For a square symmetric matrix \mathbf{A} , the minimum and maximum eigenvalues are $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ respectively. Furthermore, the following inequality expressions hold

$$\lambda_{\min}(\mathbf{A}) > \lambda_{\max}(\mathbf{B}) \Rightarrow \mathbf{A} > \mathbf{B} \quad (1)$$

$$\lambda_{\max}(\mathbf{A} + \mathbf{B}) \leq \lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B}) \quad (2)$$

$$\lambda_{\min}(\mathbf{A} + \mathbf{B}) \geq \lambda_{\min}(\mathbf{A}) + \lambda_{\min}(\mathbf{B}) \quad (3)$$

$$\lambda_{\min}(\mathbf{A})\mathbf{x}^\top \mathbf{x} \leq \mathbf{x}^\top \mathbf{A} \mathbf{x} \leq \lambda_{\max}(\mathbf{A})\mathbf{x}^\top \mathbf{x} \quad (4)$$

Theorem 1 (The Schur Complement [19]). *Consider a symmetric block matrix, partitioned as*

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^\top & \mathbf{D} \end{bmatrix} \quad (5)$$

Then $\mathbf{A} > 0$ if and only if $\mathbf{B} > 0$ and $\mathbf{D} - \mathbf{C}^\top \mathbf{B}^{-1} \mathbf{C} > 0$.

Lemma 1 ([18]). *Suppose $\mathbf{A} > 0$ is defined as in (5). Let a quadratic function with arguments \mathbf{x}, \mathbf{y} be expressed as $W = [\mathbf{x}^\top, \mathbf{y}^\top] \mathbf{A} [\mathbf{x}^\top, \mathbf{y}^\top]^\top$. Define $\mathbf{F} := \mathbf{B} - \mathbf{C} \mathbf{D}^{-1} \mathbf{C}^\top$*

and $\mathbf{G} := \mathbf{D} - \mathbf{C}^\top \mathbf{B}^{-1} \mathbf{C}$. Then there holds

$$\lambda_{\min}(\mathbf{F})\mathbf{x}^\top \mathbf{x} \leq \mathbf{x}^\top \mathbf{F} \mathbf{x} \leq W \quad (6a)$$

$$\lambda_{\min}(\mathbf{G})\mathbf{y}^\top \mathbf{y} \leq \mathbf{y}^\top \mathbf{G} \mathbf{y} \leq W \quad (6b)$$

Lemma 2 ([19]). *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the l_1 -norm and Euclidean norm respectively. Then for any vector $\mathbf{x} \in \mathbb{R}^n$ there holds $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|_2$.*

Lemma 3. *Let $g(x, y)$ be a function given as*

$$g(x, y) = ax^2 + by^2 - cxy^2 - dxy \quad (7)$$

for real positive scalars $a, c, d > 0$. Then for a given $\mathcal{Y} > 0$, there exist $b > 0$ such that $g(x, y) > 0$ for all $y \in [0, \mathcal{Y}]$ and $x \in [0, \infty)$.

Corollary 1. *Let $h(x, y)$ be a function given as*

$$h(x, y) = ax^2 + by^2 - cxy^2 - dxy - ex - fy \quad (8)$$

where the real, strictly positive scalars a, c, d, e, f and two further positive scalars ε, ϑ are fixed. Suppose that for given \mathcal{Y}, ε there holds $\mathcal{Y} - \varepsilon > 0$. Let $\mathcal{X} > 0$ be such that $\mathcal{X} - \vartheta > 0$ and $\mathcal{X} - \vartheta > e/a$. Define the sets $\mathcal{U} = \{x : x \in [\mathcal{X} - \vartheta, \mathcal{X}]\}$ and $\mathcal{V} = \{y : y \in [\mathcal{Y} - \varepsilon, \mathcal{Y}]\}$. Define the region $\mathcal{R} = \mathcal{U} \cup \mathcal{V}$. Then there exists $b > 0$ such that $h(x, y) > 0, \forall x, y \in \mathcal{R}$.

We omit the proofs to Lemma 3 and Corollary 1 due to space limitations, and will include them in a subsequent extended version of the paper.

B. Graph Theory

The agent interactions can be modelled by a weighted directed graph which is denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with the finite, nonempty set of nodes $\mathcal{V} = \{v_0, v_1, \dots, v_n\}$ with node indices $\mathcal{I} = \{0, 1, 2, \dots, n\}$, and with a corresponding set of ordered edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote an ordered edge of \mathcal{G} as $e_{ij} = (v_i, v_j)$ and as the graph is directed the assumption $e_{ij} = e_{ji}$ does not hold in general. An edge $e_{ij} = (v_i, v_j)$ is outgoing with respect to v_i and incoming with respect to v_j , i.e. the edge (v_i, v_j) indicates that v_j obtains information about v_i (the precise nature of this information will be made clear in the sequel). The weighted adjacency matrix $\mathcal{A} \in \mathbb{R}^{(n+1) \times (n+1)}$ of \mathcal{G} has nonnegative elements a_{ij} . The elements of \mathcal{A} have properties such that $a_{ij} > 0 \Leftrightarrow e_{ji} \in \mathcal{E}$ while $a_{ij} = 0$ if $e_{ji} \notin \mathcal{E}$ and it is assumed $a_{ii} = 0, \forall i$. The neighbour set of v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. The $(n+1) \times (n+1)$ Laplacian matrix, $\mathcal{L} = \{l_{ij}\}$, of the associated digraph \mathcal{G} is defined

$$l_{ij} = \begin{cases} \sum_{k=1, k \neq i}^n a_{ik} & \text{for } j = i \\ -a_{ij} & \text{for } j \neq i \end{cases}$$

A directed path is a sequence of edges of the form $(v_1, v_2), (v_2, v_3), \dots$, where $v_i \in \mathcal{V}$. A directed spanning tree is a directed graph formed by directed edges of the graph that connects all the nodes, and where every vertex apart from the root has exactly one parent [20]. A graph is said to contain a spanning tree if a subset of the edges forms a spanning tree. We provide a lemma to be used in the sequel.

Lemma 4 ([18]). *Let the graph \mathcal{G} contain a directed spanning tree, and suppose there are no edges of \mathcal{G} which are incoming to the root vertex of the tree. Without loss of generality, set the root vertex to be v_0 . Then the Laplacian associated with \mathcal{G} can be partitioned as*

$$\mathcal{L} = \begin{bmatrix} 0 & \mathbf{0} \\ \mathcal{L}_{11} & \mathcal{L}_{22} \end{bmatrix} \quad (9)$$

and there exists a positive definite diagonal matrix $\mathbf{\Gamma}$ such that $\mathbf{Q} \triangleq \mathbf{\Gamma}\mathcal{L}_{22} + \mathcal{L}_{22}^\top\mathbf{\Gamma} > 0$.

