# Quantization effects in rigid formation control

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***Abstract*— In this paper we discuss quantization effects in rigid formation control systems when target formations are described by inter-agent distances. Because of practical sensing and measurement constraints, we consider in the paper distance measurements in their quantized forms. We show that under gradient-based formation control, in the case of uniform quantization, the distance errors converge locally to a bounded set whose size depends on the quantization error, while in the case of logarithmic quantization, all distance errors converge locally to zero. A special quantizer involving the signum function is then considered with which all agents can only measure coarse distances in terms of binary information. In this case the formation converges locally to a target formation within finite time.**

## I. Introduction

Quantized control has been an active research topic in the recent decade, motivated by the fact that digital sensors and numerous industrial controllers can only generate quantized measurements or feedback signals [1], [2]. Recent years have also witnessed extensive discussions on quantized control for networked control systems. This is because data exchange and transmission over networks often occurs in a digitally quantized manner, thus giving rise to coarse and imperfect information; see e.g., [3]–[7].

In this paper we aim to discuss the quantization effect on rigid formation control. Formation control based on graph rigidity is a typical networked control problem involving inter-agent measurements and cooperation. There have been many papers in the literature focusing on control performance and convergence analysis for rigid formation control systems (see e.g. [8]–[10]), with virtually all assuming that all agents can acquire the relative position measurements to their neighbors perfectly. We remark that there are some recent works on linear-consensus-based formation control with quantized measurement. An exemplary paper along this line of research is [11], which showed that by using very coarse measurements (i.e., measurements in terms of binary information) the formation stabilization task can still be achieved. The case of coarse measurements can be seen as a special (or extreme) quantizer, which generates quantized feedbacks in the form of binary signals. However, in [11] and similar works on linear-consensus-based formation control, a common knowledge of the global coordinate frame orientation is required for all the agents to implement the control law. This is a strict assumption and is not always desirable in practical formation control systems. Actually, it has been shown in [12] that coordinate orientation mismatch may also cause undesired formation motions in linear-consensus-based formation systems. All these restrictions and disadvantages can be avoided in rigid formation control systems, in which any common knowledge of the global coordinate system is not required.

In the framework of quantized formation control, we also consider in the latter part of this paper a special quantizer described by the *signum* function. This part is motivated by the previous work [13] which discussed the triangular formation control with coarse distance measurements involving the signum function. In this paper we will consider a more general setting, which extends the discussions from the triangular case in [13] to more general rigid formations.

The aim of this paper is to explore whether the introduction of quantized measurement and feedback can still guarantee the success of formation control. The remaining parts of this paper are organized as follows. Section II briefly reviews some background on graph rigidity and two commonly-used quantizer functions. Section III discusses the convergence of the formation systems under two quantized formation controllers. In Section IV we show a special quantized formation controller with binary distance information. Section V concludes this paper.

## II. Background and preliminaries

### A. Graph rigidity and notations

Let us introduce some notations employed throughout this paper. For a given matrix $A \in \mathbb{R}^{n\times m}$, define $\overline{A} := A \otimes I_d \in \mathbb{R}^{nd\times md}$, where the symbol $\otimes$ denotes the Kronecker product and $I_d$ is the $d$-dimensional identity matrix with $d = \{2, 3\}$. We denote by $||x||$ the Euclidean norm of a vector $x$, by $\hat{x} := \frac{x}{||x||}$ the unit vector of $x \neq 0$, and by $\tilde{x} := \frac{1}{||x||}$ the reciprocal of the norm of $x \neq 0$. For a stacked vector $x := \left[x_1^T, x_2^T, \dots, x_k^T\right]^T$ with $x_i \in \mathbb{R}^l, i \in \{1, \dots, k\}$, we define the block diagonal matrix $D_x := \text{diag}\{x_i\}_{i\in\{1,\dots,k\}} \in \mathbb{R}^{kl\times k}$.

