2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

Further Analysis on Graph Rigidity

Minh Hoang Trinh†, Myoung-Chul Park†, Zhiyong Sun‡, Brian. D. O. Anderson‡, Viet Hoang Pham†, and Hyo-Sung Ahn†

***Abstract*— This paper presents novel results on the symmetric rigidity matrix. Based on these results, new concepts in rigidity theory are proposed, including the worst-case, and imbalance rigidity indices of planar frameworks. These new indices are scale free and could be useful in rigid network synthesis problems.**

I. INTRODUCTION

Recently, graph rigidity has come to play an important role in formation control and network localization problems [1], [2]. In rigidity theory, a framework is an abstract mathematical model of a physical structure, and is defined by a graph and a realization or embedding in some space, typically here $\mathbb{R}^2$. Frameworks can describe a large class of physical structures, such as bridges, organic molecules, sensor network, formation of robotic systems, etc. The question of determining whether a framework is rigid or flexible has been studied extensively in the literature [3]–[5]. The answer involves both combinatorial and geometrical aspects, and is based on the rigidity matrix and for two dimensions at least, Laman's theorem.

While rigidity may be a requirement in many situations where frameworks are relevant, the simple presence or absence of rigidity will not usually be the sole criterion for judging the efficacy of the framework. For example, among sensor networks, which are normally required to be globally rigid, there are some networks with better capacity than others in terms of robustness and reliability to recover their vertices positions in the presence of range measurement noise. Thus, there is a need for some indices to quantitatively measure formation rigidity, and to compare these quantities among different networks. The authors of [6], [7] proposed two indices called the worst-case rigidity index (WRI) and the mean rigidity index (MRI) computable from the stiffness matrix. In [7], the authors also considered an optimal formation design problem in which links between agents could vary continuously to maximize these indices. In a two-dimensional space, the matrix $M = R^T R$,[1] termed the symmetric rigidity matrix [8], [9] or the vertex rigidity Gramian, always has three zero eigenvalues when the framework is infinitesimally rigid. The fourth smallest eigenvalue of M, denoted by $\lambda_4(M)$, is another rigidity index that has been widely used in the literature. Maintaining $\lambda_4 > 0$ is crucial to the formation to maintain its infinitesimal rigidity. For example, based on a weighted version of the symmetric rigidity matrix M, the authors of [9] proposed a decentralized rigidity maintenance control law for a multi-robot system. The authors of [10] considered a rigidity optimization problem for mobile robotic systems, in which the robots recursively update their positions to maximize the eigenvalue λ_4. The authors in [11] demonstrated the connection between a localization robustness problem and a metric of observability, λ_4/λ_n. However, many properties of this proposed metric have not been yet studied.

Although the rigidity matrix R has been widely used, there are fewer analytic results involving the matrices $R^T R$ and RR^T. This paper is devoted to studying the matrix $R^T R$. As the first contribution of this paper, we present a further analysis on the eigenvalues and eigenvectors of the vertex rigidity Gramian $M = R^T R$. These results give insights on the physical meaning of the nonzero eigenvalues of M, their dependence on the framework's size and the addition or removal of edges in the framework. The second contribution proceeds from the observation that although various rigidity indices have been proposed in the literature, for almost all indices, the scale of the formation affects the value of the index. For example, each equilateral triangle has a different rigidity index varying quadratically with the length of the triangle side. This fact, to some degree, is counter-intuitive. Thus, as the second and logically separate contribution, we propose two new rigidity indices derived from the matrix M, including the worst case Euclidean rigidity index and the imbalance index. Unlike most existing indices in the literature, these new indices are scale free and can be used to effectively compare rigidity between different frameworks. In formation control and network localization problems, the new rigidity indices could be particularly useful in designing rigid formations with consideration to robustness issue.

The rest of this paper is organized as follows. Section II includes the background for this paper. Section III contains further results on the symmetric rigidity matrix. New rigidity indices are defined in Section IV. Finally, some concluding remarks and suggestions for future works are given in Section V.

II. PRELIMINARIES

In this paper, $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space, $\mathbb{R}^{m \times n}$ is the set of m by n real matrices. We refer to $[A]_k$ and $[A]_{ij}$ as the k-th row of matrix A and the

†Distributed Control & Autonomous Systems Lab., Gwangju Institute of Science and Technology (GIST), Gwangju, Republic of Korea. E-mail: {trinhhoangminh,mcpark,vietph,hyosung}@gist.ac.kr

‡Data61-CSIRO (formerly NICTA) and Research School of Engineering, The Australian National University, Canberra ACT 2601, Australia. B. D. O. Anderson is also with School of Automation, Hangzhou Dianzi University, Hangzhou 310018, China. Email: {zhiyong.sun,brian.anderson}@anu.edu.au

[1]The definitions will be clear in Section II.

978-1-5090-1836-9/16/$31.00 ©2016 IEEE

the entry in the $i-$th row and $j-$th column of matrix A, correspondingly. We use $\text{diag}(A_i)_{i=1}^m = \text{diag}(A_1, \ldots, A_m)$ to define the block diagonal matrix formed from m matrices $A_1, \ldots, A_m$. The n by n identity matrix is denoted by I_n, and $\otimes$ denotes the Kronecker product.