C. Euler-Lagrange Systems

The Euler-Lagrange equations can be used to describe the dynamics of a class of nonlinear agents, and the general form for the i^{th} agent equation of motion is:

$$\mathbf{M}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i + \mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)\dot{\mathbf{q}}_i + \mathbf{g}_i(\mathbf{q}_i) = \boldsymbol{\tau}_i \quad (10)$$

where $\mathbf{q}_i \in \mathbb{R}^p$ is a vector of the generalised coordinates, $\mathbf{M}_i(\mathbf{q}_i) \in \mathbb{R}^{p \times p}$ is the inertia matrix, $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i) \in \mathbb{R}^{p \times p}$ is the Coriolis and centrifugal force matrix, $\mathbf{g}_i(\mathbf{q}_i) \in \mathbb{R}^p$ is the vector of potential forces and $\boldsymbol{\tau}_i \in \mathbb{R}^p$ is the control input vector. This work assumes that all agents are fully-actuated. For each agent, we use superscript (j) to denote the j^{th} generalised coordinate; $\mathbf{q}_i = [q_i^{(1)}, \dots, q_i^{(p)}]^\top$. We assume that the systems described using (10) have the following properties given below (with details in [6], [7]):

- A1 The matrix $\mathbf{M}_i(\mathbf{q}_i)$ is symmetric positive definite.
- A2 There exist scalar constants $k_m, k_{\overline{M}} > 0$ such that $k_m \mathbf{I}_p \leq \mathbf{M}_i(\mathbf{q}_i) \leq k_{\overline{M}} \mathbf{I}_p, \forall i, \mathbf{q}_i$. It follows that $\sup_{\mathbf{q}_i} \|\mathbf{M}_i\|_2 \leq k_{\overline{M}}$ and $k_m \leq \inf_{\mathbf{q}_i} \|\mathbf{M}_i^{-1}\|_2^{-1}, \forall i$.
- A3 The matrix $\mathbf{C}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i)$ is defined such that $\dot{\mathbf{M}}_i - 2\mathbf{C}_i$ is skew-symmetric. It follows that $\dot{\mathbf{M}}_i = \mathbf{C}_i + \mathbf{C}_i^\top$.
- A4 There exists scalar constants $k_C, k_g > 0$ such that $\|\mathbf{C}_i\|_2 \leq k_C \|\dot{\mathbf{q}}_i\|_2, \forall i, \dot{\mathbf{q}}_i$ and $\|\mathbf{g}_i\|_2 < k_g, \forall i$.

D. Problem Statement

The leader is denoted as agent 0 with $\mathbf{q}_0(t)$ and $\dot{\mathbf{q}}_0(t)$ being the time-varying generalised coordinates and generalised velocity of the leader respectively. The control objective is to develop a model-independent, distributed algorithm which allows a directed network of Euler-Lagrange agents to synchronise and track the trajectory of the leader. The leader tracking objective is said to be achieved if:

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\mathbf{q}_i(t) - \mathbf{q}_0(t)\|_2 &= 0 \\ \lim_{t \rightarrow \infty} \|\dot{\mathbf{q}}_i(t) - \dot{\mathbf{q}}_0(t)\|_2 &= 0 \end{aligned}, \forall i = 1, \dots, n \quad (11)$$

Assumption A5: We make the following assumptions on the leader trajectory. We assume that the trajectory $\mathbf{q}_0(t)$ is Lipschitz continuous. We also assume that the derivatives $\dot{\mathbf{q}}_0$ and $\ddot{\mathbf{q}}_0$ are bounded, which can be expressed as $\|\mathbf{1}_n \otimes \dot{\mathbf{q}}_0\|_2 \leq k_p$ and $\|\mathbf{1}_n \otimes \ddot{\mathbf{q}}_0\|_2 \leq k_q$.

The *sensing* graph topology for both the relative generalised coordinates and relative generalised velocities is captured by the fixed, weighted, directed graph \mathcal{G}_A with an associated Laplacian \mathcal{L}_A . That is, if for agent i we have $a_{ij} > 0$, then agent i knows a_{ij} and can separately sense $\mathbf{q}_i - \mathbf{q}_j$ and $\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j$. We denote the neighbour set of agent i on \mathcal{G}_A

as \mathcal{N}_{Ai} . We further assume that agent i can measure its own coordinate and velocity, \mathbf{q}_i and $\dot{\mathbf{q}}_i$, respectively. A second graph, \mathcal{G}_B , exists between the followers to *communicate* information about the leader's state. It is fixed, weighted and directed with the associated Laplacian \mathcal{L}_B , and the neighbour set of agent i on \mathcal{G}_B is denoted as \mathcal{N}_{Bi} . Note that $v_j \in \mathcal{N}_{Bi}$ when agent j communicates information directly to agent i regarding the leader's state (the precise nature of this information is described in Section III-A). Further note that \mathcal{G}_A is not necessarily equal to \mathcal{G}_B and so $\mathcal{N}_{Ai} \neq \mathcal{N}_{Bi}$ in general. However, excluding the leader v_0 , the node set of \mathcal{G}_A is the same as the node set of \mathcal{G}_B .

Finally, we assume that all possible initial conditions lie in some fixed but arbitrarily large set, Ω , which is known a priori. This is not unreasonable, as many systems will have an expected operating range for \mathbf{q} and $\dot{\mathbf{q}}$.

By model-independent, we mean that the algorithm does not contain $\mathbf{M}_i, \mathbf{C}_i, \mathbf{g}_i \forall i$ and does not contain the linear parametrisation. An agent's algorithm is distributed if, during execution, the agent only needs to receive information about its neighbours.

Notice that it is possible for $\mathbf{M}_i \neq \mathbf{M}_j, \mathbf{C}_i \neq \mathbf{C}_j$ and $\mathbf{g}_i \neq \mathbf{g}_j$ for any i, j but $\mathbf{q}_i \in \mathbb{R}^p, \forall i$. In other words this work treats Euler-Lagrange agents which have heterogeneous parameters but with generalised coordinates which are defined such that $\mathbf{q}_i - \mathbf{q}_j, \forall i, j$ is meaningful.

