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Robust Control of Undirected Rigid Formations with Constant Measurement Bias in Relative Positions

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***Abstract*— This paper investigates the robust control of a class of minimally infinitesimally rigid formations in the face of constant bias in the measurements of relative positions between neighboring agents. It is shown that by appropriately modifying standard gradient controls, desired formation shape can be achieved up to small errors in relative positions.**

I. Introduction

Control of formations of mobile autonomous agents has attracted extensive attention in the past few years for its widespread applications in the military, scientific explorations, and the entertainment industry. Problems in formation control include forming, maintaining, and modifying a formation, as well as other objectives. Distributed control of formations is based on various agent sensing capabilities, such as relative position, range-only, or bearing-only sensing [1]–[8].

Using graph rigidity [4]–[6], potential function-based gradient controls [6]–[8] can locally stabilize rigid undirected multi-agent formations, provided that two neighboring agents have (1) precisely the same understanding of what the prescribed target distance between them is supposed to be and (2) precise measurements of each others' relative positions. However, small measurement biases or mismatch errors between neighboring agents are inevitable [9]. It has been shown that for this reason, standard gradient controls will almost surely result in rotations in a two-dimensional (2-D) plane or helical movements in three-dimensional (3-D) space if small constant mismatch errors occur in prescribed target distances [10]–[13].

One promising way to deal with this undesired phenomenon is to use estimator-based controllers [9], [14]–[16] to estimate the mismatch errors and cancel them. Following this approach, several methods have been proposed [9], [14], [15] to locally, exponentially stabilize triangular formations as well as other minimally infinitesimally rigid formations when the mismatch errors are small constants. If the mismatch errors are due to measurement errors and are sums of constants and sinusoidal signals with known frequencies, a controller based on the internal model principle can be used to guarantee local exponential stability [16].

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Although the literature addresses the robustness problem stemming from mismatch errors in neighboring agents' prescribed target distances, misbehavior due to bias errors in relative position measurements has not been studied. Numerical simulations show that with constant measurement bias in two or more pairs of neighboring agents' relative positions, undirected rigid formations will first rotate, then drift with constant velocities. The main contribution of this paper is to propose a method to fix this problem for those undirected minimally infinitesimally rigid 2-D formations which can be generated by a sequence of vertex additions.

The paper is organized as follows. In Section II, we review several useful concepts regarding rigid formations, introduce the notation used in this paper, formulate the problem and state our objective. In Section III, we present the controller. In Section IV, we present our main findings regarding how to achieve desired formations in the face of constant measurement bias in relative positions for a class of minimally infinitesimally rigid formations, using the aforementioned controller.

II. Background and Problem Formulation

A. Notation and Preliminaries

Let $\mathbb{G} = \{\mathcal{V}, \mathcal{E}\}$ denote the *neighbor graph* of a formation of $n \geq 2$ agents, where $\mathcal{V}$ is the set of vertices with labels $1, 2, \ldots, n$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. In 2-D space, $x_i \in \mathbb{R}^2$ denotes the global coordinate vector of agent $i \in \mathcal{V}$. A *formation* is specified by a pair $\{\mathbb{G}, x\}$, where $x = [x_1' \ x_2' \ \ldots \ x_n']' \in \mathbb{R}^{2n}$ and prime denotes transpose. We say that agents i and j are *neighbors* if $(i, j) \in \mathcal{E}$. Neighbors are responsible for maintaining the prescribed distance between them. Let $\mathcal{N}_i \subseteq \mathcal{V}$ be the set of labels of agent i's neighbors. Let m denote the number of edges in $\mathcal{E}$ and write $\mathbf{m}$ for the set of edge labels $\mathbf{m} = \{1, 2, \ldots, m\}$.

