RELATIONS BETWEEN FREQUENCY DEPENDENT CONTROL AND STATE WEIGHTING IN LQG PROBLEMS

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Abstract

The paper considers the relation between controller designs for the same linear system achieved with two different quadratic performance indices, with these indices differing only to the extent that one has frequency weighting on the control, while the other has the inverse frequency weighting on the state. The main result is that whereas the optimal controls as time functions can be the same, the optimal controllers are different. Filtering results are also obtained.

1. Introduction

Linear quadratic controller design for time-invariant linear systems of the type

$$\dot{x} = Ax + Bu \qquad (1.1)$$

has recently evolved from using performance indices of the form

$$V[x_0,u(.)] = \int_0^\infty [u'Ru + x'Qx]dt \qquad (1.2)$$

(with the usual conditions on A,B,Q,R to guarantee existence of a unique, constant, stabilizing feedback control law) to the use of indices of the form

$$V[x_0,u(.)] = \int_0^\infty [u'R(j\omega)u + x'Q(j\omega)x]dt \qquad (1.3)$$

See e.g. [1,2]. Restrictions of course are imposed on R and Q: R must be rational and uniformly positive definite hermitian for all real ω, and Q must be rational, bounded and nonnegative definite hermitian for all real ω. The notation which mixes frequency domain and time-domain quantities is to be criticised, but is suggestive.

Suppose now that the plant for which a control design is being made has associated with it high frequency uncertainty. Then one can argue that by choosing $R(j\omega)$ to be bigger for large ω than small ω, one can penalize high frequency controls more than low frequency controls and in this way, reduce the loop gain at high frequencies (thus accommodating the high frequency uncertainties). Equally, one might argue that by leaving R constant, one could achieve much the same goal by choosing $Q(j\omega)$ to be smaller for large ω than small ω. This will mean that high frequency components of the state will be more readily tolerated, and thus less the subject of corrective control action, than the low frequency components. Again, one would expect that the loop gain at high frequencies would be reduced.

This thinking at once gives rise to the question, is it better to put extra high frequency weighting on $R(j\omega)$ or reduced high frequency weighting on $Q(j\omega)$? This paper considers this question, and in fact, the more precise question, what is the difference in using, for the same system (1.1), the two performance indices

$$V_1[x_0,u(.)] = \int_0^\infty [\left|\frac{\alpha(j\omega)}{\beta(j\omega)}\right|^2 u'Ru + x'Qx]dt \qquad (1.4)$$

$$V_2[x_0,u(.)] = \int_0^\infty [u'Ru + \left|\frac{\beta(j\omega)}{\alpha(j\omega)}\right|^2 x'Qx]dt \qquad (1.5)$$

As it turns out, both indices yield the same closed-loop poles, both indices yield the same optimal control expressed as a time function (assuming a matching of initial conditions in the now dynamic controller), but they yield different control laws when these are obtained via an intuitively reasonable augmentation procedure. These points are studied in Sections 2 and 3 of the paper, and various additional points are given in Section 4.

Related questions are considered in Section 5 and 6 for filtering problems. There we find that if two filtering problems differ only in that one has colored input noise and the other colored measurement noise with the inverse spectrum, the two filters have the same poles (but are otherwise different). We are unable however to establish any equality between two time-domain quantities associated with the two filters.

2. Examples

Consider for the system $\dot{x}=u$ the minimization of

$$V_1[x(0),u(.)] = \int_0^\infty [x^2 + \left|\frac{1+\beta j\omega}{1+\alpha j\omega}\right|^2 u^2]dt \qquad (2.1)$$

When $\beta>\alpha$, this has the effect of introducing the classical notion of a lag compensation into the linear-quadratic problem; a key effect is to increase the high frequency roll-off of the closed-loop system, [3]. We should clarify two elements of imprecision about (2.1). First, the mixture of frequency domain and time domain notation, while now semistandard, means that

$$\alpha\dot{v}+v = \beta\dot{u}+u \qquad (2.2)$$

and the integrand in (2.1) should be interpreted as being x^2+v^2. We can think in fact of the underlying system equations as

$$\begin{aligned}\dot{z}+\tfrac{1}{\beta}z &= v\\ \dot{x} &= (\tfrac{1}{\beta} - \tfrac{\alpha}{\beta^2})z + \tfrac{\alpha}{\beta}v\end{aligned} \qquad (2.3)$$

[Notice that $u = (\beta^{-1}-\alpha\beta^{-2})z+\alpha\beta^{-1}v$].

