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Distributed Network Flows Solving Linear Algebraic Equations

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***Abstract*— We study distributed network flows as solvers in continuous time for the linear algebraic equation $\mathbf{z} = \mathbf{H}\mathbf{y}$. Each node i holds a row $\mathrm{h}_i^{\mathrm{T}}$ of the matrix H and the corresponding entry z_i in the vector $\mathbf{z}$. The first "consensus + projection" flow under investigation consists of two terms, one from standard consensus dynamics and the other contributing to projection onto each affine subspace specified by the h_i and z_i. The second "projection consensus" flow on the other hand simply replaces the relative state feedback in consensus dynamics with projected relative state feedback. Without dwell-time assumption on switching graphs as well as without positively lower bounded assumption on arc weights, we prove that all node states converge to a common solution of the linear algebraic equation, if there is any. The convergence is global for the "consensus + projection" flow while local for the "projection consensus" flow in the sense that the initial values must lie on the affine subspaces. If the linear equation has no exact solutions, we show that the node states can converge to a ball around the least squares solution whose radius can be made arbitrarily small through selecting a sufficiently large gain for the "consensus + projection" flow under fixed bidirectional graphs. Semi-global convergence to approximate least squares solutions is demonstrated for general switching directed graphs under suitable conditions.**

I. INTRODUCTION

In the past decade, distributed consensus algorithms have attracted a significant amount of research attention [8]–[13], due to their wide applications in distributed control and estimation [14], [15], distributed signal processing [16], and distributed optimization methods [17]–[20]. The basic idea is that a network of interconnected nodes can reach an agreement, or consensus, via local information exchange as long as the communication graph is well connected. As a generalized consensus notion, constrained consensus seeks to reach a consensus in the intersection of a few convex sets, where each set corresponds to a particular node serving as the node's supporting state space [21], [22].

On the other hand, another intriguing problem lies in how to develop distributed algorithms that solve a linear algebraic equation $\mathbf{z} = \mathbf{H}\mathbf{y}$, $\mathbf{H} \in \mathbb{R}^{N\times m}, \mathbf{z} \in \mathbb{R}^{N}$ with respect to variable $\mathbf{y} \in \mathbb{R}^{m}$, to which tremendous efforts have been devoted with many algorithms developed under different levels of information exchanges among the nodes [29]–[36]. Naturally, we can assume that a node i possesses a row vector, $\mathbf{h}_i^{\mathrm{T}}$ of $\mathbf{H}$ as well as the corresponding entry, z_i in $\mathbf{z}$. Each node will then be able to determine an affine subspace whose elements satisfy the equation $\mathbf{h}_i^{\mathrm{T}}\mathbf{y} = z_i$. This is to say, as long as the original equation $\mathbf{z} = \mathbf{H}\mathbf{y}$ has a nonempty solution set, solving the equation would be equivalent to finding a point in the intersection of all the affine subspaces and therefore can be put into the constrained consensus framework. If however the linear equation has no exact solution and its least squares solutions are of interest, we ended up with a convex optimization problem with quadratic cost and linear constraints.

In this paper, we study two distributed network flows as distributed solvers for such linear algebraic equations in continuous time. The first so-called "consensus + projection" flow consists of two additive terms, one from standard consensus dynamics and the other contributing to projection onto each affine subspace. The second "projection consensus" flow on the other hand simply replaces the relative state feedback in consensus dynamics with projected relative state feedback. Essentially only relative state information is exchanged among the nodes for both of the two flows, which justifies their full *distributedness*. To study the asymptotic behaviours of the two distributed flows with respect to the solutions of the linear equation, new challenges arise in the loss of intersection boundedness and interior points for the exact solution case as well as in the complete loss of intersection sets for the least squares solution case. As a result, the analysis cannot be simply mapped back to the studies in [21], [22].

Under mild conditions on the communication graphs (without requiring a dwell-time on switching graphs and without requiring a positively lower bound on non-zero arc weights), we prove that all node states asymptotically converge to a common solution of the linear algebraic equation for the two flows, if there is any. The convergence is global for the "consensus + projection" flow, and local for the "projection consensus" flow in the sense that the initial values must be put into the affine subspaces. We manage to characterize the node limits for balanced or fixed graphs. If the linear equation has no exact solutions, we show that the node states can be forced to converge to a ball of fixed but arbitrarily small radius surround the least squares solution by taking the gain of the consensus dynamics to be sufficiently large for "consensus + projection" flow under fixed and undirected graphs. Semi-global convergence to approximate least squares solutions is established for general switching directed graphs under suitable conditions. A minor, but more explicit result arises where it is also shown that the "projection consensus" flow drives the average of the

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node states to the least squares solution with complete communication graphs. Due to space limitation, we refer to [1] for detailed proofs of the reported results.