This section briefly summarizes some definitions from algebraic graph theory and rigidity theory [2]. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with a vertex set $\mathcal{V} = \{1, \ldots, n\}$, and an edge set $\mathcal{E} \subset \{(i,j) : i,j \in \mathcal{V}, i \neq j\}$ with $|\mathcal{E}| = m$ edges. The neighbor set of a vertex $i \in \mathcal{V}$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$. Further, $\mathcal{K}_n$ will be used to denote the complete graph in n vertices.

For each vertex i in $\mathcal{V}$, we associate i with a point $p_i = [p_i^x, p_i^y]^T \in \mathbb{R}^2$. Then, the formation shape is determined by a realization $p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{2n}$. The graph $\mathcal{G}$ and a realization $p \in \mathbb{R}^{2n}$ together define a *framework* $\mathcal{F} = (\mathcal{G}, p)$ in the two dimensional plane. Two realizations p and p' of a graph $\mathcal{G}$ are *similar* if there exists a scale factor $\zeta > 0$ such that $\|p_i - p_j\| = \zeta \|p_i' - p_j'\|$, for all $i, j \in \mathcal{V}$. Specifically, when $\zeta = 1$, p and p' are called *congruent*.

Let $H \in \mathbb{R}^{m \times n}$ be the incidence matrix of $\mathcal{G}$ with an arbitrary edge ordering and orientations and let $\mathcal{L}(\mathcal{G}) = H^T H$ be the Laplacian matrix of $\mathcal{G}$. For any edge in $\mathcal{E}$ which is consistent with the construction of H, let $z_k = p_j - p_i$ and $d_k = \|z_k\|$, $k = 1, \ldots, m$. In this paper, z_k is sometimes used to refer to a corresponding edge in $\mathcal{E}$ without ambiguity. If two edges $z_i, z_j \in \mathcal{E}$ share a common vertex and have opposite (same) directions relative to their shared vertex, we call z_i, z_j positively (negatively) adjacent and denote $z_i \sim^+ z_j$ (resp., $z_i \sim^- z_j$). Note that an edge is not adjacent to itself.

Define $z = \left[z_1^T, \ldots, z_m^T\right]^T \in \mathbb{R}^{2m}$; then we have $z = (H \otimes I_2)p$. Also, let $f_\mathcal{G} : \mathbb{R}^{2n} \mapsto \mathbb{R}^m$,

$$f_\mathcal{G}(p) := (\|p_i - p_j\|^2)_{(i,j)\in\mathcal{E}} = [\|z_1\|^2, \ldots, \|z_m\|^2]^T,$$

be the edge function of the framework $\mathcal{F} = (\mathcal{G}, p)$; The rigidity matrix $R \in \mathbb{R}^{m \times 2n}$ is defined as the Jacobian matrix $R = \frac{1}{2}\partial f_\mathcal{G}(p)/\partial p$. A framework $\mathcal{F}$ is *infinitesimally rigid* if and only if $rank(R) = 2n - 3$ [2]. Note that R can be written as

$$R = D(z)^T (H \otimes I_2), \quad (1)$$

where $D(z) = \text{diag}(z_i)_{i=1}^m \in \mathbb{R}^{2m \times m}$. The *symmetric rigidity matrix* [9] is defined as follow

$$M := R^T R \in \mathbb{R}^{2n \times 2n}.$$

The matrix M is symmetric and positively semidefinite. Although the rigidity matrix R is widely used in *infinitesimal rigidity* theory, there is a less use of the symmetric rigidity matrix. The next section focuses on investigating some properties of the matrix M.

III. The Symmetric Rigidity Matrix

A. *Physical meaning of eigenvectors of the symmetric rigidity matrix*

Consider a framework $\mathcal{F} = (\mathcal{G}, p)$ in the plane with the rigidity matrix R. Let λ_k, $k = 1, \ldots, 2n$, denote the k-th eigenvalue of M and v^k, $k = 1, \ldots, 2n$, be its corresponding eigenvector. Further, let $v^k = [v_1^{kT}, \ldots, v_n^{kT}]^T$, where each v_i^k is a 2-vector. For convenience, we assume that $2n$ eigenvalues of M are ordered as follows $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2n}$. We have the following fact.

Theorem 1: [8] Consider a framework $\mathcal{F} = (\mathcal{G}, p)$ in the plane, with the rigidity matrix $R \in \mathbb{R}^{m \times 2n}$ and let $M = R^T R \in \mathbb{R}^{2n \times 2n}$. Then,
(i) $\text{rank}(M) \leq 2n - 3$, or M has at least three zero eigenvalues;
(ii) $\mathcal{N}(M) = \mathcal{N}(R)$;
(iii) The framework $\mathcal{F}$ is infinitesimally rigid if and only if $\text{rank}(M) = 2n - 3$. Equivalently, the fourth smallest eigenvalue λ_4 of M is positive.