Consider an undirected graph with $m$ edges and $n$ vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node $i$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. We define an orientated incidence matrix $B \in \mathbb{R}^{n\times m}$ for the undirected

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graph $\mathcal{G}$ by assigning an *arbitrary* orientation for each edge. Note that for a rigid formation modelled by an *undirected* graph considered in this paper, the orientation of each edge for writing the incidence matrix can be defined arbitrarily and the stability analysis in the next sections remains unchanged. By doing this, we define the entries of $B$ as $b_{ik} = +1$, if $i = \mathcal{E}_k^{\text{tail}}$, or $b_{ik} = -1$, if $i = \mathcal{E}_k^{\text{head}}$, or $b_{ik} = 0$ otherwise, where $\mathcal{E}_k^{\text{tail}}$ and $\mathcal{E}_k^{\text{head}}$ denote the tail and head nodes, respectively, of the edge $\mathcal{E}_k$, i.e. $\mathcal{E}_k = (\mathcal{E}_k^{\text{tail}}, \mathcal{E}_k^{\text{head}})$. For a connected undirected graph, one has $\text{null}(B^T) = \text{span}\{\mathbf{1}_n\}$.

We denote by $p = [p_1^T, p_2^T, \cdots, p_n^T]^T \in \mathbb{R}^{dn}$ the stacked vector of all the agents' positions $p_i \in \mathbb{R}^d$. The pair $(\mathcal{G}, p)$ is said to be a framework of $\mathcal{G}$ in $\mathbb{R}^d$. The incidence matrix $B$ defines the sensing topology of the formation, i.e. it encodes the set of available relative positions that can be measured by the agents. One can construct the stacked vector of available relative positions by

$$z = \overline{B}^T p, \tag{1}$$

where each element $z_k \in \mathbb{R}^d$ in $z$ is the relative position vector for the vertex pair defined by the edge $\mathcal{E}_k$.

Let us now briefly recall the notions of infinitesimally rigid framework and minimally rigid framework from [14]. Define the edge function $f_{\mathcal{G}}(p) := \underset{k}{\text{col}}\left(\|z_k\|^2\right)$ where the operator col defines the stacked column vector. We denote the Jacobian of $\frac{1}{2} f_{\mathcal{G}}(p)$ by $R(z)$, which is called the *rigidity matrix*. An easy calculation shows that $R(z) = D_z^T \overline{B}^T$. A framework $(\mathcal{G}, p)$ is *infinitesimally rigid* if $\text{rank}(R(z)) = 2n-3$ when it is embedded in $\mathbb{R}^2$ or if $\text{rank}(R(z)) = 3n-6$ when it is embedded in $\mathbb{R}^3$. Additionally, if $|\mathcal{E}| = 2n-3$ in the 2D case or $|\mathcal{E}| = 3n-6$ in the 3D case then the framework is called *minimally rigid*. In this paper we assume that the target formation is infinitesimally and minimally rigid, while the convergence results obtained in this paper can be extended to non-minimally rigid target formations by following the analysis in [15] or [10].

### B. Quantizer functions

In this paper we mainly consider two types of quantizers: the uniform quantizer and the logarithmic quantizer [3], [5]–[7].

*1) Definition of the quantizers:* The *symmetric uniform* quantizer is a map $q_u : \mathbb{R} \to \mathbb{R}$ such that

$$q_u(x) = \delta_u \left( \left\lfloor \frac{x}{\delta_u} \right\rceil \right), \tag{2}$$

where $\delta_u$ is a positive number and $\lfloor a \rceil, a \in \mathbb{R}$ denotes the nearest integer to $a$. We also define $\lfloor \frac{1}{2} + h \rceil = h$ for any $h \in \mathbb{Z}$.

The *logarithmic* quantizer is an odd map $q_l : \mathbb{R} \to \mathbb{R}$ such that

$$q_l(x) = \begin{cases} \exp(q_u(\ln x)) & \text{when } x > 0; \\ 0 & \text{when } x = 0; \\ -\exp(q_u(\ln(-x))) & \text{when } x < 0. \end{cases}$$

where $\exp(\cdot)$ denotes the exponential function.

*2) Properties of the quantizers:* For the uniform quantizer, the quantization error is always bounded by $\delta_u/2$, namely $|q_u(x) - x| \le \frac{\delta_u}{2}$ for all $x \in \mathbb{R}$.

For the logarithmic quantizer, it holds that $q_l(x)x \ge 0$, for all $x \in \mathbb{R}$, and the equality holds if and only if $x = 0$. The quantization error for the logarithmic quantizer is bounded by $|q_l(x) - x| \le \delta_l |x|$, where the parameter $\delta_l$ is determined by $\delta_l = \exp(\frac{\delta_u}{2}) - 1$ (note that $\delta_l > 0$ because $\delta_u > 0$).