The remainder of this paper is organized as follows. Section II introduces the network model, presents the distributed flows under consideration, and defines the problem of interest. Section III and Section IV present the main result for exact solution and least squares solution case, respectively. Finally Section V concludes the paper with a few remarks.

Notation and Terminology

A directed graph (digraph) is an ordered pair of two sets denoted by $\mathrm{G}=(\mathrm{V},\mathrm{E})$ [2]. Here $\mathrm{V}=\{1,\dots,N\}$ is a finite set of vertices (nodes). Each element in E is an ordered pair of two distinct nodes in V, called an arc. A directed path in G with length k from v_1 to v_{k+1} is a sequence of distinct nodes, $v_1v_2\dots v_{k+1}$, such that $(v_m,v_{m+1})\in\mathrm{E}$, for all $m=1,\dots,k$. A digraph G is termed *strongly connected* if for any two distinct nodes $i,j\in\mathrm{V}$, there is a path from i to j. A digraph is called *bidirectional* when $(i,j)\in\mathrm{E}$ if and only if $(j,i)\in\mathrm{E}$ for all i and j. A strongly connected bidirectional digraph is simply called *connected*. All vectors are column vectors and denoted by bold, lower case letters, i.e., $\mathbf{a},\mathbf{b},\mathbf{c}$, etc.; matrices are denoted with bold, upper case letters, i.e., $\mathbf{A},\mathbf{B},\mathbf{C}$, etc.; sets are denoted with $\mathcal{A},\mathcal{B},\mathcal{C}$, etc. Depending on the argument, $|\cdot|$ stands for the absolute value of a real number or the cardinality of a set. The Euclidean norm of a vector is denoted as $\|\cdot\|$.

II. Problem Definition

A. Linear Equations

Consider the following linear algebraic equation:

$$\mathbf{z}=\mathbf{H}\mathbf{y} \tag{1}$$

with respect to variable $\mathbf{y}\in\mathbb{R}^m$, where $\mathbf{H}\in\mathbb{R}^{N\times m}$ and $\mathbf{z}\in\mathbb{R}^N$. We know from the basics of linear algebra that overall there are three cases.

(I) There exists a unique solution satisfying Eq. (1): $\operatorname{rank}(\mathbf{H})=m$ and $\mathbf{z}\in\operatorname{span}(\mathbf{H})$.

(II) There is an infinite set of solutions satisfying Eq. (1): $\operatorname{rank}(\mathbf{H})<m$ and $\mathbf{z}\in\operatorname{span}(\mathbf{H})$.

(III) There exists no solution satisfying Eq. (1): $\mathbf{z}\notin\operatorname{span}(\mathbf{H})$.

We denote

$$\mathbf{H}=\begin{pmatrix}\mathbf{h}_1^{\mathrm{T}}\\ \mathbf{h}_2^{\mathrm{T}}\\ \vdots\\ \mathbf{h}_N^{\mathrm{T}}\end{pmatrix},\quad \mathbf{z}=\begin{pmatrix}z_1\\ z_2\\ \vdots\\ z_N\end{pmatrix}$$

with $\mathbf{h}_i^{\mathrm{T}}$ being the i-th row vector of $\mathbf{H}$. For the ease of presentation we assume throughout the rest of the paper that

$$\big\|\mathbf{h}_i\big\|=1,\ i=1,\dots,N.$$

Introduce

$$\mathcal{A}_i:=\Big\{\mathbf{y}:\ \mathbf{h}_i^{\mathrm{T}}\mathbf{y}=z_i\Big\}$$

for each $i=1,\dots,N$, which is an affine subspace. It is then clear that Case (I) is equivalent to the condition that $\mathcal{A}:=\bigcap_{i=1}^N\mathcal{A}_i$ is a singleton, and that Case (II) is equivalent to the condition that $\mathcal{A}:=\bigcap_{i=1}^N\mathcal{A}_i$ is an affine space with a nontrivial dimension. For Case (III), a least squaress solution of (1) can be defined via the following optimization problem:

$$\min_{\mathbf{y}\in\mathbb{R}^m}\big\|\mathbf{z}-\mathbf{H}\mathbf{y}\big\|^2, \tag{2}$$

which yields a unique solution $\mathbf{y}^\star=(\mathbf{H}^{\mathrm{T}}\mathbf{H})^{-1}\mathbf{H}\mathbf{z}$ if $\operatorname{rank}(\mathbf{H})=m$.