III. MAIN RESULT

A. Finite-Time Distributed Observer

Before we show the main result of the paper, we detail a distributed finite-time observer, developed in [21], which allows each follower to obtain the leader states (i.e. \mathbf{q}_0 and $\dot{\mathbf{q}}_0$). Let $\hat{\mathbf{r}}_i$ and $\hat{\mathbf{v}}_i$ be the i^{th} agent's estimated values for the leader position and velocity respectively. Note that if agent i can directly sense the leader, v_0 , then $\hat{\mathbf{r}}_i(t) = \mathbf{q}_0(t)$ and $\hat{\mathbf{v}}_i(t) = \dot{\mathbf{q}}_0(t)$ for all $t \geq 0$ (i.e. v_0 is a parent of agent i on the graph \mathcal{G}_A). Without loss of generality, denote the r children of v_0 as v_1, \dots, v_r . Then agent $i \in \{1, \dots, r\}$ does not need to use an observer. The observer for agent $i \in \{r+1, \dots, n\}$ is given as

$$\dot{\hat{\mathbf{r}}}_i = \hat{\mathbf{v}}_i - \alpha_1 \operatorname{sgn} \left(\sum_{j \in \mathcal{N}_{Bi}} b_{ij} (\hat{\mathbf{r}}_i - \hat{\mathbf{r}}_j) \right) \quad (12a)$$

$$\dot{\hat{\mathbf{v}}}_i = -\alpha_2 \operatorname{sgn} \left(\sum_{j \in \mathcal{N}_{Bi}} b_{ij} (\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_j) \right) \quad (12b)$$

where b_{ij} are the elements of the adjacency matrix associated with graph \mathcal{G}_B and $\alpha_1, \alpha_2 > 0$ are control gains. From [21], the above observer requires the leader trajectory to satisfy two assumptions. The first is that \mathbf{q}_0 is differentiable and the second is that $\|\ddot{\mathbf{q}}_0\|_2 \leq K$ (i.e. the acceleration of the leader is bounded). Both of these assumptions are satisfied in our paper, see Assumption A5. Define agents v_1, \dots, v_r and agents v_{r+1}, \dots, v_n as the leaders and children of the graph \mathcal{G}_B respectively.

Theorem 2 (Theorem 4.1 of [21]). *If, for each child of \mathcal{G}_B , there exists a directed path from a leader of \mathcal{G}_B to that child,*

and $\alpha_2 > k_q/n$ then there holds $\hat{r}_i(t) = \mathbf{q}_0(t)$ and $\hat{v}_i(t) = \dot{\mathbf{q}}_0(t)$ for all i , for all $t \geq T_1$ where $T_1 < \infty$.

B. Model-Independent Control Law

Consider the following discontinuous, model-independent algorithm for the i^{th} agent

$$\begin{aligned} \tau_i = & - \sum_{j \in \mathcal{N}_{Ai}} a_{ij} \gamma_i (\mathbf{q}_i - \mathbf{q}_j) - \mu \sum_{j \in \mathcal{N}_{Ai}} a_{ij} \gamma_i (\dot{\mathbf{q}}_i - \dot{\mathbf{q}}_j) \\ & - \beta \operatorname{sgn} (\mu^{-1} (\mathbf{q}_i - \hat{r}_i) + (\dot{\mathbf{q}}_i - \hat{v}_i)) \end{aligned} \quad (13)$$

where a_{ij} is the weighted (i, j) entry of the adjacency matrix \mathcal{A} associated with the graph \mathcal{G}_A . The constant γ_i is defined such that $\mathbf{\Gamma} = \operatorname{diag}[\gamma_1, \dots, \gamma_n]$ satisfies $\mathbf{\Gamma} \mathcal{L}_{22} + \mathcal{L}_{22}^T \mathbf{\Gamma} > 0$. The constant control gains μ and β are strictly positive and their design will be specified later.

Let us denote the new error variable $\tilde{\mathbf{q}}_i = \mathbf{q}_i - \mathbf{q}_0$. Let $\tilde{\mathbf{q}} = [\tilde{\mathbf{q}}_1^T, \dots, \tilde{\mathbf{q}}_n^T]^T$ be the stacked column vector of all $\tilde{\mathbf{q}}_i$. The tracking objective is therefore achieved if $\tilde{\mathbf{q}}(t) = \dot{\tilde{\mathbf{q}}}(t) = 0$ as $t \rightarrow \infty$. We denote $\mathbf{g} = [\mathbf{g}_1^T, \dots, \mathbf{g}_n^T]^T$, $\mathbf{q} = [\mathbf{q}_1^T, \dots, \mathbf{q}_n^T]^T$, and $\dot{\mathbf{q}} = [\dot{\mathbf{q}}_1^T, \dots, \dot{\mathbf{q}}_n^T]^T$ as the stacked column vector of all \mathbf{g}_i , \mathbf{q}_i and $\dot{\mathbf{q}}_i$ respectively. Let $\mathbf{M}(\mathbf{q}) = \operatorname{diag}[\mathbf{M}_1(\mathbf{q}_1), \dots, \mathbf{M}_n(\mathbf{q}_n)]$, $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \operatorname{diag}[\mathbf{C}_1(\mathbf{q}_1, \dot{\mathbf{q}}_1), \dots, \mathbf{C}_n(\mathbf{q}_n, \dot{\mathbf{q}}_n)]$. Since $\mathbf{M}_i = \mathbf{M}_i^T > 0$, $\forall i$ then $\mathbf{M} \mathbf{M}^T \text{top} > 0$. Define an error vector, $\mathbf{e}_i = \hat{r}_i - \mathbf{q}_0$, $\forall i = 1, \dots, n$ and $\dot{\mathbf{e}}_i = \hat{v}_i - \dot{\mathbf{q}}_0$. Let $\mathbf{e} = [\mathbf{e}_1^T, \dots, \mathbf{e}_n^T]^T$, $\dot{\mathbf{e}} = [\dot{\mathbf{e}}_1^T, \dots, \dot{\mathbf{e}}_n^T]^T$. Note that $\mathbf{e}_i = \dot{\mathbf{e}}_i = \mathbf{0}$ for $i = 1, \dots, r$.

The definition of $\tilde{\mathbf{q}}_i$ yields $\mathbf{M}_i \dot{\tilde{\mathbf{q}}}_i = \mathbf{M}_i \dot{\mathbf{q}}_i - \mathbf{M}_i \dot{\mathbf{q}}_0$ and combining the agent dynamics (10) and the control law (13), the closed-loop system for the follower network can be expressed as

$$\begin{aligned} \dot{\tilde{\mathbf{q}}} \in & \text{a.e. } \mathcal{K} \left[-\mathbf{M}^{-1} [\mathbf{C} \dot{\tilde{\mathbf{q}}} + (\mathbf{\Gamma} \mathcal{L}_{22} \otimes \mathbf{I}_p) (\tilde{\mathbf{q}} + \mu \dot{\tilde{\mathbf{q}}}) + \mathbf{g} \right. \\ & \left. + \beta \operatorname{sgn} (\mu^{-1} \mathbf{s} + \dot{\mathbf{s}}) + \mathbf{M} (\mathbf{1}_n \times \dot{\mathbf{q}}_0) + \mathbf{C} (\mathbf{1}_n \times \dot{\mathbf{q}}_0) \right] \end{aligned} \quad (14)$$

where \mathcal{K} denotes the differential inclusion, *a.e.* stands for ‘‘almost everywhere’’ and $\mathbf{s} = \mathbf{q} - (\mathbf{1} \otimes \hat{\mathbf{q}}) = \tilde{\mathbf{q}} - \mathbf{e}$. Filippov solutions for (14) exist because the signum function is measurable and locally essentially bounded. This implies that the Filippov solutions of $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$ are absolutely continuous functions of time [22].