The above definitions for scalar-valued uniform and logarithmic quantizers can be generalized to vector-valued quantizers for a vector in a component-wise manner.

### C. Nonsmooth analysis

Consider a differential equation

$$\dot{x}(t) = X(x(t)), \tag{3}$$

where $X : \mathbb{R}^d \to \mathbb{R}^d$ is a vector field which is measurable but discontinuous. The existence of a continuously differentiable solution to (3) cannot be guaranteed due to the discontinuity of $X(x(t))$. Also, as shown in [7], the Caratheodory solutions may not exist from a set of initial conditions of measure zero in quantized control systems. In this paper, we understand the solutions to the quantized rigid formation system in the sense of Filippov [16]. We first introduce the Filippov set-valued map.

**Definition 1.** *Let $\mathcal{D}(\mathbb{R}^d)$ denote the collection of all subsets of $\mathbb{R}^d$. The Filippov set-valued map $F[X] : \mathbb{R}^d \to \mathcal{D}(\mathbb{R}^d)$ is defined by*

$$\mathcal{F}[X](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S})=0} \overline{\text{co}}\{X(\mathbb{B}(x, \delta) \backslash \mathcal{S}\}, \quad x \in \mathbb{R}^d \tag{4}$$

*where $\overline{\text{co}}$ denotes convex closure, $\mathcal{S}$ is the set of $x$ at which $X(x)$ is discontinuous, $\mathbb{B}(x, \delta)$ is the open ball of radius $\delta$ centered at $x$, and and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero.*

Because of the way the Filippov set-valued map is defined, the value of $\mathcal{F}[X]$ at a discontinuous point $x$ is independent of the value of the vector field $X$ at $x$. Filippov solutions are absolutely continuous curves that satisfy almost everywhere the differential inclusion $\dot{x}(t) \in \mathcal{F}[X](x)$ defined above. Some properties of the Filippov solution can be found in the review [17].

## III. FORMATION CONTROL WITH QUANTIZED MEASUREMENT

### A. Quantized formation controllers

In rigid formation control each agent is required to measure the relative position (i.e. bearing and range) to its neighbors via a bearing sensor and a range sensor. If one assumes perfect measurements, a typical formation controller can be written as (see e.g. [8], [18])

$$\dot{p}_i = -\sum_{k=1}^{m} b_{ik}(||z_k|| - d_k)\, \hat{z}_k, \tag{5}$$

while more general forms of formation controllers to stabilize rigid formations are discussed in [10].

In the presence of quantized sensing and measurement, the right-hand side of the above formation control system (5) needs modification. In this paper we assume that the distance measurement (with an offset, see the following equation (6)) is quantized, and the bearing measurement is unquantized. This assumption is reasonable because the bearing measurement is always bounded (described by a unit vector or by an angle in $[-\pi, \pi)$ in the 2D case). A normal digital sensor, say a 10-bit uniform quantizer, applying to bearing measurements gives rather accurate measurement with very small error to the true bearing. This is not the case for distance measurements which may have larger magnitudes. In this paper we use quantized distance measurement in the formation controller design, while in the future work this may be relaxed by considering both quantized range and bearing measurements. A quantized formation controller can be written as

$$\dot{p}_i = -\sum_{k=1}^{m} b_{ik}\, q(||z_k|| - d_k)\, \hat{z}_k, \tag{6}$$

where $q$ is a quantization function, which can be the uniform quantizer or the logarithmic quantizer. We also assume that all the agents use the same quantizer $q(\cdot)$, and their initial positions start with non-collocated positions (i.e. $z_k(0) \neq 0$ for all $k$).

**Remark 1.** *One may wonder why there is not use of the quantization feedback in the form of $q(||z_k||)$, i.e. the direct quantized distance measurement, in the control (6). We note three reasons for choosing $q(||z_k|| - d_k)$ instead of $q(||z_k||)$:*

- *In rigid formation control, the control objective is to stabilize the actual distances between neighboring agents to prescribed values. If one chooses the quantization strategy in the form of $q(||z_k||)$, then this control objective may not be achieved. To this end, the quantization strategy $q(||z_k|| - d_k)$ used in (6) can be interpreted as a digital distance sensor with an embedded or prescribed offset (where the offset is the desired distance $d_k$).*
- *In the case of non-uniform quantizers (e.g., logarithmic quantizer), the quantization accuracy usually increases when the quantizer input approaches the desired state (which is the origin in this case). Thus, when the formation approaches closer to the target formation, a higher quantization accuracy (if possible) is required, and this cannot be achieved if one uses the quantization function (e.g., logarithmic quantization) on the actual distance in the form of $q(||z_k||)$.*
- *We will further show in Section IV that the chosen quantization strategy $q(||z_k|| - d_k)$ will specialize to a simple and effective quantizer with* coarse binary *distance measurement.*