We begin by assuming that there may be a small constant *bias* in the relative position of agent j measured by agent i. The problem of formation control with constant bias errors can then be converted into one with a small slowly-varying *mismatch error* between each pair of neighboring agents' prescribed target distances. To model mismatch errors, each edge in an undirected graph $\mathbb{G}$ is oriented from a leader vertex to a follower vertex in a carefully chosen direction, as will be explained in Section IV-A. Edge orientation is chosen so that one agent estimates the mismatch error and cancel it in its update rule. We call this agent the leader of this edge and call the other agent the follower, which acts as if there were no mismatch error.

Let $\bar{k} \in \mathcal{V}$ denote the label of the leader of edge $k \in \mathbf{m}$. Let $\mathcal{N}_i^+ \subseteq \mathcal{N}_i$ be the set of $j \in \mathcal{N}_i$ such that agent i is the follower of the oriented edge between agents i and j. Let $\mathcal{N}_i^- = \mathcal{N}_i - \mathcal{N}_i^+$, the complement of $\mathcal{N}_i^+$ in $\mathcal{N}_i$. Similarly, let $\mathcal{E}_i^+ \subseteq \mathcal{E}_i$ be the set of $k \in \mathcal{E}_i$ such that agent i is the follower of the oriented edge k. Let $\mathcal{E}_i^- = \mathcal{E}_i - \mathcal{E}_i^+$, the complement of $\mathcal{E}_i^+$ in $\mathcal{E}_i$. Let $z_k \in \mathbb{R}^2$ be the vector of the oriented edge $k \in \mathbf{m}$ measured by the follower, i.e., $z_k = x_i - x_j$ if agent i is connected to agent j by edge k and $k \in \mathcal{E}_i^+$. Let $z = [z_1' \ z_2' \ \ldots \ z_m']' \in \mathbb{R}^{2n}$.

The *rigidity matrix* $R_{m\times dn}$ of the formation $\{\mathbb{G}, x\}$ is defined as

$$R = \frac{df_{\mathbb{G}}(x)}{2dx}$$

where $f_{\mathbb{G}}: \ \mathbb{R}^{2n} \longrightarrow \mathbb{R}^m$ is the *rigidity function* [6].

We shall be concerned exclusively with neighbor graphs which are minimally rigid. For such formations $\{\mathbb{G}, x\}$, $R(x)$ has rank equal to $2n-3$ in 2-D space and to $3n-6$ in 3-D space. In $\mathbb{R}^2$, any graph that can be generated by a Henneberg construction [17] is minimally rigid.

B. Problem Formulation

The desired formation is specified by an undirected minimally infinitesimally rigid [6] graph $\mathbb{G}$ with a prescribed target distance $d_k > 0$ for each edge $k \in \mathbf{m}$.

Suppose $\nu_k \in \mathbb{R}^2$ is a small constant bias in the measurement of relative position between neighboring agents of edge $k \in \mathbf{m}$. As the vector of the oriented edge k measured by the follower is z_k, the same vector measured by the leader $\bar{k}$ is $z_k + \nu_k$.

Let $e_k \in \mathbb{R}$ be the error associated with edge $k \in \mathbf{m}$ calculated by its follower, i.e., $e_k = \|z_k\|^2 - d_k^2$. Let $e = [e_1 \ e_2 \ \ldots \ e_m]' \in \mathbb{R}^m$. Taking ν_k into consideration, the error calculated by the leader $\bar{k}$ is $\|z_k + \nu_k\|^2 - d_k^2 = e_k + 2\nu_k' z_k + \nu_k'\nu_k$. Define the slowly-varying mismatch error associated with edge $k \in \mathbf{m}$ to be $\mu_k = 2\nu_k' z_k + \nu_k'\nu_k \in \mathbb{R}$. Let $\mu = [\mu_1 \ \mu_2 \ \ldots \ \mu_m]' \in \mathbb{R}^m$.