The second imprecise aspect of (2.1) is that the left side of (2.1) fails to display dependence on the initial condition associated with the frequency dependent operation in the integrand: when (2.3) is viewed as the defining equation, (2.1) becomes dependent on x(0), z(0) and v(.), or equivalently, x(0), z(0) and u(.).

The optimal control law can be computed as

$$v = -[\sqrt{(\alpha^2\beta^{-2}+\beta^{-2}+2\beta^{-1})} - \beta^{-1} - \alpha\beta^{-1} \quad 1]\begin{bmatrix} z \\ x \end{bmatrix} \tag{2.4}$$

(see figure 1). Block diagram manipulation yields, with $\gamma \triangleq \sqrt{(\alpha^2\beta^{-2}+\beta^{-2}+2\beta^{-1})}$ and in Laplace transform notation,

$$U_1^*(s) = \frac{-\beta^{-1}(\alpha s+1)}{s+\gamma-\alpha\beta^{-1}} X(s) + \frac{\alpha^2\beta^{-2}+\beta^{-1}-\alpha\beta^{-1}\gamma}{s+\gamma-\alpha\beta^{-1}} z(0) \tag{2.5}$$

Next, consider, again for $\dot{x}=u$, the minimization of

$$V_2(x(0),u(.)] = \int_0^\infty [\left| \frac{1+\alpha j\omega}{1+\beta j\omega} \right|^2 x^2 + u^2]dt \tag{2.6}$$

Define w by

$$\dot{w}+\beta^{-1}w = x. \tag{2.7}$$

Then

$$V_2[x(0),u(.)] = \int_0^\infty \{[(\beta^{-1}-\alpha\beta^{-2})w + \alpha\beta^{-1}x]^2 + u^2\}dt$$

[Of course, V_2 depends also on $w(0)$.] The optimal control is

$$u_2^* = -(\gamma-\beta^{-1})x - (\beta^{-1}+\beta^{-2}-\beta^{-1}\gamma)w \tag{2.8}$$

and so

$$U_2^*(s) = -(\gamma-\beta^{-1})X(s) - \frac{(\beta^{-1}+\beta^{-2}-\beta^{-1}\gamma)}{s+\beta^{-1}}[X(s)+w(0)]$$

$$= \frac{-(\gamma-\beta^{-1})s-\beta^{-1}}{s+\beta^{-1}} X(s) - \frac{(\beta^{-1}+\beta^{-2}-\beta^{-1}\gamma)}{s+\beta^{-1}} w(0) \tag{2.9}$$

The controllers (2.5) and (2.9) are clearly quite different. As noted in [1], the first controller has as a zero the stable pole of the frequency domain function weighting u^2 in (2.1), and the second controller has as a pole the stable pole of the frequency domain function weighting x^2 in (2.6).

We now assert that if $w(0)$, $z(0)$ are appropriately related, the actual time trajectories for u_1^* and u_2^* are the same. We use the fact that $sX(s) - x(0) = U(s)$.

$$X(s) = \frac{U(s)}{s} + \frac{x(0)}{s}.$$

Using this result in (2.5) yields

$$[s^2 + \gamma s + \beta^{-1}]U_1^*(s)$$

$$= -\beta^{-1}(\alpha s + 1)\, x(0) + s(\alpha^2\beta^{-2} + \beta^{-1} - \alpha\beta^{-1}\gamma)z(0)$$

Using it in (2.9) yields

$$[s^2+\gamma s+\beta^{-1}]U_2^*(s)$$

$$= [-(\gamma-\beta^{-1})s-\beta^{-1}]x(0) - s(\beta^{-1}+\beta^{-2}-\beta^{-1}\gamma)w(0)$$

Observe then that

$$U_1^*(s) \equiv U_2^*(s) \tag{2.10}$$

if and only if

$$-(\gamma-\beta^{-1})x(0) - (\beta^{-1}+\beta^{-2}-\beta^{-1}\gamma)w(0)$$

$$= -\beta^{-1}\alpha x(0) + (\alpha^2\beta^{-2}+\beta^{-1}-\alpha\beta^{-1}\gamma)z(0) \tag{2.11}$$

To the extent that linear optimal control design is simply a vehicle for deriving a time-invariant feedback controller, the problems define the two different feedback controllers

$$U_1^*(s) = \frac{-\beta^{-1}(\alpha s+1)}{s+(\gamma-\alpha\beta^{-1})} X(s) \tag{2.12}$$

and

$$U_2^*(s) = \frac{-\beta^{-1}[(\beta\gamma-1)s+1]}{s+\beta^{-1}} X(s) \tag{2.13}$$

despite the equality of $U_1^*(t)$ and $U_2^*(t)$. It is easily checked that the closed-loop characteristic polynomial with either controller is $s^2+\gamma s+\beta^{-1}$. Another interpretation of what we have just shown is that (2.12) and (2.13) are alternative controllers, in the sense that with the same $x(0)$, the controllers yield $u_1^*(t) = u_2^*(t)$ provided the initial conditions in the controllers are related.