Three eigenvectors associating with three zero eigenvalues of M span the null space of the rigidity matrix R and can be found as [10]:

$$\begin{aligned}
v^1 &= [1 \quad 0 \quad 1 \quad 0 \quad \ldots \quad 1 \quad 0]^T, \\
v^2 &= [0 \quad 1 \quad 0 \quad 1 \quad \ldots \quad 0 \quad 1]^T, \\
v^3 &= [-p_1^y \quad p_1^x \quad -p_2^y \quad p_2^x \quad \ldots \quad -p_n^y \quad p_n^x]^T.
\end{aligned}$$

Observe that v^1 and v^2 are orthogonal, while v^3 is not orthogonal to either v^1 or v^2. By using the Gram-Schmidt process, we construct from v^3 a vector orthogonal to both v^1 and v^2 as follows:

$$v^{3'} = v^3 + \bar{p}^y v^1 - \bar{p}^x v^2,$$

where $\bar{p}^x = \frac{1}{n}\sum_{i=1}^n p_i^x$ and $\bar{p}^y = \frac{1}{n}\sum_{i=1}^n p_i^y$ are correspondingly x-, and y-coordinates of the formation's centroid.

Intuitively, if we regard each vector v^1, v^2, $v^{3'}$ as a stacking of 2-vectors associated with n vertices and those 2-vectors are velocities at the nodes in question, the framework will translate along the $x-$, $y-$ axes with unit speed, or rotate around its centroid, respectively.

Next, we will give a physical interpretation of the eigenvectors of the matrix M based on the study of statics for frameworks. Consider the framework as an actual physical object whose edges are straight and comprise stiff rods connected by articulated joints at the vertices. A *stress* is a set of scalars $w = [w_{ij}]_{(i,j)\in\mathcal{E}}$ defined for each edge in the graph [5]. An *equilibrium stress* of a framework is a collection of scalars w_{ij}, one for each edge (i,j) of $\mathcal{G}$, such that

$$\sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = \mathbf{0}, \quad \forall i = 1, \ldots, n. \quad (2)$$

More specifically, $w_{ij}(p_i - p_j)$ can be understood as the force exerted by the rod on the vertex p_i along the rod (i,j). The scalar w_{ij} gives the magnitude of the force per unit length. The force is called a tension in the rod if $w_{ij} < 0$, and a compression if $w_{ij} > 0$.

A stress is called trivial when $w_{ij} = 0, \forall (i,j) \in \mathcal{E}$. Moreover, a framework is termed *stress free* if it admits only the trivial stress.

Remark 1: A framework is stress free if and only if $\text{rank}(R) = m$, the number of edges of the framework [4]. Let $\mathcal{F}$ be an infinitesimally rigid framework, so that we have $\text{rank}(R) = 2n - 3$. Thus, the necessary and sufficient condition for $\mathcal{F}$ to be stress free is $m = 2n - 3$, i.e. that $\mathcal{F}$ is minimally infinitesimally rigid.

For each eigenvector $v^k = [v_1^{kT}, \ldots, v_n^{kT}]^T \in \mathbb{R}^{2n}$ of the matrix M,

$$Mv^k = R^T R v^k = \lambda_k v^k.$$

Let $w = Rv^k \in \mathbb{R}^m$, it follows that

$$Mv^k = R^T(Rv^k) = R^T w = \lambda_k v^k, \quad k = 1, \ldots, 2n. \quad (3)$$

We can write $w = [w_{ij}]_{(i,j)\in\mathcal{E}} \in \mathbb{R}^m$, and (3) can be expressed as

$$\sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = \lambda_k v_i^k, \quad (4)$$

where $i = 1, \ldots, n$ and $k = 1, \ldots, 2n$. Now, we regard each vector v^k as a stacking of 2-vector forces associated with each of the n joints of the framework.

For each eigenvector v^k, $k = 1, 2, 3$, corresponding to a zero eigenvalue of M, we have $Mv^k = 0v^k = \mathbf{0}$, $k = 1, 2, 3$. It follows from $\mathcal{N}(M) = \mathcal{N}(R)$ that $w = Rv^k = \mathbf{0}$. Therefore, w immediately satisfies (2) and we have a trivial stress in this case.

Next, supposing that $\mathcal{F}$ is infinitesimally rigid, we focus on the nonzero eigenvalues λ_k, $k = 4, \ldots, 2n$, and their corresponding eigenvectors of M. A vector $F = [F_1^T, \ldots, F_n^T]^T \in \mathbb{R}^{2n}$ is an *equilibrium force* if

$$\sum\nolimits_{i=1}^{n} F_i = \mathbf{0}, \quad (5)$$

$$\sum\nolimits_{i=1}^{n} p_i \times F_i = \mathbf{0}, \quad (6)$$

where $\times$ denotes the cross product. The two conditions together imply that the sum of the forces F_i is zero and the moments about any axis is zero. We say that F is a *resolvable force* for $\mathcal{F} = \mathcal{G}(p)$ if there exist scalars w_{ij}, one for each edge (i, j) of $\mathcal{F}$, such that

$$F_i + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = \mathbf{0}, \quad \forall i = 1, \ldots, n. \quad (7)$$

There is a connection between the concepts of equilibirium force and resolvable force.

Lemma 1: [4, Proposition 4.3] Suppose $\mathcal{F} = (\mathcal{G}, p)$ is a framework in $\mathbb{R}^2$ where $p_1, \ldots, p_n$ are not collinear. Then $\mathcal{F}$ is infinitesimally rigid in $\mathbb{R}^2$ if and only if every equilibrium force for p is a resolvable force for $\mathcal{F}$. Moreover, each equilibrium force is uniquely resolvable if and only if $\mathcal{F}$ is minimally infinitesimally rigid.