Taking into account slowly varying mismatch errors, for agent i, the standard gradient control studied in [6] is of the form:

$$\begin{aligned} \dot{x}_i &= -\sum_{k\in\mathcal{E}_i^+} z_k e_k + \sum_{k\in\mathcal{E}_i^-} (z_k+\nu_k)(e_k + 2\nu_k' z_k + \nu_k'\nu_k) \\ &= -\sum_{k\in\mathcal{E}_i^+} z_k e_k + \sum_{k\in\mathcal{E}_i^-} (z_k+\nu_k)(e_k + \mu_k) \end{aligned} \quad (1)$$

Note that with the mismatch error as defined above, if e is bounded, then z is bounded so that μ is a bounded, smooth, and slowly-varying signal. In particular, there exist constants δ_μ, $\bar{\delta}_\mu$, $\delta_{\dot{\mu}} > 0$ such that

$$\bar{\delta}_\mu \le \|\mu\| \le \delta_\mu , \qquad \|\dot{\mu}\| \le \delta_{\dot{\mu}}$$

The main idea for stabilizing undirected formations with mismatch errors is to estimate the mismatch errors and cancel them. Let $\widehat{\mu}_k \in \mathbb{R}$ denote the estimated mismatch error for edge $k \in \mathbf{m}$ by its leader $\bar{k}$ and let $\bar{\mu}_k = \mu_k - \widehat{\mu}_k \in \mathbb{R}$. For agent i, the modified gradient control equation with the estimated mismatch error is

$$\begin{aligned} \dot{x}_i &= -\sum_{k\in\mathcal{E}_i^+} z_k e_k + \sum_{k\in\mathcal{E}_i^-} (z_k+\nu_k)(e_k + \mu_k - \widehat{\mu}_k) \\ &= -\sum_{k\in\mathcal{E}_i^+} z_k e_k + \sum_{k\in\mathcal{E}_i^-} (z_k+\nu_k)(e_k + \bar{\mu}_k) \end{aligned} \quad (2)$$

Let $\bar{\mu} = [\bar{\mu}_1 \ \bar{\mu}_2 \ \ldots \ \bar{\mu}_m]' \in \mathbb{R}^m$. It can be shown that (2) is equivalent to the following state space form

$$\dot{x} = (-R' + V')e + (S' + V')\bar{\mu} \quad (3)$$

where $R_{m\times 2n}$ is the rigidity matrix of the formation, $S_{m\times 2n}$ [15] is a matrix derived from R in the sense that $(k, 2j-1)^{th}$ and $(k, 2j)^{th}$ entries are the same as $-R$ if $\bar{k} = j$ and all other entries equal zero. $V_{m\times 2n}$ is a constant matrix related to bias errors in relative position measurements and has the same non-zero structure as S in the sense that $(k, 2j-1)^{th}$ and $(k, 2j)^{th}$ entries equal ν_k' if $\bar{k} = j$ and all other entries equal zero.

Our objective is to update $\widehat{\mu}_k$ based only on local measurements of agent $\bar{k} \in \mathcal{V}$ to achieve a desired formation. Due to biased measurements, achieving a desired formation means that when t is sufficiently large, the following inequality holds.

$$\|e\| < \|V\|$$

That is, the norm of error is bounded by the norm of measurement bias in relative positions.

Local measurements available to agent $\bar{k}$ are

$$z_l, \quad e_l, \quad l \in \mathcal{E}_{\bar{k}}^+; \qquad z_l + \nu_l, \quad e_l + \mu_l, \quad \widehat{\mu}_l, \quad l \in \mathcal{E}_{\bar{k}}^-$$

If agent $\bar{k}$ is the follower of edge l, it knows the edge vector z_l and the error $e_l = \|z_l\|^2 - d_l^2$ associated with edge l. If agent $\bar{k}$ is the leader of edge l, it knows the biased edge vector $z_l + \nu_l$, the sum of the error e_l and the mismatch $\mu_l = 2\nu_l' z_l + \nu_l'\nu_l$ but not their individual values, and the estimated mismatch error $\widehat{\mu}_l$ for edge l.