Of course, the robustness properties (which bear on the effect of plant uncertainty) are different for the two controllers. Some discussion of this point appears in [3], which shows that high frequency robustness properties are most easily affected by putting frequency dependence in R rather than Q.

In the next section, we show that the conclusion in this example is a general one.

3. Main Results

Following is the main result. A detailed proof is omitted because of length constraints.

Theorem. Consider the system

$$\dot{x}=Fx+Gu \qquad y=H'x \tag{3.1}$$

and the two performance indices

$$V_1 = \int_0^\infty [\left| \frac{\beta(j\omega)}{\alpha(j\omega)} \right|^2 u'u + y'y]dt \tag{3.2}$$

$$V_2 = \int_0^\infty [u'u + \left| \frac{\alpha(j\omega)}{\beta(j\omega)} \right|^2 y'y]dt \tag{3.3}$$

Suppose further that

$[F,G]$ is completely controllable and $[F,H]$ completely observable (3.4)

$\frac{\alpha(s)}{\beta(s)}$ is proper (3.5)

$\beta(s)$, $\alpha(s)$ are coprime and have all zeros in

$\mathrm{Re}[s]<0$ (3.6)

There is no zero s_0 of $H'(sI-F)^{-1}G$ such that

$\alpha(s_0)=0$, $\mathrm{Re}[s_0]=0$ (3.7)

There is no eigenvalue s_0 of F with

$\beta(s_0)=0$, $\mathrm{Re}[s_0]=0$ (3.8)

Suppose optimal control laws

$u_1=k_1(s)x$, $u_2=k_2(s)x$ are determined (by following the state augmentation procedures illustrated in the last section). Then if the initial states for the controllers are suitably related and the plant (3.1) has the same initial state for both optimization problems, the optimal controls are equal, not as laws, but as time functions:

$$u_1^*(t) = u_2^*(t) \qquad (3.9)$$

Proof of Theorem. In outline, the proof will proceed as follows:
(a) the structure of the optimal controllers is established (b) closed-loop systems are defined (using these optimal controllers) and with an external input (c) it is shown that if these two closed-loop systems have zero initial conditions and the same time functions for their inputs, then the inputs to the part of each closed-loop system which is the original plant are the same (d) the external inputs will be viewed as setting an initial state, which then allows concluding of the main result.

4. Further Issues

Collection of controllers giving the same closed-loop modes. In Section 2, we exhibited for the system with transfer function 1/s two feedback controllers with transfer functions

$$\frac{\beta^{-1}(\alpha s+1)}{s+(\gamma-\alpha\beta^{-1})}, \quad \frac{-\beta^{-1}[(\beta\gamma-1)s+1]}{s+\beta^{-1}}$$

and these controllers gave rise to the same closed-loop modes, in fact the same state trajectories, given the right initial conditions. While the fact that such a result might be possible is perhaps counterintuitive, there is a simple explanation, see Figure 2. (The external controls should be disregarded at this point.) As the block diagram manipulations suggest, with zero external signal the plant $W(s)$ with feedback controllers

$$K(s), \quad \frac{\delta(s)K(s)}{1+[1-\delta(s)]K(s)W(s)}$$

gives rise to two closed loop systems with modes that are closely related. More precisely, if

$$W = \frac{b}{a}, \; K = \frac{d}{c}, \; \delta = \frac{f}{e} \qquad (a,b,\ldots,f \text{ polynomial})$$

the closed-loop modes for the first controller are the zeros of $ac+bd$. For the second controller, they are the zeros $ae(ac+bd)$, assuming no pole-zero cancellations within $\delta K[1+(1-\delta)KW]^{-1}$. The selection of

$$\delta(s) = \frac{(\beta\gamma-1)s+1}{\alpha s+1} \qquad (4.1)$$

provides the connection between the two controllers of the example of Section 2 (and the necessary pole-zero cancellations are guaranteed to occur so that the closed-loop modes are the same for both controllers).

Of course, there are an infinite number of possible selections of $\delta(s)$ and thus an infinite number of possible controllers giving the same closed-loop trajectories; the design method based on augmentation selects in each case a particular controller. In the conventional regulator problem, with no frequency dependent weighting, the design method selects the one controller which is memoryless.