In the presence of quantized measurement and feedback, the right-hand side of (6) is discontinuous and we will consider the following differential inclusion

$$\dot{p}_i \in \mathcal{F}\left[\sum_{k=1}^{m} b_{ik}\, q(||z_k|| - d_k)\, \hat{z}_k\right]. \tag{7}$$

In the following, we calculate the differential inclusion $\mathcal{F}(q(e_k))$ which will be used in later analysis. In the case of a symmetric uniform quantizer, one has $\mathcal{F}(q_u(e_k)) = h\delta_u$, if $e_k \in \left((h-\frac{1}{2})\delta_u, (h+\frac{1}{2})\delta_u\right), h \in \mathbb{Z}$, or $\mathcal{F}(q_u(e_k)) = [h\delta_u, (h+1)\delta_u]$, if $e_k = (h+\frac{1}{2})\delta_u, h \in \mathbb{Z}$. Note that $e_k\mathcal{F}(q_u(e_k)) \geq 0$ for all $e_k$, and $e_k\mathcal{F}(q_l(e_k)) = 0$ if and only if $e_k \in [-\frac{\delta_u}{2}, \frac{\delta_u}{2}]$.

In the case of a logarithmic quantizer, the differential inclusion $\mathcal{F}(q_u(e_k))$ can be calculated as $\mathcal{F}(q_l(e_k)) = \text{sign}(e_k)\text{exp}(q_u(\ln|e_k|))$, if $e_k \neq \text{exp}((h+\frac{1}{2})\delta_u), h \in \mathbb{Z}$, or $\mathcal{F}(q_l(e_k)) = [\text{exp}(h\delta_u), \text{exp}((h+1)\delta_u)]$, if $e_k = \text{exp}((h+\frac{1}{2})\delta_u), h \in \mathbb{Z}$. Also note that $e_k\mathcal{F}(q_l(e_k)) > 0$ for all $e_k \neq 0$, and $e_k\mathcal{F}(q_u(e_k)) = 0$ if and only if $e_k = 0$.

We define the distance error vector as $e = [e_1, e_2, \ldots, e_m]^T$. Then in a compact form, one can rewrite the dynamics of (7) as

$$\dot{p} \in \mathcal{F}\Big[-\overline{B}D_{\hat{z}}\, q\Big(e(\operatorname*{col}_k(\|z_k\|))\Big)\Big], \tag{8}$$

In order not to overload the notation, here by $\hat{z}$ we exclusively mean the vector-wise normalization of $z$, therefore $D_{\hat{z}}$ in the above equation and in the sequel is defined as $D_{\hat{z}} = \text{diag}\{\hat{z}_1, ..., \hat{z}_m\}$. This notation rule also applied to $\tilde{z}$ in the sequel. Note that the differential inclusion $\mathcal{F}(q(e))$ with the vector $e$ is defined according to the product rule of Filippov's calculus properties (see [19]).

## B. Properties of quantized formation control systems

We first discuss the solution issue of the formation control system (7). However, it is more convenient to focus on the dynamics of the relative position vector $z$, which can be derived from (7) as follows

$$\dot{z} = \overline{B}^T\dot{p} \in \mathcal{F}\Big[-\overline{B}^T\overline{B}D_{\hat{z}}\, q\Big(e(\operatorname*{col}_k(\|z_k\|))\Big)\Big]. \tag{9}$$

First note that at any non-collocated finite initial point $p(0)$, the right-hand side of (7) and of (9) is measurable and locally essentially bounded. Thus, the existence of a local Filippov solution of (7) and of (9) starting at such initial points is guaranteed.