Consider a network with nodes indexed in the set $\mathrm{V}=\big\{1,\dots,N\big\}$. Each node i has access to the value of $\mathbf{h}_i$ and z_i without the knowledge of $\mathbf{h}_j$ or z_j from other nodes. Each node i holds a state $\mathbf{x}_i(t)\in\mathbb{R}^m$ and exchanges this state information with other nodes. We are interested in distributed flows for the $\mathbf{x}_i(t)$ that asymptotically solve the equation (1), i.e., $\mathbf{x}_i(t)$ approaches some solution of (1) as t grows.

B. Network Communication Structures

Let Θ denote the set of all directed graphs associated with node set V. Node interactions are described by a signal $\sigma(\cdot):\mathbb{R}^{\geq 0}\mapsto\Theta$. The digraph that $\sigma(\cdot)$ defines at time t is denoted as $\mathrm{G}_{\sigma(t)}=\big(\mathrm{V},\mathrm{E}_{\sigma(t)}\big)$, where $\mathrm{E}_{\sigma(t)}$ is the set of arcs. The neighbor set of node i at time t, denoted $\mathrm{N}_i(t)$, is given by

$$\mathrm{N}_i(t):=\Big\{j:\ (j,i)\in\mathrm{E}_{\sigma(t)}\Big\}.$$

This is to say, at any given time t, node i can only receive information from the nodes in the set $\mathrm{N}_i(t)$.

Associated with each ordered pair (j,i) there is a function $a_{ij}(\cdot):\mathbb{R}^{\geq 0}\to\mathbb{R}^+$ representing the weight of the possible connection (j,i). We impose the following assumption, which will be adopted throughout the paper without specific further mention.

Weights Assumption. The function $a_{ij}(\cdot)$ is continues except for at most a set with measure zero over $\mathbb{R}^{\geq 0}$ for all $i,j\in\mathrm{V}$; there exists $a^*>0$ such that $a_{ij}(t)\leq a^*$ for all $t\in\mathbb{R}^{\geq 0}$ and all $i,j\in\mathrm{V}$.

Denote $\mathbb{I}_{(j,i)\in\mathrm{E}_{\sigma(t)}}$ as an indicator function, where $\mathbb{I}_{(j,i)\in\mathrm{E}_{\sigma(t)}}=1$ if $(j,i)\in\mathrm{E}_{\sigma(t)}$ and $(j,i)\in\mathrm{E}_{\sigma(t)}=0$ otherwise. We impose the following definition on the connectivity of the network communication structures.

Definition 1: (i) An arc (j,i) is said to be a δ-arc of $\mathrm{G}_{\sigma(t)}$ for the time interval $[t_1,t_2)$ if

$$\int_{t_1}^{t_2}a_{ij}(t)\mathbb{I}_{(j,i)\in\mathrm{E}_{\sigma(t)}}dt\geq\delta.$$

(ii) $\mathrm{G}_{\sigma(t)}$ is δ-uniformly jointly strongly connected (δ-UJSC) if there exists $T>0$ such that the δ-arcs of $\mathrm{G}_{\sigma(t)}$ on time interval $[s,s+T)$ form a strongly connected digraph for all $s\geq 0$;

(iii) $\mathrm{G}_{\sigma(t)}$ is δ-bidirectionally infinitely jointly connected (δ-BIJC) if $\mathrm{G}_{\sigma(t)}$ is bidirectional for all $t\geq 0$ and the δ-arcs of $\mathrm{G}_{\sigma(t)}$ on time interval $[s,\infty)$ form a connected graph for all $s\geq 0$.