III. CONTROLLER

It is shown in [15] that $\dot{e} = 2R\dot{x}$, so the error system satisfies

$$\dot{e} = 2(-RR' + RV')e + 2(RS' + RV')\bar{\mu} \quad (4)$$

Now, let

$$\dot{\widehat{\mu}}_k = 2\beta(e_k + \mu_k - \widehat{\mu}_k) \quad (5)$$

where $\beta > 0$ is a parameter to be chosen. Clearly, the controller of $\widehat{\mu}_k$ is based only on local measurements of agent $\bar{k}$. Let $\widehat{\mu} = [\widehat{\mu}_1 \ \widehat{\mu}_2 \ \ldots \ \widehat{\mu}_m]' \in \mathbb{R}^m$, then

$$\dot{\widehat{\mu}} = 2\beta(e + \mu - \widehat{\mu}) = 2\beta(e + \bar{\mu})$$

and

$$\dot{\bar{\mu}} = -\dot{\widehat{\mu}} + \dot{\mu} = -2\beta e - 2\beta\bar{\mu} + \dot{\mu} \quad (6)$$

Combine (4) and (6) in the state space form,

$$\begin{bmatrix} \dot{e} \\ \dot{\bar{\mu}} \end{bmatrix} = -2 \begin{bmatrix} RR' - RV' & -RS' - RV' \\ \beta I & \beta I \end{bmatrix} \begin{bmatrix} e \\ \bar{\mu} \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\mu} \end{bmatrix} \quad (7)$$

Using the controller in (5) enables us to apply the singular perturbation techniques here to study the asymptotic behavior of e.

IV. Result

In this section, the resulting behavior of a class of minimally infinitesimally rigid formations under the modified gradient control in (2) with the controller of estimated mismatch errors given in (5) will be analyzed.

A. A Class of Minimally Infinitesimally Rigid Formations

In this subsection, a class of minimally infinitesimally rigid formations is studied, membership of which turns out to be a sufficient condition for the $m \times m$ matrix $-RR' - RS'$ associated with the formation to be stable. The desire for $-RR' - RS'$ to be stable is to ensure the stability of the system (7), which will be clear in Subsection B.

Let $\mathcal{F}$ denote the set of minimally infinitesimally rigid 2-D formations whose graphs can be generated by a sequence of vertex additions. For any formation in $\mathcal{F}$, its edge orientation is chosen as follows: If edges are labeled in the sequence of Henneberg construction, the first edge between the first two vertices can be oriented in either direction. From the third vertex onward, each newly added vertex is the follower of its two newly added edges.

Lemma 1 ([16]): For any formation in $\mathcal{F}$, with the above chosen edge orientation, the matrix $-RR' - RS'$ is stable.

Proof of Lemma 1: For completeness, a slightly simpler proof is given as below. Lemma 1 will be proved by induction. Let R_i and S_i denote matrices R and S for any formation in $\mathcal{F}$ consisting of $i \geq 2$ agents, respectively.

Without loss of generality, it is assumed that agent 1 is at the origin, the coordinates of agent 2 are (1,0), and agent 1 is the follower of the edge between them.

$$R_2 = \begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}$$

So $-R_2R_2' - R_2S_2' = -1$, which is stable.