We then derive a dynamical system from (8) to describe the evolution of the distance error vector $e$. According to the definition of the distance error $e_k$, $e_k$ is a smooth function of $z_k$. Thus, by using the calculus property (see [17], [19]), one can show $\dot{e}_k$ exists and $\dot{e}_k = \frac{1}{\|z_k\|}z_k^T\dot{z}_k$ holds almost everywhere. The dynamics for the distance error vector $e$ can be obtained in a compact form as

$$\begin{aligned}\dot{e} &= -D_{\tilde{z}}R(z)\dot{p}, \quad \text{a. e.}\\ &\in -\mathcal{F}\left[D_{\tilde{z}}R(z)R^T(z)D_{\hat{z}}q(e)\right], \quad \text{a. e.}\end{aligned} \tag{10}$$

Again, the existence of a local Filippov solution of (10) starting with a non-collocated finite initial point $p(0)$ is guaranteed. In the next section, we will also show that the solutions to (10) (as well as the solutions to (7) and (9)) are *bounded* and and can be extended to $t \to \infty$ when agents' initial positions are chosen non-collocated and close to a target formation shape. Also, as shown in [15], when the formation shape is close to the desired one, the entries of the matrix $R(z)R^T(z)$ are continuously differentiable functions of $e$. Since the nonzero entries of the diagonal matrix $D_{\tilde{z}}$ are of the form $\frac{1}{\|z_k\|}$ which are also continuously differentiable functions of $e$, we conclude that the system described in (10) is a self-contained system, and we will call it *the distance error* system in the sequel.

Finally, we show some additional properties of the formation control system with quantized information. The proofs are omitted due to space limit.

**Lemma 1.** *In the presence of the uniform/logarithmic quantizer, the formation centroid remains stationary.*

**Lemma 2.** *To implement the control, each agent can use its own coordinate system to measure the relative position (quantized distance and unquantized bearing) and a global coordinate system is not involved.*

### C. Convergence analysis

In this section we aim to prove the following convergence result.

**Theorem 1.** *Suppose the target formation is infinitesimally and minimally rigid and the initial formation shape is close to the target formation shape.*

- *In the case of a uniform quantizer, the formation converges locally to an approximate and static shape defined by the set* $F_{approx} = \{e | e_k \in [-\frac{\delta_u}{2}, \frac{\delta_u}{2}], k \in \{1, \dots, m\}\}$*;*
- *In the case of a logarithmic quantizer, the formation converges locally to a static target formation shape.*

In the proof we will use the Lyapunov theory of nonsmooth analysis, for which we construct a Lyapunov function candidate as

$$V(e) = \sum_{k=1}^{m} V_k(e_k), \text{ with } V_k(e_k) = \int_0^{e_k} q(s)\mathrm{d}s. \tag{11}$$

Before giving the proof of Theorem 1, we first show some properties of the function $V$ defined in (11). For the definition of function regularity in nonsmooth analysis, see [20].

**Lemma 3.** *The function $V$ constructed in (11) is positive semidefinite, and is regular everywhere.*

The proof is omitted here due to space limit and will be provided in the full version of this paper. We remark that the proof of regularity follows similarly to the proof of [6, Lemma 6]. An observation that supports the regularity statement of $V$ is that $V$ is continuously differentiable almost everywhere, while at the nondifferentiable points $V$ has corners of convex type (see discussions in [20, Page 200]). Furthermore, according to the definition of generalized derivative [20], one can calculate the generalized derivative of $V_k$ (for the case of a uniform quantizer) as

$$\partial V_k = \begin{cases} [h\delta_u, (h+1)\delta_u], & e_k = (h+\frac{1}{2})\delta_u, h \in \mathbb{Z} \\ q(e_k), & \text{elsewise} \end{cases}$$

Similarly, one can also calculate the generalized derivative of $V_k(e_k)$ for the case of a logarithmic quantizer (which is omitted here). The generalized derivative of $V(e)$ can be obtained by the product rule (see [17]). Now we are ready to prove Theorem 1.

*Proof.* We choose the Lyapunov function constructed in (11) for the distance error system (10) with discontinuous right-side hand. Note that $R(z)R^T(z)$ and $D_{\tilde{z}}$ are positive definite matrices at the desired formation shape. Similarly to the analysis in [10] (or in [18]), we define a sub-level set $\mathcal{B}(\rho) = \{e : V(e) \leq \rho\}$ for some suitably small $\rho$, such that when $e \in \mathcal{B}(\rho)$ the formation is infinitesimally minimally rigid, and $R(z)R^T(z)$ and $D_{\tilde{z}}$ are positive definite. Note that the defined sub-level set $\mathcal{B}(\rho)$ is compact. Note also that the matrix $Q(e) := D_{\tilde{z}}R(z)R^T(z)D_{\tilde{z}}$ is also positive definite when $e \in \mathcal{B}(\rho)$, and we rewrite the distance error system as $\dot{e} \in \mathcal{F}[-Q(e)q(e)]$.