Equation (4) can be rewritten as

$$-\lambda_k v_i^k + \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = \mathbf{0}, \quad \forall i = 1, \ldots, n. \quad (8)$$

Let $F_i = -\lambda_k v_i^k, i = 1, \ldots, n$ and $F = [F_1^T, \ldots, F_n^T]^T = -\lambda_k v^k$. Based on Lemma 1 and above analysis, we have proved the following result.

Theorem 2: Given an infinitesimally rigid framework $\mathcal{F}$ in a plane. Each vector $F = -\lambda_k v^k$ ($k = 4, \ldots, 2n-3$) is a resolvable force, where v^k is the eigenvector corresponding to a nonzero eigenvalue of the symmetric rigidity matrix M of $\mathcal{F}$.

Observe further that (8) can be rewritten as

$$-v_i^k + \lambda_k^{-1} \sum_{j\in\mathcal{N}_i} w_{ij}(p_i - p_j) = \mathbf{0}, \quad \forall i = 1, \ldots, n, \quad (9)$$

where w_{ij} depends on v^k, i.e. $w_{ij} = w_{ij}(v^k)$. Thus, if a force is applied to the framework along the eigenvector direction v^k, it yields a corresponding stress in the framework. The magnitude of the stress is proportional to λ_k^{-1}. Loosely speaking, λ_k characterizes the robustness of the framework along the eigenvector direction.

Further, consider an arbitrary external force vector $v \in \mathbb{R}^{2n}$. Since the nullspace and the row space of M are always orthogonal and they together span whole $\mathbb{R}^{2n}$, v can always be decomposed into two orthogonal vectors:

$$v = v_{\text{null}} + v_{\text{row}}, \quad (10)$$

where v_{null} is in $\mathcal{N}(M)$ and v_{row} is in the row space of M. Then, v_{null} is associated with trivial stresses and congruent motions of the framework, while v_{row} is an equilibrium force and can be decomposed further into $2n - 3$ eigenvector directions.

Example 1: Consider a $\mathcal{K}_3$ equilateral triangle framework (each side's length is 1). Since the graph is $\mathcal{K}_3$, it is minimally infinitesimally rigid. Thus, M has six linearly independent eigenvectors. Three eigenvectors corresponding to zero eigenvalues are in $\mathcal{N}(M)$. We focus on three eigenvectors $v^k = [v_1^{kT}, v_2^{kT}, v_3^{kT}]^T \in \mathbb{R}^6$ corresponding to $\lambda_4, \lambda_5, \lambda_6 > 0$. The force vectors $F_i = -\lambda_k v_i^k$ corresponding to p_i, $i = 1, 2, 3$, are depicted in Fig. 1.

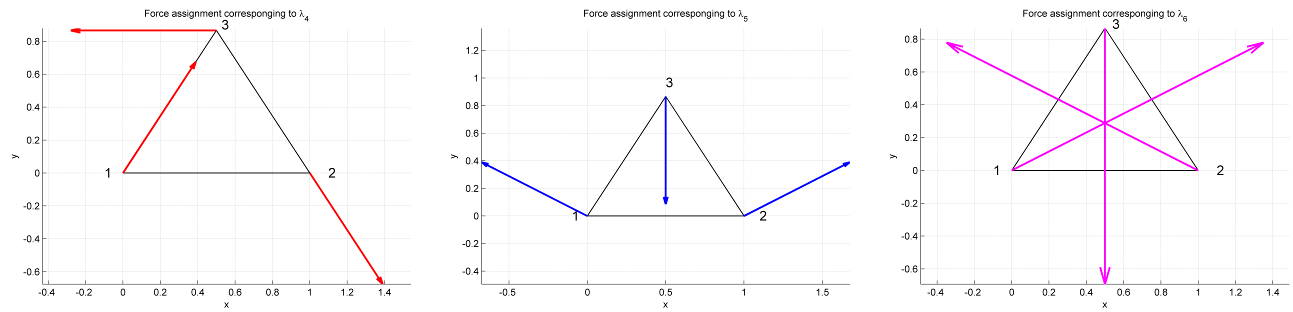
In equilateral triangle frameworks, we have $\lambda_4 = \lambda_5 = \frac{1}{2}\lambda_6$ (see Fig. 2). Suppose that each bar of the framework has a same stiffness limit independence of the length. Then, if we provide a force along the direction as depicted in Fig. 1c to break the framework, we need more effort than the directions of Fig. 1a and Fig. 1b. Consequently, the minimization of $\|\lambda_i - \lambda_j\|$, $4 \le i < j \le 2n$, implies that the sensitivity of the realized rigid graph along each eigenvector direction are distributed in a more balanced way.

B. Influence of the framework's scale, translation, orientation, and edge addition on the eigenvalues of the symmetric rigidity matrix

The following Lemma characterizes the influence of the eigenvalues of the matrix M.

Lemma 2: The eigenvalues of the symmetric rigidity matrix M associated with a framework $(\mathcal{G}, p)$ are invariant to translation, rotation and reflection of the framework.
The proof of Lemma 2 is trivial and will be omitted. We will investigate the influence of the scale and the edge addition in the remainder of the subsection.