Robustness The different controllers certainly give different robustness properties, and we have analyzed the effect of the frequency dependent weighting in (2.1) elsewhere, [3]. To make the point clearly, suppose that $\alpha=0$. Then the loop gains for the two controllers become

$$\frac{1}{\beta s(s+\gamma)}, \quad \frac{(\beta\gamma-1)s+1}{\beta s(s+\beta^{-1})}$$

Obviously as $\omega=s/j$ becomes very large, the first loop gain decreases far faster than the second, so that robustness in the face of high frequency uncertainty will be different.

Identical weighting on control and state. Let us consider two optimization problems for $\dot{x}=Fx+Gu$, $y=H'x$, with $\{F,G,H\}$ minimal. These require minimization of

$$V_1 = \int_0^\infty [u'u + y'y]dt \qquad (4.2a)$$

and

$$V_2 = \int_0^\infty \left[\left|\frac{\beta(j\omega)}{\alpha(j\omega)}\right|^2 u'u + \left|\frac{\beta(j\omega)}{\alpha(j\omega)}\right|^2 y'y\right]dt \qquad (4.2b)$$

where β,α are coprime with all zeros in $\mathrm{Re}[s]<0$, $H'(sI-F)^{-1}G$ has no zero such that $\alpha(s_0)=0$, $\mathrm{Re}[s_0]=0$, and F has no eigenvalue s_0 with $\beta(s_0)=0$, $\mathrm{Re}[s_0]=0$. We shall exhibit a result on the equality of trajectories. Let $\bar{y}$ be the output of a system with input y, transfer function matrix $\beta\alpha^{-1}I$, and zero initial state. Then if

$$V_2 = \int_0^\infty \left[\left|\frac{\beta(j\omega)}{\alpha(j\omega)}\right|^2 u'u + \bar{y}'\bar{y}\right]dt$$

and the main result of Section 3 implies that the same trajectories are associated with minimizing V_2 as are associated with minimizing

$$V_3 = \int_0^\infty \left[u'u + \left|\frac{\alpha(j\omega)}{\beta(j\omega)}\right|^2 \bar{y}'\bar{y}\right]dt$$

(provided initial conditions in the controllers are satisfactorily matched). However, $y=\alpha\beta^{-1}\bar{y}$, so that V_3 and V_1 take the same values for all u (given initial condition matching). Consequently, (4.2a) and (4.2b) give rise to the same optimal controls as time functions. The optimal controller obtained for (4.2b) using augmentation at the input and output will not be the same as that obtained for (4.2a).

Tracking It might be thought that the nonuniqueness of the controllers is a consequence of the fact that the optimal control problem posed does not give rise to any external as opposed to feedback component of the optimal control. A lengthy analysis, omitted because of space constraints shows this is not the case.

5. Filtering Example

Consider two problems, illustrated in Figure 3. The white noise sources are independent, zero mean, and unit variance. We can ask: what are the relations between filters 1 and 2 and estimates $\hat{y}_1,\hat{y}_2$ based on the measurements z_1,z_2. By analogy with the control result, one must expect some relationship between the filters.

Problem 1 is straightforward to answer. Observing that

$$1+\frac{(-s+2)}{(-s+3)(-s+1)}\frac{(s+2)}{(s+3)(s+1)}$$

$$=\left[1+\frac{-\alpha s+\beta}{(-s+3)(-s+1)}\right]\left[1+\frac{\alpha s+\beta}{(s+3)(s+1)}\right] \quad (5.1)$$

where

$$\alpha=\sqrt{(11+2\sqrt{13})}-4=0.2674$$

$$\beta=\sqrt{13}-3=0.6056 \quad (5.2)$$

and $(s+3)(s+1)+\alpha s+\beta$ is Hurwitz, we conclude that filter 1 has transfer function $\bar{w}_1(s)=w_1(s)/[1+w_1(s)]$ where

$$w_1(s)=\frac{\alpha s+\beta}{(s+3)(s+1)} \quad (5.3)$$

i.e.

$$\bar{w}_1(s)=\frac{\alpha s+\beta}{s^2+(4+\alpha)s+3+\beta}$$

For problem 2, let us define

$$z_3=\frac{s+2}{s+3}z_2. \quad (5.4)$$

and seek the transfer function of the filter which processes z_3 to yield $\hat{y}_2$. It is clear that this is equivalent to searching for filter 3 in Figure 4, which estimates y_3 from z_3.