The regularity of $V$ shown in Lemma 3 allows us to employ the nonsmooth Lyapunov theorem [17], [21] to develop the stability analysis. We calculate the set-valued derivative of $V$ along the trajectory of the distance error system (10). According to the definition of set-valued derivative in nonsmooth analysis [17], [21], one can obtain

$$\begin{aligned} \dot{V}(e)_{(10)} \in \tilde{\mathcal{L}}_{(10)}V(e) = \{a \in \mathbb{R} | \exists v \in \dot{e}_{(10)}, \\ \text{such that } \zeta^T v = a, \forall \zeta \in \partial V(e)\}. \end{aligned} \tag{12}$$

Note that the set $\tilde{\mathcal{L}}_{(10)}V(e)$ could be empty, and in this case we adopt the convention that $\max(\emptyset) = -\infty$. When it is not empty, there exists $v \in -Q(e)q(e)$ such that $\zeta^T v = a$ for all $\zeta \in \partial V(e)$. A natural choice of $v$ is to set $v \in -Q(e)\zeta$, with which one can obtain $a = -q^T(e)Q(e)q(e)$. Let $\bar{\lambda}_{\text{min}}$ denote the smallest eigenvalue of $Q(e)$ when $e(p)$ is in the compact set $\mathcal{B}$ (i.e. $\bar{\lambda}_{\text{min}} = \min\limits_{e \in \mathcal{B}} \lambda(Q(e)) > 0$). Then if the set $\tilde{\mathcal{L}}_{(10)}V(e)$ is not empty, one can show

$$\max(\tilde{\mathcal{L}}_{(10)}V(e)) \leq -\bar{\lambda}_{\text{min}} q(e)^T q(e) \tag{13}$$

and if the set $\tilde{\mathcal{L}}_{(10)}V(e)$ is empty, one has $\max(\tilde{\mathcal{L}}_{(10)}V(e)) = -\infty$. Note that both cases imply that $V$ is non-increasing, and consequently the Filippov solution $e(t)$ of (10) is bounded. Thus, all solutions to (10) (as well as the solutions to (7)) are bounded and can be extended to $t = \infty$ (i.e. there is no finite escape time).

We now divide the rest of the proof into two parts, according to different quantizers:

- The case of uniform quantizer: it can be seen that $\max(\tilde{\mathcal{L}}_{(10)}V(e)) \leq 0$ for all $e \in \mathcal{B}(\rho)$ and $0 \in \max(\tilde{\mathcal{L}}_{(10)}V(e))$ if and only if $e \in F_{\text{approx}}$. Also note that $F_{\text{approx}}$ is compact, and is positively invariant for the distance error system (10) (i.e. if the initial formation is such that $e(0) \in F_{\text{approx}}$, then all agents are static and $e(t) \in F_{\text{approx}}$ for all $t$). According to the nonsmooth invariance principle [17], [21], the first part of the convergence result is proved. Since this is a convergence to a bounded set, the convergence is achieved within finite time. Note from (6) the final formation is stationary.
- The case of logarithmic quantizer: it can be seen that $\max(\tilde{\mathcal{L}}_{(10)}V(e)) \leq 0$ for all $e \in \mathcal{B}(\rho)$ and $0 \in$

$\max(\tilde{\mathcal{L}}_{(10)}V(e))$ if and only if $e = 0$. According to the nonsmooth invariance principle [17], [21], the second part of the convergence result is proved. Also note from (6) the final formation is stationary.

□

## IV. A SPECIAL QUANTIZER: FORMATION CONTROL WITH BINARY DISTANCE INFORMATION

### A. Rigid formation control with coarse measurement

In this section we consider the special case in which each agent uses very coarse distance measurement, in the sense that it only needs to detect whether the current distance to each of its neighbors is greater or smaller than the desired distance. This gives rise to a special quantizer defined by the following *signum* function: $\text{sign}(x) = 1$ when $x > 0$, or $\text{sign}(x) = -1$ when $x < 0$, or $\text{sign}(x) = 0$ when $x = 0$. Accordingly, we obtain the following rigid formation control system with binary distance measurement:

$$\dot{p}_i = -\sum_{k=1}^{m} b_{ik}\text{sign}(||z_k|| - d_k)\, \hat{z}_k \tag{14}$$