Now suppose $-R_iR_i' - R_iS_i'$ is stable for $i \geq 2$. By vertex addition,

$$R_{i+1} = \begin{bmatrix} R_i & 0 \\ U & W \end{bmatrix}, \qquad S_{i+1} = \begin{bmatrix} S_i & 0 \\ -U & 0 \end{bmatrix}$$

So

$$-R_{i+1}R_{i+1}' - R_{i+1}S_{i+1}' = \begin{bmatrix} -R_iR_i' - R_iS_i' & 0 \\ -UR_i' - US_i' & -WW' \end{bmatrix} \tag{8}$$

where W is a 2×2 matrix and U is a $2 \times 2i$ matrix. It must be true that $\mathrm{rank}(W) = 2$ as the two newly added edges are not collinear. This implies that $-WW'$ is negative definite and therefore stable. By (8), the spectrum of $-R_{i+1}R_{i+1}' - R_{i+1}S_{i+1}'$ is the union of the spectrum of $-R_iR_i' - R_iS_i'$ and the spectrum of $-WW'$. Now $-R_iR_i' - R_iS_i'$ is stable by assumption and $-WW'$ is also stable, so that $-R_{i+1}R_{i+1}' - R_{i+1}S_{i+1}'$ is stable as well.

By induction, $-R_iR_i' - R_iS_i'$ is stable for $i \geq 2$. ■

Remarks: The proof implies that Lemma 1 also holds for certain 2-D formations that are not infinitesimally rigid, provided that from the third vertex onward, the two newly added edges associated with each new vertex are not collinear.

B. Choice of β

In this subsection, we will explain how to choose β in (5) so that $\|e\| < \|V\|$.

Let us start by analyzing the asymptotic behavior of e. From (7), we can see that when β takes a large positive value, $\dot{\bar{\mu}}$ can be much larger than $\dot{e}$ in magnitude. Here it is assumed that $\|V\|$ is sufficiently small so that $\dot{\bar{\mu}}$ is much larger than $\ddot{\mu}$ in magnitude when e is bounded. This implies that the term $-2\beta e + \dot{\mu}$ in (6) can vary much more slowly than $\bar{\mu}$. By the singular perturbation techniques, there exists a constant $\beta_1 > 0$, such that for $\forall \beta > \beta_1$,

$$\bar{\mu} \to -e + \frac{\dot{\mu}}{2\beta} \tag{9}$$

asymptotically. If we plug (9) into (4), e will behave like

$$\dot{e} = -2(RR' + RS')e + \frac{(RS' + RV')\dot{\mu}}{\beta} \tag{10}$$

If μ is constant, (10) becomes

$$\dot{e} = -2(RR' + RS')e \tag{11}$$

Given that $-2(RR' + RS')$ is stable at the desired formation, by the robustness of local exponential stability to small perturbations, there exists a domain of attraction of the exponentially stable equilibrium of the system (11) in a neighborhood of the desired formation, i.e., there exist constants $r, \lambda > 0$, such that if the initial position $x(0)$ satisfies $\|e(0)\| \leq r$, then $\|e(t)\| \leq e^{-\lambda t}\|e(0)\|$. It means that for the system (11), if the formation starts within this neighborhood of the desired formation, it converges to the desired formation exponentially fast.

Remarks: With reference to (7), let

$$A = -2\begin{bmatrix} RR' - RV' & -RS' - RV' \\ \beta I & \beta I \end{bmatrix}$$

From (9), if μ is constant, $e \to 0$ implies $\bar{\mu} \to 0$, so that A is stable. It suggests that $\beta > \beta_1$ is a sufficient condition for A to be stable. However, it is not at all a necessary condition. In a great number of formations in $\mathcal{F}$, many other values of β such as values comparable to $\|RR'\|$ or $\|RS'\|$ make A stable as well, which has been verified by numerical simulations.

By the singular perturbation techniques, if βI in A is replaced by βM, where $-M$ can be any stable matrix subject to the constraint on local measurements of each agent for the controller of $\widehat{\mu}$, A is still stable. For example,

$$M = \mathrm{diag}\{\|z_1 + \nu_1\|^2,\ \|z_2 + \nu_2\|^2,\ \ldots,\ \|z_m + \nu_m\|^2\}$$

Lemma 2: Suppose the desired formation is in $\mathcal{F}$ and the initial position $x(0)$ is such that $\|e(0)\| \leq r$, then there exists a constant $\beta_0 > 0$ such that for $\forall \beta > \beta_0$, there exists $t_1 > 0$, so that when $t > t_1$, $\|e(t)\| < \|V\|$.