Let us construct a state-variable model for the signal process:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u \quad (5.5a)$$

$$z_3=[\,1\quad -1\,]\begin{bmatrix}x_1\\ x_2\end{bmatrix}+v \quad (5.5b)$$

(Here, u and v are zero mean, unit intensity independent white noise processes and $x_1=y$.)

Filter 3 is defined by

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 1 & -3\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}+\begin{bmatrix}k_1\\ k_2\end{bmatrix}\left\{z_3-[1\quad -1]\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}\right\} \quad (5.6a)$$

$$\hat{y}=\hat{x}_1 \quad (5.6b)$$

with k_1, k_2 determined either by the standard Riccati equation approach, or by using the spectral factorization appropriate to the problem; as reference to Figure 3 and (5.1) shows, (5.1) is the correct spectral factorization for problem 2 as well as problem 1. This means that

$$[1\quad -1]\left\{sI-\begin{bmatrix}-1 & 0\\ 1 & -3\end{bmatrix}\right\}^{-1}\begin{bmatrix}k_1\\ k_2\end{bmatrix}=\frac{\alpha s+\beta}{(s+3)(s+1)}$$

i.e.

$$k_1=\beta-\alpha$$

$$k_2=\beta-2\alpha$$

Then (5.6) defines a transfer function from z_3 to $\hat{y}_3$ of

$$\bar{w}_2=[1\quad 0]\left\{sI-\begin{bmatrix}-1+\alpha-\beta & \beta-\alpha\\ 1+2\alpha-\beta & -3+\beta-2\alpha\end{bmatrix}\right\}^{-1}\begin{bmatrix}\beta-\alpha\\ \beta-2\alpha\end{bmatrix}$$

$$=\frac{(s+3)(\beta-\alpha)}{s^2+(4+\alpha)s+3+\beta}$$

and so filter 2 has transfer function

$$\frac{(\beta-\alpha)(s+2)}{s^2+(4+\alpha)s+3+\beta}.$$

Notice that filter 2 has the same poles as filter 1, while the zero is different. (This fact will be exposed as a general property in the next section.) The zero in fact is identical with the pole of the shaping filter producing the coloured measurement noise. The magnitude of the filter 2 gain is greater at all frequencies $s=j\omega$, ω real.

Had our task been to find a filter to estimate y_4 (in Figure 4) from z_3, then we should have found the same filter as for problem 1. Herein lies the connection between the two problems. However, y_4 in Figure 4 does not correspond to any variable of interest in Figure 3b.

6. Filtering/Control Duality

The result of the last section, to the effect that two related filtering problems give rise to filters with the same poles but different zeros, is a general one. In this section, we shall demonstrate this point by setting up the duality between filtering and control problems with frequency dependent weighting, and then appealing to the earlier control results.

Frequency weighted measurement noise. Suppose the signal model is $\dot{x}=Fx+Gu$, $y=H'x$ with u a zero mean, white noise process with $E[u(t)u'(s)]=Q\delta(t-s)$, and suppose measurements $z=y+n$ are available where n is a zero mean stationary process with power spectrum $R(s)=W_r(s)W_r'(-s)$ for some invertible stable, minimum phase $W_r(s)$ with $W_r^{-1}(s)=J_r+H_r'(sI-F_r)^{-1}G_r$. Figure 5 illustrates the arrangement; Figure 5b strongly suggests that, in order to obtain a filter for x, z_1 be regarded as the measurement for an augmented model defined by

$$F_1=\begin{bmatrix}F & 0\\ G_rH' & F_r\end{bmatrix}\qquad G_1=\begin{bmatrix}G\\ 0\end{bmatrix}$$

$$H_1'=[J_rH'\quad H_r']\qquad Q_1=Q\qquad R_1=I \quad (6.1)$$

Let Π^{1f} be the solution of the associated steady-state Riccati equation. It is not hard to check that the following control problem gives rise to the same steady state Riccati equation when the approach of Section 3 to controller design is used for the system $\dot{x}=F'x+Hu$, $y=G'x$: minimize

$$\int_0^\infty \{\| W_R'(j\omega)u\|^2+y'Qy\}dt$$

(Thus the duality has $H'(sI-F)^{-1}G$ and $R(s)$ for the filtering problem becoming $G'(sI-F')^{-1}H$ and $R'(s)$ for the control problem.)