**Remark 2.** *Formation control using the signum function has been discussed in several previous papers. In [22], a finite time convergence was established for stabilization of cyclic formations using binary bearing-only measurements. The paper [13] studied the stabilization control of a cyclic triangular formation with the controller* (14)*. Here we extend such controllers to stabilize general undirected formation which are minimally and infinitesimally rigid. The above controller* (14) *can also be seen as a high dimensional extension of the one-dimensional formation controller studied in [23]. Also, note that the right-hand side of* (14) *is composed of the sum of a unit vector multiplied by a signum function. This implies that the formation controller* (14) *is of special interest in practice since the control action is explicitly upper bounded by the cardinality of the set of neighbors for each agent, which prevents potential implementation problems due to saturation.*

Again, we consider the Filippov solution to the formation control system (14). The differential inclusion $\mathcal{F}(\text{sign}(e_k))$ can be calculated as $\mathcal{F}(\text{sign}(e_k)) = 1$ if $\|z_k\| > d_k$, or $\mathcal{F}(\text{sign}(e_k)) = [-1, 1]$ if $\|z_k\| = d_k$, or $\mathcal{F}(\text{sign}(e_k)) = -1$ if $\|z_k\| < d_k$. In a compact form, the rigid formation system (14) can be rewritten as

$$\dot{p} \in \mathcal{F}[-R^T(z)D_{\tilde{z}}\text{sign}(e)], \tag{15}$$

where $\text{sign}(e)$ is defined in a component-wise way.

Note that the right-hand side of (15) is measurable and essentially bounded at any non-collocated and finite point $p$, and the existence of a local Filippov solution to (15) is guaranteed from such an initial point $p(0)$. In the following analysis we will also show that the solutions are bounded and complete. Similar to the analysis in deriving the distance error system shown in Section III-B, the distance error system with binary distance information can be obtained as

$$\dot{e} \in \mathcal{F}[-D_{\tilde{z}}R(z)R^T(z)D_{\tilde{z}}\text{sign}(e)], \quad \text{a. e.} \tag{16}$$

Again, similar to the analysis for (10), one can also show that (16) is a self-contained system when $e$ takes values locally around the origin.

### B. Convergence analysis

The main result in this section is stated in the following convergence theorem for the formation controller (15) with binary distance information.

**Theorem 2.** *Suppose the target formation is infinitesimally and minimally rigid, the initial formation shape is close to the target formation shape, and the formation controller* (14) *with binary distance information is applied.*

- *The formation converges locally to a static target formation shape;*
- *The convergence is achieved within a finite time upper bounded by $T^* = \frac{\|e(0)\|_1}{\bar{\lambda}_{min}}$ where $\bar{\lambda}_{min}$ is defined in the proof.*

*Proof.* Part of the proof for this theorem is similar to the proof of Theorem 1. Choose the Lyapunov function defined as $V = \sum_{k=1}^{m} V_k(e_k)$ with $V_k(e_k) = |e_k|$ for the distance error system (16). Note that $V$ is a convex and regular function of $e$. Also $V$ is locally Lipschitz at $e = 0$ and is continuously differentiable at all other points. The generalized derivative of $V_k(e_k)$ can be calculated as

$$\partial V_k = \begin{cases} 1, & e_k > 0; \\ [-1, 1], & e_k = 0; \\ -1, & e_k < 0. \end{cases}$$

and the generalized derivative of $V$ can be calculated similarly via the product rule (see [17]). We define a sub-level set $\mathcal{B}(\rho) = \{e : V(e) \leq \rho\}$ for some suitably small $\rho$, such that when $e \in \mathcal{B}(\rho)$ the formation is infinitesimally minimally rigid and $R(z)R^T(z)$ and $D_{\tilde{z}}$ are positive definite. Now the matrix $Q(e) := D_{\tilde{z}}R(z)R^T(z)D_{\tilde{z}}$ is also positive definite when $e \in \mathcal{B}(\rho)$. Let $\bar{\lambda}_{\text{min}}$ denote the smallest eigenvalue of $Q(e)$ when $e(p)$ is in the compact set $\mathcal{B}$ (i.e. $\bar{\lambda}_{\text{min}} = \min_{e \in \mathcal{B}} \lambda(Q(e)) > 0$).