C. Distributed Flows

Denote mapping $\mathcal{P}_{\mathcal{A}_i}: \mathbb{R}^m \to \mathbb{R}^m$ as the projection onto the affine subspace $\mathcal{A}_i$. Let $K > 0$ be a given constant. We consider the following continuous-time network flows:

[**"Consensus + Projection" Flow**]:

$$\dot{\mathbf{x}}_i = K\Big(\sum_{j\in \mathrm{N}_i(t)} a_{ij}(t)\big(\mathbf{x}_j - \mathbf{x}_i\big)\Big) + \mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_i) - \mathbf{x}_i, \quad i \in \mathrm{V}; \tag{3}$$

[**"Projection Consensus" Flow**]:

$$\dot{\mathbf{x}}_i = \sum_{j\in \mathrm{N}_i(t)} a_{ij}(t)\big(\mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_j) - \mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_i)\big), \quad i \in \mathrm{V}. \tag{4}$$

The first term of (3) is a standard consensus flow [8], while along

$$\dot{\mathbf{x}}_i = \mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_i) - \mathbf{x}_i,$$

$x_i(t)$ will be asymptotically projected onto $\mathcal{A}_i$. The flow (4) simply replaces the relative state in standard consensus flow with relative projective state.

Notice that a particular equilibrium point of these equations is given by $\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N = \mathbf{y}$, where $\mathbf{y}$ is a solution of Eq. (1). The aim of the two flows is to ensure that the $\mathbf{x}_i(t)$ asymptotically tend to a solution of Eq. (1). Note that, the projection consensus flow (4) can be rewritten as

$$\dot{\mathbf{x}}_i = \mathcal{P}_{\mathcal{A}_i}\Big(\sum_{j\in \mathrm{N}_i(t)} a_{ij}(t)(\mathbf{x}_j - \mathbf{x}_i)\Big) - \sum_{j\in \mathrm{N}_i(t)} a_{ij}(t)\cdot \mathcal{P}_{\mathcal{A}_i}(0). \tag{5}$$

Therefore, in both of the "consensus + projection" and the "projection consensus" flows, a node i essentially only receives the information

$$\sum_{j\in \mathrm{N}_i(t)} a_{ij}(t)\big(\mathbf{x}_j(t) - \mathbf{x}_i(t)\big)$$

from its neighbors, and the flows can then be utilized in addition with the $\mathbf{h}_i$ and z_i it holds. In this way, the flows (3) and (4) are *distributed*.

Without loss of generality we assume the initial time is $t_0 = 0$. We denote the trajectories of the two flows as $\mathbf{x}(t) = (\mathbf{x}_1^{\mathrm{T}}(t) \cdots \mathbf{x}_N^{\mathrm{T}}(t))^{\mathrm{T}}$.

D. Discussions

1) Geometries: The two network flows are intrinsically different in their geometries. In fact, the "consensus + projection" flow is a special case of the optimal consensus flow proposed in [22] consisting of two parts, a consensus part and another projection part. The "projection consensus" flow, first proposed and studied in [36] for fixed bidirectional graphs, is the continuous-time analogue of the projected consensus algorithm proposed in [21]. Because it is a gradient descent on a Riemannian manifold if the communication graph is undirected and fixed, there is guaranteed convergence to an equilibrium point. The two flows are closely related to the alternating projection algorithms, first studied by von Neumann in the 1940s [24]. We refer to [28] for a thorough survey on the developments of alternating projection algorithms. We illustrate the intuitive difference between the two flows in Figure 1.

2) Relation with Previous Work: The dynamics described in systems (3) and (4) are linear, in contrast to the nonlinear dynamics due to general convex set studied in [21], [22]. However, we would like to emphasize that new challenges arise with the systems (3) and (4) compared to the work of [21], [22]. First of all, for both of the Cases (I) and (II), $\mathcal{A} := \bigcap_{i=1}^N \mathcal{A}_i$ contains no interior point. This interior point condition however is essential to the results in [21]. Next, for Case (II) where $\mathcal{A} := \bigcap_{i=1}^N \mathcal{A}_i$ is an affine space with a nontrivial dimension, the boundedness condition for the intersection of the convex sets no longer holds, which plays a key role in the analysis of [21], [22]. Finally, for Case (III), $\mathcal{A} := \bigcap_{i=1}^N \mathcal{A}_i$ becomes an empty set. The least squares solution case then completely falls out from the discussions of constrained consensus in [21], [22].