Assertion. The optimal filter can be represented in the form of Figure 6 where

$$\begin{bmatrix} K_{f_1} \\ \\ K_{f_2} \end{bmatrix} = \Pi^{1f} H_1 R^{-1} \tag{6.2}$$

$$M_f(s) = J_r + H_r'[sI - F_r + K_{f2}H_r']^{-1}(G_r - K_{f2}J_r) \tag{6.3}$$

Proof. Let x_1 be the state vector of the augmented model, with $x_1 = [x' \quad x_r']'$. With K_{f1} as above, standard Kalman filter theory [4] yields

$$\dot{\hat{x}} = F\hat{x} + K_{f1}z_1 - K_{f1}J_rH'\hat{x} - K_{f1}H_r'\hat{x}_r$$

$$\dot{\hat{x}}_r = F_r\hat{x}_r + G_rH'\hat{x} + K_{f2}z_1 - K_{f2}J_rH'\hat{x} - K_{f2}H_r'\hat{x}_r$$

Use the fact that

$$z_1 = J_rz + H_r'x_r \qquad \dot{x}_r = F_rx_r + G_rz \tag{6.4}$$

to obtain

$$\dot{\hat{x}} = F\hat{x} + K_{f1}J_rz - K_{f1}H_r'(\hat{x}_r - x_r) - K_{f1}J_rH'\hat{x} \tag{6.5}$$

and

$$\frac{d}{dt}(\hat{x}_r - x_r) = (F_r - K_{f2}H_r')(\hat{x}_r - x_r) + (G_r - K_{f2}J_r)H'\hat{x}$$
$$+ (K_{f2}J_r - G_r)z$$

so that

$$(\hat{x}_r - x_r) = -[j\omega I - F_r + K_{f2}H_r']^{-1}(G_r - K_{f2}J_r)(z - H'\hat{x})$$

Combining this with (6.5) yields the desired result.

Several points can be noted. First, $M_f(s)$ is the transfer function matrix obtained by applying output to state feedback of $-K_{f2}$ to the state-variable realization of $W_r^{-1}(s)$ in (6.4); hence $M_f(s)$ has the same zeros as $W_r^{-1}(s)$ [which are also the poles of $W_r(s)$], and these will also be zeros of the filter. Second, application of the ideas of Section 3 to the control problem yields for the controller

$$u(t) = -[(G_r' - J_r'K_{f2})(sI - F_r' + H_rK_{f2}')^{-1}H_r + J_r']K_{f1}' \, x(t) \tag{6.6}$$

and so the control/filtering duality is maintained: as the Assertion above makes clear, the effective filter gain is
$K_{f1}[J_r + H_r'(sI - F_r + K_{f2}H_r')^{-1}(G_r - K_{f2}J_r)]$.
The poles of the filter are obviously the same as the poles of the closed-loop *system* resulting from the above controller.

Frequency weighted input noise. Suppose the signal model is $\dot{x} = Fx + Gu$, $y = H'x$, measurements $z = y + n$ are available with $n(.)$ zero mean gaussian white noise, $E[n(t)n'(s)] = R\delta(t-s)$. Suppose that the power spectrum of u is $Q(j\omega)$, with $Q(s) = W_q(s)W_q'(-s)$ for some $W_q(s) = J_q + H_q'(sI - F_q)^{-1}G_q$, with $W_q(s)$ stable and minimum phase. To estimate x, it is obvious that one should define an augmented system [as the cascade of $W_q(s)$ and $H'(sI - F)^{-1}G$]. In obvious notation,

$Q_2 = I \qquad R_2 = R$

$$F_2 = \begin{bmatrix} F & GH_q' \\ 0 & F_q \end{bmatrix} \qquad G_2 = \begin{bmatrix} GJ_q \\ G_q \end{bmatrix} \qquad H_2' = [H' \quad 0]$$

Suppose the steady state Riccati equation has a solution Π^{2f}.

It is readily checked that the following control problem gives rise to the same steady state Riccati equation when the approach of Section 3 to controller design is used; for the system $\dot{x} = F'x + Hu$, $y = G'x$, minimize

$$\int_0^\infty [u'Ru + \| W_q'(j\omega)y \|^2]dt$$

(Thus the duality has $H'(sI-F)^{-1}G$ and $Q(s)$ for the filtering problem becoming $G'(sI-F')^{-1}H$ and $Q'(s)$ for the control problem.)

Figure 7a illustrates the implementation of the filter and Figure 7b illustrates a rearrangement in which

$$K_{f2}(s) = [\Pi_{11}^{2f} + GH_q'(sI - F_q)^{-1}\Pi_{21}^{2f}]HR^{-1}$$

[The control law for the control problems is, using the method of Section 2,

$$u = K_{c2}(s)x = -R^{-1}H'[\Pi_{11}^{2f} + \Pi_{21}^{2f'}(sI - F_q')^{-1}H_qG']x$$

which is consistent with the duality.]