Since a framework is defined based on a graph and a realization in the space, the eigenvalues of M depends on both the vertex positions and the associated graph. The following fact characterizes the influence of framework's scale on the eigenvalues of the symmetric rigidity matrix.



(a) λ_4 (b) λ_5 (c) λ_6

Fig. 1: The resolvable force assigned to vertices of an equilateral triangular framework corresponding to $\lambda_k, k = 4, 5, 6$.

Theorem 3: Consider two frameworks $\mathcal{F}_1 = (\mathcal{G}, p)$ and $\mathcal{F}_2 = (\mathcal{G}, q)$, which have the same graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Let R_1, R_2 be correspondingly rigidity matrices of $\mathcal{F}_1$ and $\mathcal{F}_2$. Denote the eigenvalues of $M_1 = R_1^T R_1$ and $M_2 = R_2^T R_2$ by $\lambda_1 \leq \ldots \leq \lambda_{2n}$ and $\lambda'_1 \leq \ldots \leq \lambda'_{2n}$. If $\mathcal{F}_1$ is similar to $\mathcal{F}_2$ by a scale factor $\zeta > 0$, we have

$$\lambda_i = \zeta^2 \lambda'_i, \quad \forall i = 1, \ldots, 2n. \tag{11}$$

Proof: Since $\mathcal{F}_2$ is similar to $\mathcal{F}_1$ by a scale factor $\zeta > 0$, there exists a matrix $\Omega \in \mathbb{R}^{2\times 2}$, $\Omega^T\Omega = I_2$, such that

$$z_k^T = \zeta \Omega z_k'^T, \quad k = 1, \ldots, m. \tag{12}$$

From (1) and (12), we can write

$$\begin{aligned} R_1 &= \text{diag}(z_i^T)_{i=1}^m (H \otimes I_2) \\ &= \zeta(\Omega \otimes I_m)\text{diag}(z_i'^T)_{i=1}^m (H \otimes I_2) \\ &= \zeta(\Omega \otimes I_m) R_2. \end{aligned} \tag{13}$$

Since $\Omega^T\Omega = I_2$, it follows that $M_1 = R_1^T R_1 = \zeta^2 R_2^T R_2 = \zeta^2 M_2$. The result is then immediate. ∎

Now, we analyze the influence of adding or removing bars on the framework. In combinatorial rigidity theory, we know that adding bars may convert a nonrigid framework to a rigid one. Also, removing a bar from a rigid framework may make it flexible. In order to quantify the effect of adding or removing bars to the symmetric rigidity matrix, we have the following Theorem.

Theorem 4: Consider an infinitesimally rigid framework in the plane $\mathcal{F} = (\mathcal{G}, p)$ with $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{E}| = m$. Consider the action of adding a new edge (i, j) to $\mathcal{E}$, where $(i, j) \notin \mathcal{E}$, $(i, j) \in \mathcal{V}\times\mathcal{V}$, and $i \neq j$. That action yields a new framework $\mathcal{F}' = (\mathcal{G}', p)$ with $\mathcal{G}' = (\mathcal{V}, \mathcal{E} \cup z_{m+1})$, having the same realization $p \in \mathbb{R}^{2n}$. The symmetric rigidity matrices of two frameworks M and M' correspondingly have eigenvalues $\{\lambda_i\}_{i=1,\ldots,2n}$ and $\{\lambda'_i\}_{i=1,\ldots,2n}$. Then, $\mathcal{F}'$ is more rigid than $\mathcal{F}$, in the sense that $\lambda_{i+1} \geq \lambda'_i \geq \lambda_i, \forall i = 1, \ldots, 2n$.

Proof: Since $\mathcal{F}$ is infinitesimally rigid, the framework $\mathcal{F}'$ obtained by adding an edge (i, j) to $\mathcal{E}$, is also infinitesimally rigid. Let H be the incidence matrix of $\mathcal{F}$, then the incidence matrix H' of $\mathcal{F}'$ can be written as

$$H' = \begin{bmatrix} H \\ h_{m+1}^T \end{bmatrix},$$

where $h_{m+1} \in \mathbb{R}^n$. The $i-$th and the $j-$th entries of h_{m+1} is 1 and -1 correspondingly, while other entries are all 0. Then, the matrix M' can be expressed as

$$M' = M + (h_{m+1} \otimes I_2) z_{m+1} z_{m+1}^T (h_{m+1}^T \otimes I_2). \tag{14}$$

Equation (14) implies that the addition of a new edge is equivalent to perturb the symmetric rigidity matrix M by a rank-1 symmetric positive semidefinite matrix $(h_{m+1} \otimes I_2) z_{m+1} z_{m+1}^T (h_{m+1}^T \otimes I_2)$. Since any symmetric matrix perturbed by a positive semidefinite matrix will result in a new matrix with non-descreasing eigenvalues and these eigenvalues are interlacing [12], the claim of Theorem 4 follows immediately. ∎

The intuitive physical meaning of Theorem 4 is that adding a new bar to a rigid framework makes the framework more rigid. Also, the removal of a bar makes a redundantly rigid framework become less rigid. The following result follows immediately from Theorem 4.