Notice that the system (14) is non-autonomous in the sense that it is not a self-contained system (since the arguments of \mathbf{M} and \mathbf{C} depend on \mathbf{q} and $\dot{\mathbf{q}}$). Furthermore, the system is not self-contained in the sense that the terms associated with the leader, $\mathbf{M} (\mathbf{1}_n \times \dot{\mathbf{q}}_0)$ and $\mathbf{C} (\mathbf{1}_n \times \dot{\mathbf{q}}_0)$, may be seen as an external input. Although the system (14) is not a self-contained system, it turns out that using arguments like those of usual Lyapunov theory, we will be able to prove a stability result for (14).

C. An Upper Bound Using Initial Conditions

Before we proceed with main proof, we provide a method to calculate a not necessarily tight upper bound on the initial states expressed as $\|\tilde{\mathbf{q}}(0)\|_2 < \mathcal{X}$ and $\|\dot{\tilde{\mathbf{q}}}(0)\|_2 < \mathcal{Y}$ using the set of initial conditions, Ω . In the sequel, we show that these bounds hold for all time, and exponential convergence results. Due to spatial limitations, we show only the bound

on $\dot{\tilde{\mathbf{q}}}$ and leave the reader to follow an identical process for $\tilde{\mathbf{q}}$. In keeping with the model-independent nature of the paper, define a function as

$$\tilde{V}_\mu = \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix}^T \begin{bmatrix} \lambda_{\max}(\mathbf{X}) \mathbf{I}_{np} & \frac{1}{2} \mu^{-1} (k_{\overline{M}} + \delta) \mathbf{I}_{np} \\ \frac{1}{2} \mu^{-1} (k_{\overline{M}} + \delta) \mathbf{I}_{np} & \frac{1}{2} (k_{\overline{M}} + \delta) \mathbf{I}_{np} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix} \quad (15)$$

where, $\mathbf{X} = (\mathbf{\Gamma} \mathcal{L}_{22} + \mathcal{L}_{22}^T \mathbf{\Gamma}) \otimes \mathbf{I}_p > 0$ from Lemma 4, and $\delta > 0$ is arbitrarily small and fixed. Note that $(k_{\overline{M}} + \delta) \mathbf{I}_{np} > \mathbf{M}$ and that \tilde{V}_μ is *not* a Lyapunov function. Let the matrix in (15) be \mathbf{L} and suppose that it is positive definite (we show in the sequel that this is true for a sufficiently large μ). Let μ_1^* be such that $\mathbf{L} > 0$ and observe the positive definiteness then also holds for $\mu \geq \mu_1^*$. While \tilde{V}_μ is a function of $\tilde{\mathbf{q}}(t)$ and $\dot{\tilde{\mathbf{q}}}(t)$, we use $\tilde{V}_\mu(t)$ to denote $\tilde{V}_\mu(\tilde{\mathbf{q}}(t), \dot{\tilde{\mathbf{q}}}(t))$. Next, let

$$V_{L\mu} = \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix}^T \begin{bmatrix} \frac{1}{4} \lambda_{\min}(\mathbf{X}) \mathbf{I}_{np} & \frac{1}{2} \mu^{-1} (k_{\underline{m}} - \delta) \mathbf{I}_{np} \\ \frac{1}{2} \mu^{-1} (k_{\underline{m}} - \delta) \mathbf{I}_{np} & \frac{1}{2} (k_{\underline{m}} - \delta) \mathbf{I}_{np} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix} \quad (16)$$

Call the matrix in (16) \mathbf{N} . Let the arbitrarily small δ be such that $(k_{\underline{m}} - \delta) > 0$. As above, we will show in the sequel that there is a μ sufficiently large which ensures that $\mathbf{N} > 0$. Let μ_2^* be such that $\mathbf{N} > 0$; for $\mu \geq \mu_2^*$ there continues to hold $\mathbf{N} > 0$. Set $\mu_3^* = \max\{\mu_1^*, \mu_2^*\}$. Define

$$\rho_1(\mu) = \frac{1}{2} k_{\overline{M}} + \delta - \frac{1}{4} \mu^{-2} (k_{\overline{M}} + \delta)^2 \lambda_{\max}(\mathbf{X})^{-1} \quad (17)$$

$$\rho_2(\mu) = \frac{1}{2} (k_{\underline{m}} - \delta) - \mu^{-2} (k_{\underline{m}} - \delta)^2 \lambda_{\min}(\mathbf{X})^{-1} \quad (18)$$

and observe that for sufficiently large μ there holds $\rho_1 > \rho_2$. Assume without loss of generality that $\rho_1 > \rho_2$ (if not, one can always find a $\mu_4^* > \mu_3^*$ such that $\rho_1 > \rho_2$). Note that for any $\mu \geq \mu_3^*$ there holds $\tilde{V}_\mu \leq \tilde{V}_{\mu_3^*}$ and $\rho_2(\mu_3^*) \leq \rho(\mu)$. It follows from Lemma 1 and (6b) that

$$\|\dot{\tilde{\mathbf{q}}}(0)\|_2 \leq \sqrt{\frac{\tilde{V}_\mu(0)}{\rho_1(\mu)}} \leq \sqrt{\frac{\tilde{V}_{\mu_3^*}(0)}{\rho_1(\mu_3^*)}} < \sqrt{\frac{\tilde{V}_{\mu_3^*}(0)}{\rho_2(\mu_3^*)}} \quad (19)$$

From Ω , one obtains a bounded set of possible values for $\tilde{V}_{\mu_3^*}(0)$ and define the maximal value of the set as $\tilde{V}^* = \sup_{\Omega} \tilde{V}_{\mu_3^*}(0)$. There holds

$$\|\dot{\tilde{\mathbf{q}}}(0)\|_2 \leq \sqrt{\frac{\tilde{V}^*}{\rho_1(\mu_3^*)}} < \sqrt{\frac{\tilde{V}^*}{\rho_2(\mu_3^*)}} = \mathcal{Y}_1, \forall \dot{\tilde{\mathbf{q}}}(0) \in \Omega \quad (20)$$