The poles of the filter are the zeros of $\det[sI - F - K_{f2}(s)H']$ and the poles of the closed loop system with the above controller are the zeros of $\det[sI - F' + HK_{c2}'(s)]$. These are clearly the same.

Because of the duality, we see that if two control problems give rise to the same closed-loop poles, then the two dual filtering problems must also give rise to the same closed-loop poles. Duality does not however carry through to the point where equality of the time-trajectories in two related control problems implies an interesting equality in the filtering problem.

7. Conclusion

If the ultimate justification of a linear-quadratic design is really to minimize a performance index or to achieve certain closed-loop poles we have shown that frequency dependence in the index can be shuffled between the state and control weighting in the performance index. On the other hand, if the aim is in part to obtain a robust design that will offer some insensitivity to variation in the plant, then there is a very real difference in having frequency dependent weighting on the control as opposed to the state. Frequency dependent weighting on the control at least allows the moving in the frequency domain of robustness from one frequency band to another [3]; this only seems possible to a much more limited extent with frequency dependent weighting in the state term.

For the filter, we have seen that closed-loop poles can be retained when frequency dependent weighting is shifted from the process noise to the measurement noise, but otherwise the filter changes.

References

[1] N.K. Gupta, "Frequency-shaped cost functionals: extensions of linear-quadratic-gaussian design methods", AIAA J. Guidance and Control, Nov/Dec 1980, pp 529-535.

[2] N.K. Gupta, M.G. Lyons, J.M. Auburn and G. Margulies, "Frequency-shaping methods in large space structures control", Proc. AIAA Guidance and Control Conference, Albuquerque, NM, 1981.

[3] B.D.O. Anderson and D.L. Mingori, "Use of frequency dependence in linear-quadratic control problems to frequency-shape robustness", submitted for publication.

[4] B.D.O. Anderson and J.B. Moore, Linear Optimal Control, Prentice-Hall, New Jersey, 1971.

Figure Captions

1. Controller structure with frequency weighting of input.
2. Manipulating one closed loop into another closed loop.
3. Augmented signal model for colored (a) input noise (b) output noise.
4. Alternative structure for studying colored output noise.
5. General structure for studying colored output noise.
6. Filter structure with colored output noise.
7. Filter structures with colored input noise.