Corollary 1: Consider a framework $\mathcal{F} = (\mathcal{G}, p)$ and the set of frameworks $\mathcal{F}' = (\mathcal{G}', p)$ with the same vertex set as $\mathcal{V}$, then the framework with $\mathcal{G}' = \mathcal{K}_n$ is the most rigid in this set.

IV. THE NEW RIGIDITY INDICES

In the literature, there are several works on rigidity maintenance that considered the fourth smallest eigenvalue of M, λ_4, as an *index of rigidity* and tried to maximize λ_4. However, it appears from one point of view at least that λ_4 is not the best variable to measure framework's rigidity. This argument will be made clear from the following example.

Let us consider two triangle frameworks which are similar with the scale factor ζ. Theorem 2 suggests that by increasing ζ, λ_4 will be quadratically increased. If we consider λ_4 as the measure of rigidity, it follows that a bigger triangle framework is more rigid than a smaller one. Thus, λ_4 alone is not a good index to compare the degree of rigidity between different frameworks, since λ_4 depends on framework's size. Instead of λ_4, let us define a new measure of rigidity.

Definition 1: The worst-case rigidity index of a framework $\mathcal{F} = (\mathcal{G}, p)$, denoted by χ , is the ratio between the fourth smallest eigenvalue to the summation of all eigenvalues of the symmetric rigidity matrix M,

$$\chi = \frac{\lambda_4}{\sum_{i=1}^{2n} \lambda_i} = \frac{\lambda_4}{\text{tr}(M)}. \qquad (15)$$

It follows from Definition 1, for any framework, $\chi = 0$ means that the framework is not infinitesimally rigid. Moreover, χ is scale free as stated in the following Proposition.

Proposition 1: Consider two frameworks $\mathcal{F}_1 = (\mathcal{G}, p)$ and $\mathcal{F}_2 = (\mathcal{G}, q)$ satisfying condition (11) in Theorem 3. Let χ_1, χ_2 be worst-case rigidity indices of $\mathcal{F}_1$ and $\mathcal{F}_2$ accordingly. Then $\chi_1 = \chi_2$.

Proof: It follows from Theorem 3 that the eigenvalues of M and M' satisfy $\lambda_i = \zeta^2 \lambda'_i$, $\forall i = 1, \ldots, 2n$. Hence,

$$\chi_1 = \frac{\lambda_4}{\sum_{i=1}^{2n} \lambda_i} = \frac{\zeta^2 \lambda'_4}{\sum_{i=1}^{2n} \zeta^2 \lambda'_i} = \frac{\lambda'_4}{\sum_{i=1}^{2n} \lambda'_i} = \chi_2,$$

or i.e., all similar frameworks have the same degree of rigidity. ■

It is worth mentioning that $\sum_{i=1}^{2n} \lambda_i = \sum_{i=4}^{2n} \lambda_i = tr(M) = \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i} d_{ij}^2 = 2 \sum_{i=1}^{m} d_i^2$. As discussed in Section 3, each eigenvector of M represents a direction to decompose an applied force and the inverse of each nonzero eigenvalue of M represents the sensitivity of M along each eigenvector direction of the framework. Consequently, $\chi = \frac{\lambda_4}{\text{tr}(M)}$ characterizes how rigid $\mathcal{F}$ is along the most sensitive direction. Since χ is scale free, we can compare the worst-case rigidity between frameworks.

Motivated from the observation in Example 1, it is reasonable to search for a graph that has balanced sensitivities along all eigenvector directions of the framework. To this end, we now propose the concept of imbalance index.

Definition 2: The the imbalance index of a framework is defined as the ratio between the fourth eigenvalue and the largest eigenvalue of the symmetric rigidity matrix M,

$$\xi = \frac{\lambda_4}{\lambda_{2n}}. \qquad (16)$$

The imbalance index ξ is also scale-free. Furthermore, $\xi = 0$ when $\mathcal{F}$ is not rigid and $0 < \xi < 1$, otherwise.

Now, we try to find a bound for eigenvalues of M. For that, we recall two useful lemmas.

Lemma 3: [12, Theorem 1.3.22] The symmetric rigidity matrix $N = RR^T$ and the matrix $M = R^T R$ have the same nonzero eigenvalues with the same multiplicities.

Lemma 4: [13, Theorem 2] Consider a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Furthermore, let λ_{min} and λ_{max} be the smallest and largest eigenvalue of A, respectively. Then,

$$|\lambda_{max} - \lambda_{min}| \geq \max_{1 \leq i \neq j \leq n} \sqrt{([A]_{ii} - [A]_{jj})^2 + 4[A]_{ij}^2}.$$

Proposition 2: Consider an infinitesimally rigid framework with n vertices and $m \geq 2n-3$ edges. Then the largest eigenvalue of the matrix M, λ_{2n}, satisfies

$$\lambda_{2n} \geq 2 \max_{1 \leq i \neq j \leq m} \sqrt{(d_i^2 - d_j^2)^2 + (d_i d_j \cos\theta_{ij})^2}, \qquad (17)$$

$$\lambda_{2n} \leq d_{\max} \lambda_{\max}(\mathcal{L}(\mathcal{G})), \qquad (18)$$

where $\cos\theta_{ij} = \frac{z_i^T z_j}{\|z_i\| \|z_j\|} = \frac{z_i^T z_j}{d_i d_j}$, $d_{\max} = \max_{i=1,\ldots,m} d_i$, and $\lambda_{\max}(\mathcal{L}(\mathcal{G}))$ is the largest eigenvalue of $\mathcal{L}(\mathcal{G})$.