3) Grouping of Rows: We have assumed that each node i only has access to the value of $\mathbf{h}_i$ and z_i from the equation (1) and therefore there are a total of N nodes. Alternatively, there can be $n \le N$ nodes with node i holding $n_i \ge 1$ rows, denoted $(\mathbf{h}_{i_1}\ z_{i_1}), \dots, (\mathbf{h}_{i_{n_i}}\ z_{i_{n_i}})$ of $(\mathbf{H}\ \mathbf{z})$. In this case, we can revise the definition of $\mathcal{A}_i$ to

$$\mathcal{A}_i := \Big\{\mathbf{y}:\ \mathbf{h}_{i_k}^{\mathrm{T}}\mathbf{y} = z_{i_k},\ k = 1, \dots, n_k\Big\},$$

which is nonempty if (1) has at least one solution. Then the two flows (3) and (4) can be defined in the same manner for the n nodes.

Let[1]

$$\bigcup_{i=1}^n \Big\{(\mathbf{h}_{i_1}\ z_{i_1}), \dots, (\mathbf{h}_{i_{n_i}}\ z_{i_{n_i}})\Big\} = \Big\{(\mathbf{h}_1\ z_1), \dots, (\mathbf{h}_N\ z_N)\Big\}.$$

Note that, the $\mathcal{A}_i$ are still affine subspaces, while solving (1) exactly continues to be equivalent to finding a point in $\bigcap_{i=1}^n \mathcal{A}_i$. Consequently, all our results for Cases (I) and (II) apply also to this new setting with row grouping.

III. Main Results: Exact Solutions

In this section, we show how the two distributed flows asymptotically solve the equation (1) under quite general conditions for Cases (I) and (II).

A. Singleton Solution Set

We first focus on the case when Eq. (1) admits a unique solution $\mathbf{y}_*$, or equivalently,

$$\mathcal{A} := \bigcap_{i=1}^N \mathcal{A}_i = \{\mathbf{y}_*\}$$

is a singleton. For the "consensus + projection" flow (3), the following theorem holds.

Theorem 1: Let (I) hold with $\mathbf{y}_*$ being the unique solution of (1). Then along the "consensus + projection" flow (3), there holds

$$\lim_{t\to\infty} \mathbf{x}_i(t) = \mathbf{y}_*,\ \ i \in \mathrm{V}$$

[1]Note that, it is not necessary to require the $\big\{(\mathbf{h}_{i_1}\ z_{i_1}), \dots, (\mathbf{h}_{i_{n_i}}\ z_{i_{n_i}})\big\}$ to be disjoint.

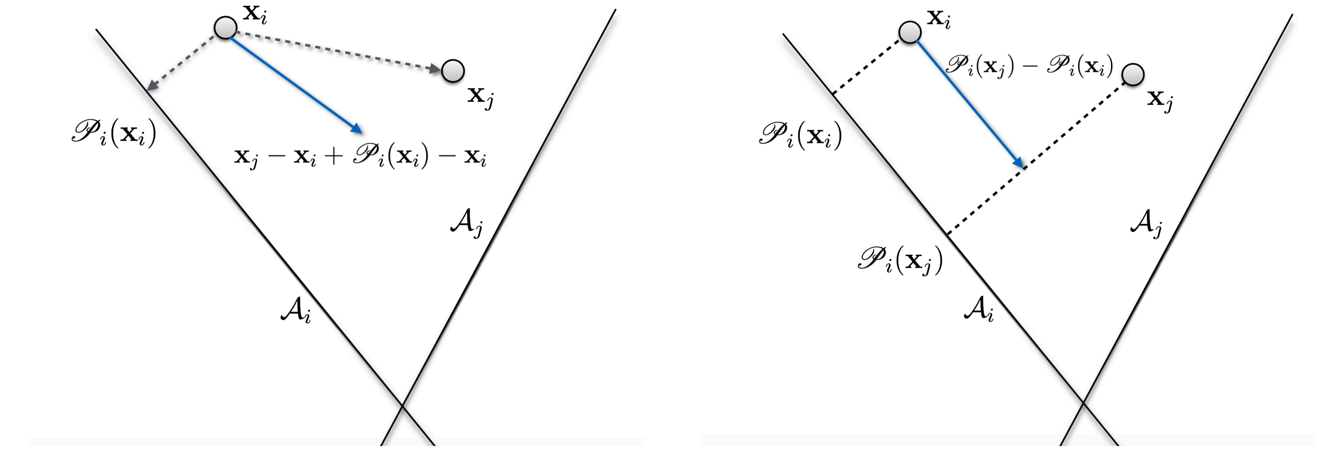

Fig. 1. An illustration of the "consensus + projection" flow (left) and the "projection consensus" flow (right). The blue arrows mark the vector of $\dot{\mathbf{x}}_i$ for the two flows, respectively.

for all initial values if $\mathrm{G}_{\sigma(t)}$ is either δ-UJSC or δ-BIJC.