Similarly, one can also obtain a bound $\|\tilde{\mathbf{q}}(0)\|_2 \leq \mathcal{X}_1$ using $\tilde{V}_{\mu_3^*}$ and $\rho_2(\mu_3^*)$. Compute

$$\hat{V} = \lambda_{\max}(\mathbf{X}) \mathcal{X}_1^2 + \frac{1}{2} (k_{\overline{M}} + \delta) \mathcal{Y}_1^2 + (\mu_3^*)^{-1} (k_{\overline{M}} + \delta) \mathcal{X}_1 \mathcal{Y}_1 \quad (21)$$

and observe that $\tilde{V}^* \leq \hat{V}$. Finally, compute the bound $\mathcal{Y} = \sqrt{\hat{V} / \rho_2(\mu_3^*)}$. Observe that there holds

$$\|\dot{\tilde{\mathbf{q}}}(0)\|_2 < \sqrt{\frac{\tilde{V}^*}{\rho_2(\mu_3^*)}} \leq \sqrt{\frac{\hat{V}}{\rho_2(\mu_3^*)}} = \mathcal{Y}, \forall \dot{\tilde{\mathbf{q}}}(0) \in \Omega \quad (22)$$

and notice that $\mathcal{Y} \geq \mathcal{Y}_1$. One can similarly obtain \mathcal{X} using (6a). Note both sides of (22) are independent of μ . Thus $\|\dot{\tilde{\mathbf{q}}}(0)\|_2 < \mathcal{Y}$ (and similarly $\|\tilde{\mathbf{q}}(0)\|_2 < \mathcal{X}$) can be used for all $\mu \geq \mu_3^*$. We now proceed to the main result.

D. Stability Proof

Theorem 3. *The equilibrium of system (14) is semi-globally exponentially stable if 1) the network \mathcal{G}_A contains a directed spanning tree with the leader as the root node, and 2) the control gains μ, β satisfy lower bounding inequalities. These inequalities are computable from the bounds on the matrices of the agent dynamics, bounds on the graph Laplacian, the set of initial conditions Ω and assumptions on the leader trajectory, A5. For a given \mathcal{G}_A containing a directed spanning tree, there always exists μ, β which satisfy the inequalities.*

Proof. The proof will be presented in four parts. In *Part 1*, we present a Lyapunov-like candidate function, V and show it is positive definite for a sufficiently large μ . In *Part 2*, we analyse the derivative \dot{V} and show that it is upper bounded by the summation of a continuous function and a set-valued function. *Part 3* analyses the upper bound on \dot{V} . In performing the associated calculations, we will assume that the above computed values \mathcal{X}, \mathcal{Y} are such that $\|\tilde{\mathbf{q}}(t)\|_2 < \mathcal{X}$ and $\|\dot{\tilde{\mathbf{q}}}(t)\|_2 < \mathcal{Y}$ hold for some interval $[0, T_2)$. We show that a finite T_2 creates a contradiction, i.e. T_2 can be taken to be infinite. With T_2 infinite, *Part 4* concludes that the trajectories remain bounded for all time; a final step concludes the tracking objective is achieved exponentially fast.

Part 1: Consider the Lyapunov-like candidate function

$$V = \frac{1}{2}\tilde{\mathbf{q}}^\top \mathbf{X}\tilde{\mathbf{q}} + \mu^{-1}\tilde{\mathbf{q}}^\top \mathbf{M}\dot{\tilde{\mathbf{q}}} + \frac{1}{2}\dot{\tilde{\mathbf{q}}}^\top \mathbf{M}\dot{\tilde{\mathbf{q}}} = V_1 + V_2 + V_3 \quad (23)$$

where \mathbf{X} was given below (15). In quadratic form, V is

$$V = \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix}^\top \begin{bmatrix} \frac{1}{2}\mathbf{X} & \frac{1}{2}\mu^{-1}\mathbf{M} \\ \frac{1}{2}\mu^{-1}\mathbf{M} & \frac{1}{2}\mathbf{M} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}} \\ \dot{\tilde{\mathbf{q}}} \end{bmatrix} \quad (24)$$

The function V is positive definite in the variables $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$ if the matrix in (24), which depends on \mathbf{q} , is positive definite. From Theorem 1, and the assumed properties of \mathbf{M}_i , the matrix is positive definite if and only if $\mathbf{X} - \mu^{-2}\mathbf{M}$ is positive definite, which is implied by $\lambda_{\min}(\mathbf{X}) - \mu^{-2}k_{\overline{M}} > 0$ since there holds $k_{\overline{M}} \geq \sup_{\mathbf{q}} \lambda_{\max}(\mathbf{M})$. Then V is positive definite if $\mu > \sqrt{k_{\overline{M}}/\lambda_{\min}(\mathbf{X})}$ and because $\mathbf{X} > 0$, then there always exists such a $\mu > 0$. While V is a function of $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$, for simplicity, let $V(t)$ denote $V(\tilde{\mathbf{q}}(t), \dot{\tilde{\mathbf{q}}}(t))$. Call the matrix in (24) \mathbf{H} . It is straightforward to verify that $\mathbf{L} > \mathbf{H}$ for any μ satisfying $\mu > \sqrt{k_{\overline{M}}/\lambda_{\min}(\mathbf{X})}$ and thus $V(t) < \bar{V}(t), \forall t$. Similar calculations show that $\mathbf{N} > 0$ if

$$\mu > \sqrt{\frac{2k_{\overline{M}}}{\lambda_{\min}(\mathbf{X})}} \quad (25)$$

and notice that any μ satisfying (25) also ensures that $\mathbf{L} > \mathbf{H} > \mathbf{N} > 0$, and thus (22) holds. *From here onwards, we assume μ has been selected to satisfy (25) and therefore V is positive definite in the variables $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$ and V is radially unbounded. Observe that by utilising expression (4) and the properties of the spectral norm, we obtain*

$$V \leq \frac{1}{2}\lambda_{\max}(\mathbf{X})\|\tilde{\mathbf{q}}\|_2^2 + \frac{1}{2}k_{\overline{M}}\|\dot{\tilde{\mathbf{q}}}\|_2^2 + \mu^{-1}k_{\overline{M}}\|\tilde{\mathbf{q}}\|_2\|\dot{\tilde{\mathbf{q}}}\|_2 \quad (26)$$