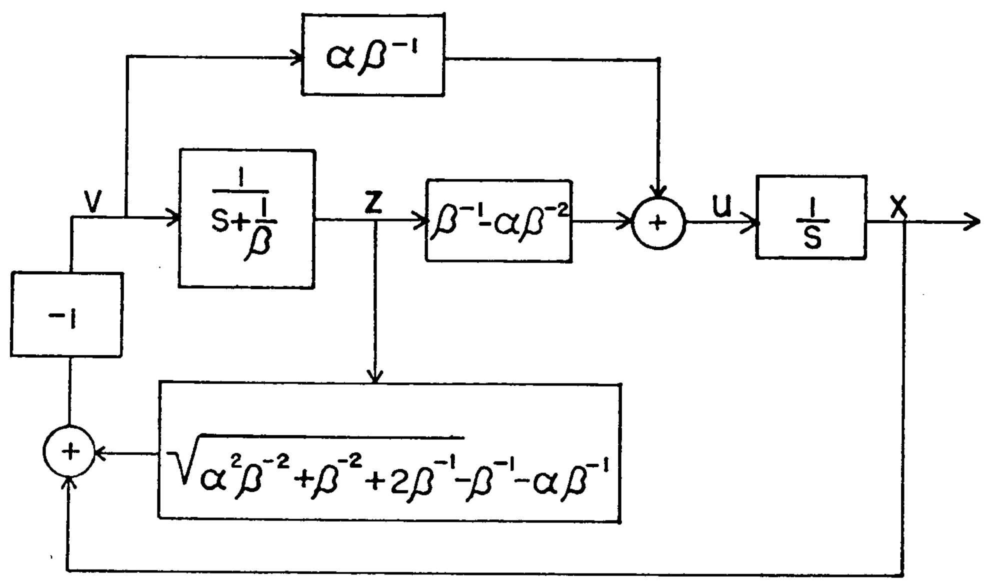

Figure I

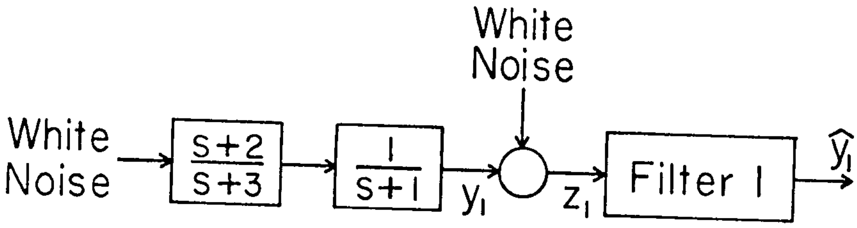

Figure 3a

RJG 23-2-83

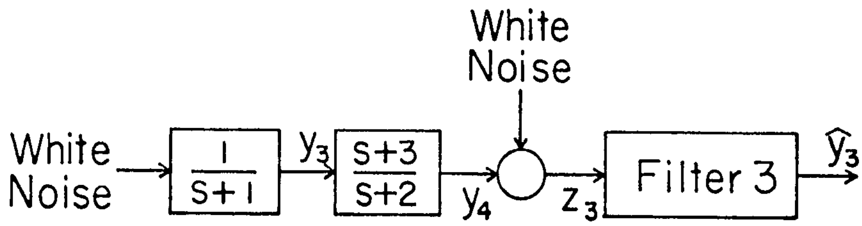

Figure 4

RJG 23-2-83

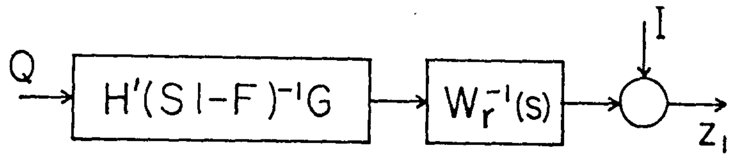

Figure 5b

AJG 23-2-83

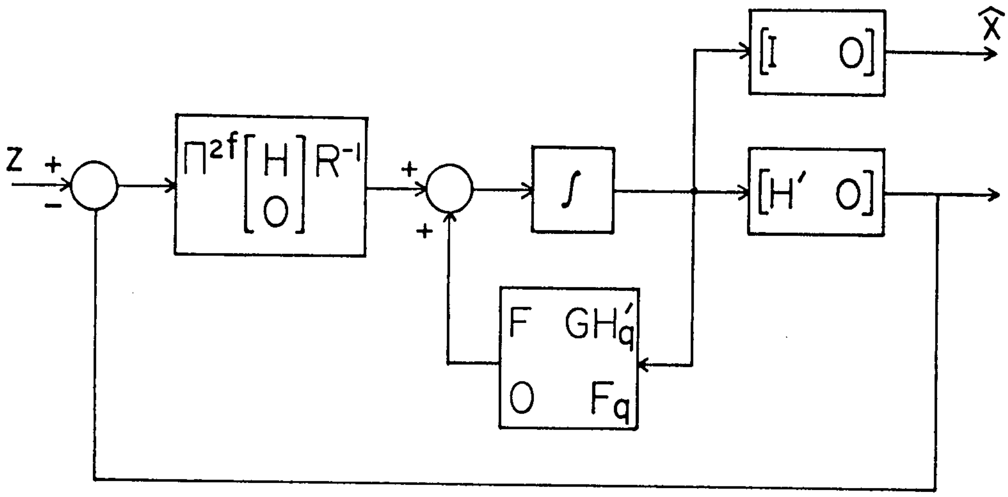

Figure 7a

RJG 22-2-83

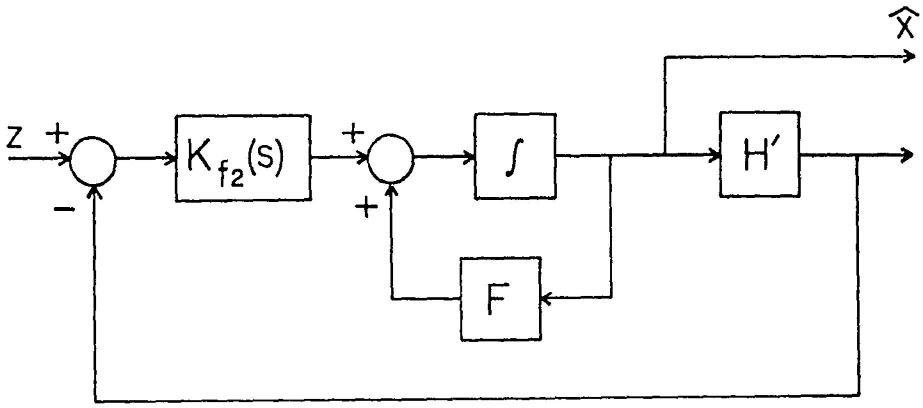

Figure 7b

RJG 22-2-83