Proof of Lemma 2: If the desired formation is in $\mathcal{F}$, by Lemma 1, $-2(RR'+RS')$ is stable at the desired formation. From the discussion above, let $\Phi(t,\tau)$ be the state transition matrix of $-2(RR'+RS')$, then $\|\Phi(t,\tau)\| \leq e^{-\lambda(t-\tau)}$ for $t-\tau > 0$.

A bounded e gives rise to a bounded z and thus a bounded $RS'+RV'$, so there exists a constant $\delta_{RS'+RV'} > 0$ such that $\|RS'+RV'\| < \delta_{RS'+RV'}$ for $\|e\| < r+\|V\|$. Similarly, we can choose $\delta_{\dot{\mu}}$ such that when $\|e\| < r+\|V\|$, $\|\dot{\mu}\| < \delta_{\dot{\mu}}$. Clearly, $\|e(0)\| \leq r < r+\|V\|$, so $\|RS'+RV'\| < \delta_{RS'+RV'}$ and $\|\dot{\mu}\| < \delta_{\dot{\mu}}$ at $t=0$. Since $\|RS'+RV'\|$ and $\|\dot{\mu}\|$ are both continuous functions of t, let $T = \sup\{t : \|R(\tau)S'(\tau)+R(\tau)V'\| < \delta_{RS'+RV'}$ and $\|\dot{\mu}(\tau)\| < \delta_{\dot{\mu}}$ for $\tau \in [0,t]\}$.

From (10), by the variation of constants formula, when $0 \leq \tau \leq t$,

$$e(t) = \Phi(t,0)e(0) + \int_0^t \Phi(t,\tau)\frac{(R(\tau)S'(\tau)+R(\tau)V')\dot{\mu}(\tau)}{\beta}d\tau \quad (12)$$

By (12) and the inequalities of vector and matrix norm,

$$\begin{aligned}
\|e(T)\| &= \|\Phi(T,0)e(0) + \int_0^T \Phi(T,\tau)\frac{(R(\tau)S'(\tau)+R(\tau)V')\dot{\mu}(\tau)}{\beta}d\tau\| \\
&\leq \|\Phi(T,0)\|\|e(0)\| + \int_0^T \|\Phi(T,\tau)\|\frac{\|(R(\tau)S'(\tau)+R(\tau)V')\|\|\dot{\mu}(\tau)\|}{\beta}d\tau \\
&< \|e(0)\|e^{-\lambda T} + \frac{\delta_{RS'+RV'}\delta_{\dot{\mu}}}{\beta}\int_0^T e^{-\lambda(T-\tau)}d\tau \\
&= \|e(0)\|e^{-\lambda T} + \frac{\delta_{RS'+RV'}\delta_{\dot{\mu}}}{\beta}\frac{1-e^{-\lambda T}}{\lambda} \\
&< \|e(0)\|e^{-\lambda T} + \frac{\delta_{RS'+RV'}\delta_{\dot{\mu}}}{\beta\lambda} \quad (13)
\end{aligned}$$

Let

$$\beta_2 = \frac{\delta_{RS'+RV'}\delta_{\dot{\mu}}}{\|V\|\lambda} > 0, \qquad \beta_0 = \max\{\beta_1, \beta_2\} > 0$$

$$\beta > \beta_0 \quad \Longrightarrow \quad \frac{\delta_{RS'+RV'}\delta_{\dot{\mu}}}{\beta\lambda} < \|V\|$$