Part 2: Let \dot{V} be the set-valued derivative of V with respect to time, along the system trajectory. From (23) we obtain $\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3$. Using $\dot{\tilde{\mathbf{q}}}$ from (14) and Assumption A3, we obtain

$$\begin{aligned} \dot{V} \in \mathcal{K} \left[-\mu^{-1} \left(\frac{1}{2}\tilde{\mathbf{q}}^\top \mathbf{X}\dot{\tilde{\mathbf{q}}} + \frac{1}{2}\mu^2\dot{\tilde{\mathbf{q}}}^\top \mathbf{X}\dot{\tilde{\mathbf{q}}} - \dot{\tilde{\mathbf{q}}}^\top \mathbf{M}\dot{\tilde{\mathbf{q}}} \right. \right. \\ \left. \left. - \tilde{\mathbf{q}}^\top \mathbf{C}^\top \dot{\tilde{\mathbf{q}}} + (\tilde{\mathbf{q}} + \mu\dot{\tilde{\mathbf{q}}})^\top \mathbf{C}(\mathbf{1} \otimes \dot{\mathbf{q}}_0) \right) - \mathbf{x}^\top \Delta \right. \\ \left. - \beta \mathbf{x}^\top \text{sgn}(\mathbf{x} - \mathbf{y}) \right] \quad (27) \end{aligned}$$

where $\Delta = \mathbf{g} + \mathbf{M}(\mathbf{1}_n \times \ddot{\mathbf{q}}_0)$, $\mathbf{x} = \mu^{-1}\tilde{\mathbf{q}} + \dot{\tilde{\mathbf{q}}}$ and $\mathbf{y} = \mu^{-1}\mathbf{e} + \dot{\mathbf{e}}$ (recall that $\mathbf{s} = \tilde{\mathbf{q}} - \mathbf{e}$). The detailed calculations shall be included in the extended version of the paper. From the bounds on \mathbf{g} , \mathbf{M} and $\mathbf{1} \otimes \ddot{\mathbf{q}}_0$, it follows that $\mathbf{x}^\top \Delta \leq \xi \|\mathbf{x}\|_2$ where $\xi = k_g + k_{\overline{M}}k_q$. From Assumption A4, the property of norms and the definition of $\tilde{\mathbf{q}}$, it follows that $\|\mathbf{C}\|_2 = \|\mathbf{C}^\top\|_2 \leq k_C \|\dot{\tilde{\mathbf{q}}}\|_2 \leq k_C(\|\tilde{\mathbf{q}}\|_2 + k_p)$. Thus

$$\tilde{\mathbf{q}}^\top \mathbf{C}^\top \dot{\tilde{\mathbf{q}}} \leq k_C k_p \|\tilde{\mathbf{q}}\|_2 \|\dot{\tilde{\mathbf{q}}}\|_2 + k_C \|\tilde{\mathbf{q}}\|_2 \|\dot{\tilde{\mathbf{q}}}\|_2^2 \quad (28a)$$

$$\begin{aligned} (\tilde{\mathbf{q}} + \mu\dot{\tilde{\mathbf{q}}})^\top \mathbf{C}(\mathbf{1} \otimes \dot{\mathbf{q}}_0) &\leq k_C k_p \|\tilde{\mathbf{q}}\|_2 \|\dot{\tilde{\mathbf{q}}}\|_2 + \mu k_C k_p \|\dot{\tilde{\mathbf{q}}}\|_2^2 \\ &+ \mu k_C k_p^2 \|\mu^{-1}\tilde{\mathbf{q}} + \dot{\tilde{\mathbf{q}}}\|_2 \end{aligned} \quad (28b)$$

Let $\varphi(\mu) = \frac{1}{2}\mu^2 \lambda_{\min}(\mathbf{X}) - \mu k_C k_p - k_{\overline{M}}$. Define functions \dot{V}_A (absolutely continuous) and \dot{V}_B (set-valued) as

$$\begin{aligned} \dot{V}_A &= -\mu^{-1}(\varphi(\mu)\|\dot{\tilde{\mathbf{q}}}\|_2^2 + \frac{1}{2}\lambda_{\min}(\mathbf{X})\|\tilde{\mathbf{q}}\|_2^2 \\ &- k_C k_p \|\tilde{\mathbf{q}}\|_2 \|\dot{\tilde{\mathbf{q}}}\|_2 - k_C \|\tilde{\mathbf{q}}\|_2 \|\dot{\tilde{\mathbf{q}}}\|_2^2) \end{aligned} \quad (29a)$$

$$\dot{V}_B \in \mathcal{K} \left[-\beta \mathbf{x}^\top \text{sgn}(\mathbf{x} - \mathbf{y}) + k_C k_p^2 \|\mathbf{x}\|_2 + \xi \|\mathbf{x}\|_2 \right] \quad (29b)$$

Applying the inequalities in (28), and the eigenvalue inequalities noted in Section II-A, the derivative \dot{V} in (27) is upper bounded as $\dot{V} \leq \dot{V}_A + \dot{V}_B$.

Part 3: Here, there are three sub-parts. In *Part 3.1*, we show $\dot{V}_A < 0$ within a given region of $\tilde{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}$ space if μ is sufficiently large. Next in *Part 3.2*, we study \dot{V}_B . Lastly, *Part 3.3* studies $\dot{V}_A + \dot{V}_B$ and we prove an inequality which, if valid on an infinite interval, ensures the trajectories are bounded for all time.

Part 3.1: Consider the region of the state variables given by $\|\tilde{\mathbf{q}}(t)\|_2 \in [0, \infty)$ and $\|\dot{\tilde{\mathbf{q}}}(t)\|_2 \in [0, \mathcal{Y}]$ where $\mathcal{Y} > 0$ was computed in Section III-C.

It is straightforward to compute a $\mu_5^* \geq \mu_3^*$ such that $\varphi > 0, \forall \mu \geq \mu_5^*$. Note the definition of μ_5^* ensures $\mathbf{L} > \mathbf{H} > \mathbf{N} > 0$. Apply Lemma 3 to the function \dot{V}_A in (29a) where $x = \|\tilde{\mathbf{q}}\|_2$ and $y = \|\dot{\tilde{\mathbf{q}}}\|_2$. With $b = \varphi > 0$, it is straightforward to conclude that for any $\mu \geq \mu_5^*$ such that

$$\varphi(\mu) > \frac{k_C^2 \mathcal{Y}^2 + k_C^2 k_p^2 + 2k_C^2 k_p \mathcal{Y}}{2\lambda_{\min}(\mathbf{X})} \quad (30)$$

the function \dot{V}_A in (29a) is negative definite in the region.

Part 3.2 Now consider \dot{V}_B over two time intervals, $t_p = [0, T_1)$ and $t_Q = [T_1, T_2)$, where T_1 is given in Theorem 2 and T_2 is the infimum of those values of t for which one of the inequalities $\|\tilde{\mathbf{q}}(t)\|_2 < \mathcal{X}$, $\|\dot{\tilde{\mathbf{q}}}(t)\|_2 < \mathcal{Y}$ fails. In other words, $[0, T_2)$ is maximal. At the start of *Part 3.3*, we argue that without loss of generality, it is possible to take $T_2 > T_1$.