Proof: From Lemma 3, the nonzero eigenvalues of $M = R^T R$ are the same as those of the matrix

$$\begin{aligned} N = RR^T &= D(z)(H \otimes I_2)(H^T \otimes I_2)D(z)^T \\ &= \text{diag}(z_i)_{i=1}^m (HH^T \otimes I_2) \text{diag}(z_i^T)_{i=1}^m. \end{aligned}$$

The entries of N are given by

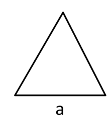
$$[N]_{ij} = \begin{cases} 2z_i^T z_i, & i = j, \\ z_i^T z_j, & z_i \sim^+ z_j, \\ -z_i^T z_j, & z_i \sim^- z_j, \\ 0, & \text{otherwise.} \end{cases}$$

Based on Lemma 4, since $z_i^T z_i = d_i^2, \|z_i^T z_j\| = d_i d_j \cos\theta_{ij}$, we have $\lambda_{max}(N) - \lambda_{min}(N) \geq 2 \max_{1 \leq i \neq j \leq m} \sqrt{(d_i^2 - d_j^2)^2 + (d_i d_j \cos\theta_{ij})^2}$.

Observe that when $m > 2n-3$, since N has rank $2n-3$, it has at least one zero eigenvalue 0. Thus, $\lambda_{min}(N) = 0$, $\lambda_{max}(N) = \lambda_{max}(M) = \lambda_{2n}(M)$, and it follows that

$$\lambda_{2n}(M) \geq 2 \max_{1 \leq i \neq j \leq m} \sqrt{(d_i^2 - d_j^2)^2 + (d_i d_j \cos\theta_{ij})^2}.$$

Finally, the inequality (18) is from [14, Corollary 1]. ■



$$\sum_{i=1}^{6} \lambda_i = 6a^2$$

k	λ_k	$\lambda_k / \sum_{i=1}^{6} \lambda_i$
4	$3a^2 / 2$	$1 / 4$
5	$3a^2 / 2$	$1 / 4$
6	$3a^2$	$2 / 4$

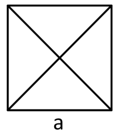
Fig. 2: Eigenvalues and rigidity indices of a $\mathcal{K}_3$ equilateral triangle framework side a.

To end this section, we will use Proposition 2 to roughly estimate χ in two simple examples. Consider an equilateral triangle framework with $\mathcal{K}_3$ underlying graph and each side's length is a as depicted in Fig. 2. Then, $m = 3 = 2 \times 3 - 3$. It follows from Proposition 2 that $\lambda_6(M) - \lambda_4(M) \geq a^2$. In this case, $tr(M) = 6a^2$. Thus, $6a^2 = \lambda_6(M) + \lambda_5(M) + \lambda_4(M) > a^2 + 3\lambda_4(M)$. It follows that $\lambda_4(M) < 5a^2/3$ and $\chi(M) < 5/18$. Calculation shows that $\chi(M) = 1/4$.

Next, consider a square framework (each side's length is a) with $\mathcal{K}_4$ graph as in Fig 3. Since $m = 6 > 2 \times 4 - 3 = 5$, applying Proposition 2 yields, $\lambda_8(M) \geq 2\sqrt{2}a^2$. Thus, $\chi = \lambda_4/tr(M) \leq (tr(M) - \lambda_8(M))/(4tr(M)) = (8 - \sqrt{2})/32$. Exact calculation shows that in this case $\chi(M) = 1/8$. We can compare χ with two other cases, where the graph is $\mathcal{K}_4$ removing one edge and have the same square realization, see Fig. 4–Fig. 5. Observe the eigenvalues are decreased in these cases, which is consistent with Theorem 4.

Given that the number of vertices in the graph is given, it is desired to find the framework with the largest worst-case rigidity indices. Then, the network design process could be done in a look-up manner. We state here a conjecture on determining the framework with the largest worst-case rigidity index $\chi_{\max}$.

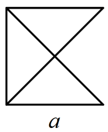
Conjecture 1: If the framework $\mathcal{F} = (\mathcal{K}_n, p)$ is infinitesimally rigid and its realization forms a convex polygon of n



$\sum_{i=1}^{8} \lambda_i = 16a^2$

k	λ_k	$\lambda_k / \sum_{i=1}^{8} \lambda_i$
4	$2a^2$	1 / 8
5	$2a^2$	1 / 8
6	$2a^2$	1 / 8
7	$4a^2$	2 / 8
8	$6a^2$	3 / 8

Fig. 3: Eigenvalues and rigidity indices of a $\mathcal{K}_4$ square framework side a.