In the following, we calculate the set-valued derivative of $V$ along the trajectory described by the differential inclusion (16). The argument follows similarly to the analysis in the proof of Theorem 1. According to the definition of set-valued derivative in nonsmooth analysis [17], [21], the set-valued derivative is described by

$$\begin{aligned} \dot{V}(e)_{(16)} \in \tilde{\mathcal{L}}_{(16)}V(e) = \{a \in \mathbb{R} | \exists v \in \dot{e}_{(16)}, \\ \text{such that } \zeta^T v = a, \forall \zeta \in \partial V(e)\}. \end{aligned} \tag{17}$$

If the set $\tilde{\mathcal{L}}_{(16)}V(e)$ is not empty, there exists $v \in -Q(e)\text{sign}(e)$ such that $\zeta^T v = a$ for all $\zeta \in \partial V(e)$. A natural choice of $v$ is to set $v \in -Q(e)\zeta$, with which one can obtain $a = -\text{sign}^T(e)Q(e)\text{sign}(e)$. Then one can further show

$$\max(\tilde{\mathcal{L}}_{(16)}V(e)) \leq -\bar{\lambda}_{\text{min}}\text{sign}(e)^T\text{sign}(e) \tag{18}$$

if the set is not empty, while if it is empty we adopt the convention $\max(\tilde{\mathcal{L}}_{(16)}V(e)) = -\infty$. Note that this implies that $V$ is non-increasing, and consequently the Filippov solution $e(t)$ is bounded. Thus, all solutions to (16) (as well as the solutions to (15)) are complete and can be extended to $t = \infty$ (i.e. there is no finite escape time). It can be seen that $\max(\tilde{\mathcal{L}}_{(16)}V(e)) \leq 0$ for all $e \in \mathcal{B}(\rho)$ and $0 \in \max(\tilde{\mathcal{L}}_{(16)}V(e))$ if and only if $e = 0$. According to the nonsmooth invariance principle [17], [21], the asymptotic convergence is proved.

We then prove the stronger convergence result, i.e., the finite time convergence. From the definition of the *sign* function in the beginning of this section, there holds $\text{sign}(e)^T\text{sign}(e) > 1$ for any $e \neq 0$, which implies

$$\max(\tilde{\mathcal{L}}_{(16)}V(e)) \leq -\bar{\lambda}_{\min} \tag{19}$$

for any $e \neq 0$. Thus, by applying finite time Lyapunov theorem in nonsmooth analysis (see e.g. [17]), any solution starting at $e(0) \in \mathcal{B}(\rho)$ reaches the origin in finite time, and the convergence time is upper bounded by $T^* = V(e(0))/\bar{\lambda}_{\min} = \|e(0)\|_1/\bar{\lambda}_{\min}$. □

**Remark 3.** ***(Dealing with chattering)*** *In the controller (14) the sign function is used, which may cause chattering when the formation is very close to the desired one (i.e. when $e$ is very close to the origin). Possible solutions to eliminate the chattering include the following:*

- *Add deadzone to the sign function around the origin (similar to the case of uniform quantizer; see Part 1 of Theorem 1);*
- *Use the hysteresis principle in the quantization function design.*

*The adoption of the above techniques to avoid chattering will be discussed in future research.*

## V. Conclusions and Future Work

In this paper we consider the rigid formation control problem with quantized distance measurements. We have discussed in detail the quantization effect on the convergence of rigid formation shapes under two commonly-used quantizers. In the case of the symmetric uniform quantizer, all distances will converge locally to a bounded set, the size of which depends on the quantization error. In the case of the logarithmic quantizer, all distances converge locally to the desired values. We also consider a special quantizer with a signum function, which allows each agent to use very coarse distance measurements (i.e. binary information on whether it is close or far away to neighboring agents with respect to the desired distances). We show in this case the formation shape can still be achieved.

We mention several interesting problems for future research. In this paper we assume the quantization is applied to the distance measurement, and it will be more practical to assume both quantized distance and quantized bearing measurement. It is also interesting to investigate the case that different agents employ non-identical quantizers in their controllers. This suggests a typical scenario of *distance mismatch*, but it is not clear whether it gives rise to *steady-state rigid motions*, as we have observed in [15] when agents have *unmatched distance measurements*. Another topic deserving future attention is to extend the results for single-integrator formation systems with quantized measurement in this paper to rigid formation systems with more general dynamics.

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