Consider firstly $t \in t_p$. Observe that the set-valued function $-\beta \mathbf{x}^\top \text{sgn}(\mathbf{x} - \mathbf{y})$ is upper bounded by the single-valued function $\beta \|\mathbf{x}\|_1$. Recalling the definition of \dot{V}_B in (29b) and Lemma 2, we obtain

$$\dot{V}_B \leq (\sqrt{n}\beta + k_C k_p^2 + \xi)(\mu^{-1} \|\tilde{\mathbf{q}}\|_2 + \|\dot{\tilde{\mathbf{q}}}\|_2) := \dot{V}_{\bar{B}} \quad (31)$$

For $t \in t_Q$, Theorem 2 yields that $e(t) = \dot{e}(t) = \mathbf{0}$, which implies that $\mathbf{y} = \mathbf{0}$. It then follows that

$$\begin{aligned} \dot{V}_B &\in \mathcal{K} \left[-\beta \mathbf{x}^\top \text{sgn}(\mathbf{x}) + k_C k_p^2 \|\mathbf{x}\|_2 + \xi \|\mathbf{x}\|_2 \right] \\ &= -\beta \|\mathbf{x}\|_1 + k_C k_p^2 \|\mathbf{x}\|_2 + \xi \|\mathbf{x}\|_2 \end{aligned} \quad (32)$$

The above conclusion relies on the fact that $\mathcal{K}[\mathbf{x}^\top \text{sgn}(\mathbf{x})] = \{\|\mathbf{x}\|_1\}$ is a singleton (i.e. \dot{V}_B for $t \in t_Q$ is a continuous, single-valued function in the variables $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$). By designing $\beta > k_C k_p^2 + \xi$ and from Lemma 2 it is straightforward to conclude $\dot{V}_B \leq -(\beta - k_C k_p^2 - \xi) \|\mu^{-1} \tilde{\mathbf{q}} + \dot{\tilde{\mathbf{q}}}\|_1 < 0$.

Part 3.3: Consider firstly \dot{V} for $t \in t_p$. Specifically, let $\dot{V}_{t_p} := \dot{V}_A + \dot{V}_{\bar{B}}$, defined from (29a) and (31) respectively. Note that $\dot{V} \leq \dot{V}_{t_p}$. Observe that $-\dot{V}_{t_p}$ is of the form of $h(x, y)$ in Corollary 1 with $x = \|\tilde{\mathbf{q}}\|_2$ and $y = \|\dot{\tilde{\mathbf{q}}}\|_2$. Here, $b = \varphi(\mu)$, $a = \frac{1}{2} \mu^{-1} \lambda_{\min}(\mathbf{X})$ and $e = \mu^{-1}(\sqrt{n}\beta + k_C k_p^2 + \xi)$. Thus, for some given $\vartheta, \varepsilon, \mathcal{X}, \mathcal{Y}$ satisfying the requirements detailed in Corollary 1, one can find a μ such that $\dot{V}_{t_p} < 0$ in the region \mathcal{R} . Note that ϑ, ε can be selected by the designer. The specifics of choosing $\vartheta, \varepsilon > 0$ will be detailed in an extended version of the paper.

Define the sets \mathcal{U}, \mathcal{V} and the region \mathcal{R} as in Corollary 1 with $x = \|\tilde{\mathbf{q}}\|_2$ and $y = \|\dot{\tilde{\mathbf{q}}}\|_2$. Define further sets $\bar{\mathcal{U}} = \{\|\tilde{\mathbf{q}}\|_2 : \|\tilde{\mathbf{q}}\|_2 > \mathcal{X}\}$ and $\bar{\mathcal{V}} = \{\|\dot{\tilde{\mathbf{q}}}\|_2 : \|\dot{\tilde{\mathbf{q}}}\|_2 > \mathcal{Y}\}$. Define the compact region $\mathcal{S} = \mathcal{U} \cup \mathcal{V} \setminus \bar{\mathcal{U}} \cup \bar{\mathcal{V}}$. Since $\mathcal{S} \subset \mathcal{R}$, there exists a $\mu_6^* \geq \mu_5^*$ such that $\varphi(\mu_6^*)$ satisfies the requirement on b in Corollary 1 and thus $\dot{V}_{t_p} < 0$ in the region \mathcal{S} and this holds for any $\mu \geq \mu_6^*$. Further define the region $\|\tilde{\mathbf{q}}(t)\|_2 \in [0, \mathcal{X} - \vartheta]$, $\|\dot{\tilde{\mathbf{q}}}(t)\|_2 \in [0, \mathcal{Y} - \varepsilon]$ as \mathcal{T} .

Now we justify the fact that we can assume $T_2 > T_1$. In fact, in doing so, we show that T_2 is not maximal i.e. T_2 is infinite. Suppose that the trajectory of (14) is such that in the interval $[0, \min\{T_1, T_2\}]$, it spends time in both \mathcal{S} and \mathcal{T} . Although \dot{V} is sign indefinite in \mathcal{T} (i.e. V can grow), notice from (26) that, in \mathcal{T} there holds

$$\begin{aligned} V(t) &\leq \frac{1}{2} \lambda_{\max}(\mathbf{X})(\mathcal{X} - \vartheta)^2 + \frac{1}{2} k_{\bar{M}} (\mathcal{Y} - \varepsilon)^2 \\ &\quad + \mu^{-1} k_{\bar{M}} (\mathcal{X} - \vartheta)(\mathcal{Y} - \varepsilon) := \mathcal{Z} \end{aligned} \quad (33)$$

Recall $\dot{V} < 0$ in \mathcal{S} . This implies that for all trajectories of (14) where the trajectory spends time in both \mathcal{S} and \mathcal{T} , we have $V(t) \leq \max\{\mathcal{Z}, V(0)\}$. The correct choice of ϑ, ε ensures that $\mathcal{X}_1 > \mathcal{X} - \vartheta$ and $\mathcal{Y}_1 > \mathcal{Y} - \varepsilon$. Recalling that $\delta > 0$ is arbitrarily small, we have

$$\mathcal{Z} < \lambda_{\max}(\mathbf{X}) \mathcal{X}_1^2 + (k_{\bar{M}} + \delta) \left(\frac{1}{2} \mathcal{Y}_1^2 + \mu^{-1} \mathcal{X}_1 \mathcal{Y}_1 \right) \quad (34)$$

i.e. $\mathcal{Z} < \hat{V}$. This implies that for all trajectories of (14) which spend time in both \mathcal{S} and \mathcal{T} in the interval $[0, \min\{T_1, T_2\}]$

we have $V(t) \leq \max\{\mathcal{Z}, V(0)\} < \hat{V}$. We draw this conclusion from (34) and the fact that in Section III-C we show $V(0) < \bar{V}_{\mu_3^*}(0) \leq \hat{V}$. More specifically, we have

$$\|\dot{\tilde{\mathbf{q}}}(T_2)\|_2 \leq \sqrt{\frac{V(T_2)}{\chi}} < \sqrt{\frac{\hat{V}}{\chi}} < \sqrt{\frac{\hat{V}}{\rho_2(\mu_3^*)}} = \mathcal{Y} \quad (35)$$

where, in accordance with Lemma 1, $\chi = \lambda_{\min}(\frac{1}{2} \mathbf{M} - \frac{1}{2} \mu^{-2} \mathbf{M} \mathbf{X}^{-1} \mathbf{M})$. Using the properties of spectral norms and (3), we easily compute that $\chi \geq \frac{1}{2} k_{\bar{m}} - \frac{1}{2} \mu^{-2} k_{\bar{m}}^2 \lambda_{\max}(\mathbf{X})^{-1} > \rho_2(\mu_3^*)$. Using an argument similar to the one which leads to (35), but which we omit due to spatial limitations, one can also show that $\|\tilde{\mathbf{q}}(T_2)\|_2 < \mathcal{X}$. The existence of (35) and a similar expression for $\|\tilde{\mathbf{q}}(T_2)\|_2$ contradicts the definition of T_2 . Since T_2 is not maximal (i.e. T_2 can be taken to be infinity) then $T_2 > T_1$. Note that our argument studies the case where the trajectory of (14) spends time in both \mathcal{S} and \mathcal{T} . This covers the two other possible scenarios: the trajectory is such that in $[0, T_1)$ it remains solely in either \mathcal{S} or \mathcal{T} .