$\sum_{i=1}^{8} \lambda_i = 14a^2$

k	λ_k	$\lambda_k / \sum_{i=1}^{8} \lambda_i$
4	$0.3542a^2$	0.0253
5	$2a^2$	0.1429
6	$2a^2$	0.1429
7	$4a^2$	0.2857
8	$5.6458a^2$	0.4033

Fig. 4: Eigenvalues and rigidity indices of a $\mathcal{K}_4$ removing one edge square framework side a.

vertices, χ and ξ are maximized when the realization is an n-regular polygon.

It is noted that Conjecture 1 rules out all cases when the framework's realization is not a convex n-polygon. An analytic solution for Conjecture 1 requires treatment for all possible realizations of $\mathcal{F}$ in the plane.

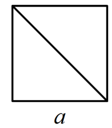
V. Conclusions and future work

In this paper, we developed some properties of the rigidity matrix $M = R^T R$ and proposed two new rigidity indices from M, namely the worst-case rigidity index χ and the imbalance index ξ. These indices are scale-free and can be used as a tool to compare rigidity between frameworks.

For future work, we are now working to unite the analysis on the eigenvalues of the matrix M with these new rigidity indices; in particular, we would like to understand how the new indices vary when an edge is added to a graph. All results in this paper are valid in two dimensions and could be extended into three dimensions. Also, the inequalities in Proposition 2 still need to be improved. A possible direction is looking at the gap between the two largest eigenvalues of the symmetric rigidity matrix. Conjecture 1 is difficult to prove completely, however, it may be experimentally verifiable by exhaustive testing by computers. Finally, applications of the new rigidity indices will be further studied, especially in optimal network design problems.

Acknowledgements

The work of B. D. O. Anderson and Z. Sun was supported by NICTA, which is funded by the Australian Government through the ICT Centre of Excellence program, and by the Australian Research Council under grant DP130103610 and DP160104500. Z. Sun is also supported by the Prime Minister's Australia Asia Incoming Endeavour Postgraduate Award from Australian Government.



$\sum_{i=1}^{8} \lambda_i = 12a^2$

k	λ_k	$\lambda_k / \sum_{i=1}^{8} \lambda_i$
4	$0.7639a^2$	0.0637
5	$2a^2$	0.1667
6	$2a^2$	0.1667
7	$2a^2$	0.1667
8	$5.2361a^2$	0.4363

Fig. 5: Eigenvalues and rigidity indices of a $\mathcal{K}_4$ removing one edge square framework side a.

The work of M. H. Trinh and H.-S. Ahn was supported by Ministry of Culture, Sports and Tourism (MCST) and Korea Creative Content Agency (KOCCA) in the Culture Technology (CT) Research & Development Program 2014.

References

[1] J. Aspnes, T. Eren, D. K. Goldenberg, A. S. Morse, W. Whiteley, Y. R. Yang, B. D. Anderson, and P. N. Belhumeur, "A theory of network localization," *IEEE Transactions on Mobile Computing*, vol. 5, no. 12, pp. 1663–1678, 2006.

[2] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Control Systems Magazine*, pp. 48–63, 2008.

[3] G. Laman, "On graphs and rigidity of plane skeletal structures," *Journal of Engineering Mathematics*, vol. 4, no. 4, pp. 331–340, 1970.

[4] B. Roth, "Rigid and flexible frameworks," *The American Mathematical Monthly*, vol. 88, no. 1, pp. 6–21, 1981.

[5] R. Connelly, "Generic global rigidity," *Discrete & Computational Geometry*, vol. 33, no. 4, p. 549–563, 2005.

[6] G. Zhu and J. Hu, "Stiffness matrix and quantitative measure of formation rigidity," in *Proc. of the Joint 48th IEEE CDC and 28th CCC, Shanghai, China*, 2009, pp. 3057–3062.

[7] ——, "Link resource allocation for maximizing the rigidity of multi-agent formations," in *Proc. of the 2011 50th IEEE Conference on Decision and Control and European Control Conference (CDC-ECC), Orlando, FL, USA*, 2011, pp. 2920–2925.

[8] D. Zelazo, A. Franchi, F. Allgower, H. H. Bulthoff, and P. R. Giordano, "Rigidity maintenance control for multi-robot systems," in *2012 Robotics: Science and Systems Conference, Sydney, Australia*, 2012.

[9] D. Zelazo, A. Franchi, H. H. Bullthoff, and P. R. Giordano, "Decentralized rigidity maintenance control with range measurements for multi-robot systems," *The International Journal of Robotics Research*, vol. 34, no. 1, pp. 105–128, 2015.

[10] Z. Sun, C. Yu, and B. D. O. Anderson, "Distributed optimization on proximity network rigidity via robotic movements," in *Proceedings of the 34th Chinese Control Conference, Hangzhou, China*, 2015, pp. 6954–6960.

[11] R. K. Williams and G. S. Sukhatme, "Observability in topology-constrained multi-robot target tracking," in *2015 IEEE International Conference on Robotics and Automation (ICRA), Seattle, Washington*, 2015, pp. 1795–1801.

[12] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1990.

[13] L. Mirsky, "Inequalities for normal and hermitian matrices," *Duke Math. J.*, vol. 24, no. 4, pp. 591–599, 1957.

[14] O. Rozenheck, S. Zhao, and D. Zelazo, "Distance-constrained formation tracking control," in *Israel 55th Annual Conference on Aerospace Sciences*, 2015.