So when $\beta > \beta_0$, $\|e(T)\| < r+\|V\|$, which implies $\|RS'+RV'\| < \delta_{RS'+RV'}$ and $\|\dot{\mu}\| < \delta_{\dot{\mu}}$ at T. Now suppose T is finite. As $\|RS'+RV'\|$ and $\|\dot{\mu}\|$ are both continuous functions of t, there exists $\epsilon > 0$, such that $\|RS'+RV'\| < \delta_{RS'+RV'}$ and $\|\dot{\mu}\| < \delta_{\dot{\mu}}$ for $t \in [T, T+\epsilon]$, which contradicts the definition of T. This implies that $T = \infty$. It means that with $\beta > \beta_0$, for $t \in [0,\infty)$, $\|RS'+RV'\| < \delta_{RS'+RV'}$, $\|\dot{\mu}\| < \delta_{\dot{\mu}}$, and $\|e(t)\| < re^{-\lambda t} + \|V\|$. It follows that for $\forall\beta > \beta_0$, there exists $t_1 > 0$, such that when $t > t_1$, $\|e(t)\| < \|V\|$. ■

Therefore β should be chosen such that $\beta > \beta_0$.

C. Main Result

If we plug (9) into (3), x will behave like

$$\dot{x} = -(R'+S')e + \frac{(S'+V')\dot{\mu}}{2\beta} \quad (14)$$

By (14) and the inequalities of vector and matrix norm, when conditions in Lemma 2 are satisfied with $\beta > \beta_0$, $t > t_1$,

$$\begin{aligned}
\|\dot{x}\| &= \|-(R'+S')e + \frac{(S'+V')\dot{\mu}}{2\beta}\| \\
&\leq \|R'+S'\|\|e\| + \frac{\|S'+V'\|\|\dot{\mu}\|}{2\beta} \\
&< \|R'+S'\|\|V\| + \frac{\|S'+V'\|\delta_{\dot{\mu}}}{2\beta} \quad (15)
\end{aligned}$$

We can tell from (15) that as $\|V\|$ and $\delta_{\dot{\mu}}$ are small constants, β is relatively large, $\|\dot{x}(t)\|$ is relatively small for $t > t_1$, which means the formation is moving much more slowly than if no corrective action is taken with respect to those biased measurements. If at $t_2 > t_1$, $\dot{x}(t_2)$ is forced to be 0, then the formation stops at the desired formation with $\|e(t_2)\| < \|V\|$. Thus, we can summarize the main result of the paper with the following theorem.

Theorem 1: Given an undirected formation with any number of sufficiently small constant biases in the measurements of neighboring agents' relative positions, suppose at $e=0$, the formation is minimally infinitesimally rigid and its graph can be generated by a sequence of vertex additions. Suppose $x(0)$ is such that $e(0)$ is in the domain of attraction of the exponentially stable equilibrium of the system (11). Under the control of (2) with $\widehat{\mu}_k$ given in (5) where $\beta > \beta_0$, there exists $t_1 > 0$ such that when $t > t_1$, $\|e(t)\| < \|V\|$.

V. Conclusion

This paper proposes a method to address the robustness problem which arises with gradient controls in the presence of small constant bias errors in neighboring agents' measurements of each others' relative positions, for the class of undirected minimally infinitesimally rigid 2-D formations that can be generated by a sequence of vertex additions. By redefining mismatch errors, the problem is converted into one with small slowly-varying mismatch errors between neighboring agents' prescribed target distances. By the singular perturbation techniques, the distributed control of the estimated mismatch error $\widehat{\mu}$ can be achieved in various ways subject to the constraint on local measurements of each agent, such as the one given in the remarks before Lemma 2.

In the future, we are interested in exploring how to obtain similar results for minimally infinitesimally rigid formations generated by the combination of vertex additions and edge splittings. Numerical simulations show that without any corrective action, rigid formations in the face of two or more biased measurements of neighboring agents' relative positions will eventually drift with constant velocities. It is not clear why this is so and we would like to analyze this phenomenon in our future research. Also, it will be helpful to determine how to replace the large unknown parameter β in (5) with a known parameter such as one comparable to $\|RR'\|$ in magnitude.

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