Part 4: Note that \dot{V}_B changes at $t = T_1$ to become negative definite. Recall that $\dot{V} \leq \dot{V}_A + \dot{V}_B$ in the region $\mathcal{D} := \mathcal{S} \cup \mathcal{T}$. The argument applied to the interval $[0, \min\{T_1, T_2\}]$ above, culminating in (35), can now be used to show that the trajectory of (14) remains in \mathcal{D} and $\dot{V} < 0$ for $t \in [T_1, \infty)$. We leave the details for an extended version of the paper. Recall that V is upper bounded as in (26), which implies that there holds $a_1 \|\tilde{\mathbf{q}}^\top, \dot{\tilde{\mathbf{q}}}^\top\|_2 \leq V \leq a_2 \|\tilde{\mathbf{q}}^\top, \dot{\tilde{\mathbf{q}}}^\top\|_2$ for some strictly positive and finite scalars a_1, a_2 . Because \mathcal{D} is compact, one can find a scalar $a_3 > 0$ such that $\dot{V} \leq -a_3 \|\tilde{\mathbf{q}}^\top, \dot{\tilde{\mathbf{q}}}^\top\|_2$. It follows that $\dot{V} \leq -\psi V$ in \mathcal{D} , where $\psi = a_3/a_2 > 0$. From this, it follows that V decays exponentially fast to zero, with a minimum rate $e^{-\psi t}$. Since V is positive definite in $\tilde{\mathbf{q}}$ and $\dot{\tilde{\mathbf{q}}}$, one can then conclude that $\|\tilde{\mathbf{q}}^\top, \dot{\tilde{\mathbf{q}}}^\top\|_2$ decays to zero exponentially fast and the leader tracking objective is achieved. \square

IV. SIMULATIONS

A simulation is provided to demonstrate the distributed algorithm (13). Each agent is a two-link robotic arm (including the term \mathbf{g}_i) and five follower agents must track the trajectory the leader agent. The equations of motion, and a picture of the manipulator, are given in [6], pp. 259-262. The generalized coordinates for agent i are $\mathbf{q}_i = [q_i^{(1)}, q_i^{(2)}]^\top$, which are the angles of each link in radians. The agent parameters and initial conditions are given in Table I of [18]. The graphs \mathcal{G}_A and \mathcal{G}_B are shown in Fig. 1. Each edge of \mathcal{G}_A has a weight of 50, while \mathcal{G}_B has edge weights of 1. The control gain pair is $\mu = 10$, $\beta = 10$. For the Laplacian \mathcal{L}_A , $\mathbf{\Gamma} = \mathbf{I}_n$ satisfies Lemma 4. For the observer, set $\alpha_1 = \alpha_2 = 5$. The leader trajectory is

$$\mathbf{q}_0(t) = \begin{bmatrix} 0.5 \sin(0.5t) - 0.8 \sin(0.1t) \\ 0.65 \sin(0.3t + \frac{\pi}{2}) \end{bmatrix}$$

Figure 2 shows the generalised coordinates $q^{(1)}$ and $q^{(2)}$. The generalised velocities, $\dot{q}^{(1)}$ and $\dot{q}^{(2)}$ are shown in Fig.

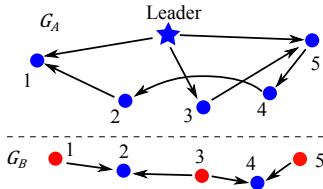


Fig. 1. Graph topologies used in the simulation. The red agents in \mathcal{G}_B do not need observers because they can directly sense the leader via \mathcal{G}_A .

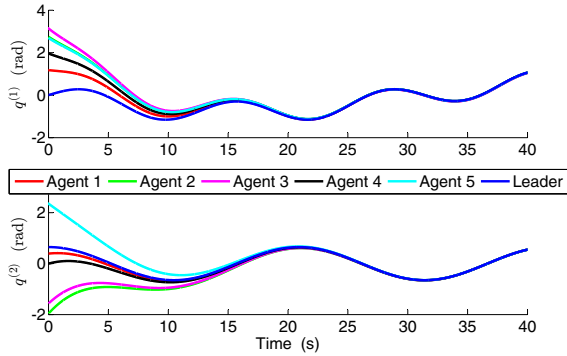


Fig. 2. Plot of generalised coordinates vs. time

3. The well studied observer results are omitted [21]. Note that the generalised coordinates synchronise at a slower rate than the generalised velocities, which is reflective of the μ term in the controller (13).

V. CONCLUSION

In this paper, we proposed a distributed model-independent algorithm which allows a directed network of agents with Euler-Lagrange self-dynamics to track the trajectory of a leader with nonconstant velocity. To facilitate the objective, we use an observer which allows each follower to obtain the states of the leader in finite-time. We then prove that the tracking objective is achieved exponentially fast and requires the directed network to contain a directed spanning tree with the leader as the root node. Furthermore, stability requires two control gain scalars to satisfy a set of lower bounding inequalities. The centralised design of the

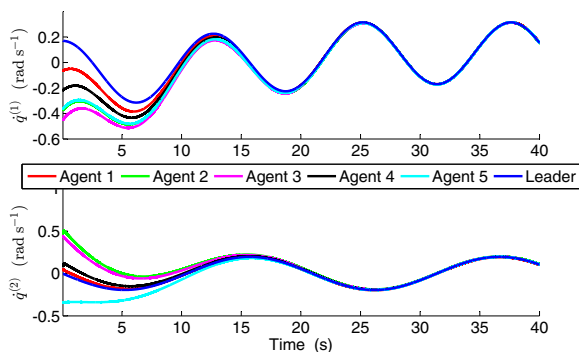


Fig. 3. Plot of generalised velocities vs. time

algorithm requires some limited knowledge on the bounds of the parameters describing the agent self-dynamics, a fixed topology and a fixed but arbitrarily large set of initial conditions. Numerical simulations are provided to show the effectiveness of the algorithm.

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