The "projection consensus" flow (4), however, can only guarantee local convergence for a particular set of initial values. The following theorem holds.

Theorem 2: Let (I) hold with $\mathbf{y}_*$ being the unique solution of (1). Suppose $\mathbf{x}_i(0) \in \mathcal{A}_i$ for all i. Then along the "projection consensus" flow (4), there holds

$$\lim_{t\to\infty} \mathbf{x}_i(t) = \mathbf{y}_*,\ i \in \mathrm{V}$$

if $\mathrm{G}_{\sigma(t)}$ is either δ-UJSC or δ-BIJC.

Remark 1: Convergence along the "projection consensus" flow relies on specific initial values due to the existence of equilibriums other than the desired consensus states within the set $\mathcal{A}$: if $\mathbf{x}_i(0)$ are all equal, then obviously they will stay there for ever along the flow (4). It was suggested in [36] that one can add another term in the "projection consensus" flow and arrive at

$$\dot{\mathbf{x}}_i = \sum_{j\in \mathrm{N}_i(t)} a_{ij}(t)\big(\mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_j) - \mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_i)\big) + \mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_i) - \mathbf{x}_i \tag{6}$$

for $i \in \mathrm{V}$, then convergence will be global under (6). We would like to point out that (6) has a similar structure as the "consensus + projection" flow with the consensus dynamics being replaced by projection consensus.

B. Infinite Solution Set

We now turn to the scenario when Eq. (1) has an infinite set of solutions, i.e., $\mathcal{A} := \bigcap_{i=1}^N \mathcal{A}_i$ is an affine space with a nontrivial dimension. We note that in this case $\mathcal{A}$ is no longer a bounded set; nor does it contain interior points. This is in contrast to the situation studied in [21], [22].

For the "consensus + projection" flow (3), we present the following result.

Theorem 3: Let (II) hold. Then along the "consensus + projection" flow (3) and for any initial value $\mathbf{x}(0)$, there exists $\mathbf{y}^\flat(\mathbf{x}(0))$, which is a solution of (1), such that

$$\lim_{t\to\infty} \mathbf{x}_i(t) = \mathbf{y}^\flat(\mathbf{x}(0)),\ i \in \mathrm{V}$$

if $\mathrm{G}_{\sigma(t)}$ is either δ-UJSC or δ-BIJC.

For the "projection consensus" flow, convergence relies on restricted initial nodes states.

Theorem 4: Let (II) hold. Then along the 'projection consensus" flow (4) and for any initial value $\mathbf{x}(0)$ with $\mathbf{x}_i(0) \in \mathcal{A}_i$ for all i, there exists $\mathbf{y}^\flat(\mathbf{x}(0))$, which is a solution of (1), such that

$$\lim_{t\to\infty} \mathbf{x}_i(t) = \mathbf{y}^\flat(\mathbf{x}(0)),\ i \in \mathrm{V}$$

if $\mathrm{G}_{\sigma(t)}$ is either δ-UJSC or δ-BIJC.

C. Discussion: Convergence Speed/The Limits

For any given graph signal $\mathrm{G}_{\sigma(t)}$, the value of $\mathbf{y}^\flat(\mathbf{x}(0))$ in Theorems 3 and 4 depends only on the initial value $\mathbf{x}(0)$. We manage to provide a characterization to $\mathbf{y}^\flat(\mathbf{x}(0))$ for balanced switching graphs or fixed graphs. Denote $\mathcal{P}_{\mathcal{A}}$ as the projection operator over $\mathcal{A}$.

Theorem 5: The following statements hold for both the "consensus + projection" and the "projection consensus" flows.

(i) Suppose $\mathrm{G}_{\sigma(t)}$ is balanced, i.e., $\sum_{j\in\mathrm{N}_i(t)} a_{ij}(t) = \sum_{i\in\mathrm{N}_j(t)} a_{ji}(t)$ for all $t \geq 0$. Suppose in addition that $\mathrm{G}_{\sigma(t)}$ is either δ-UJSC or δ-BIJC. Then

$$\lim_{t\to\infty} \mathbf{x}_i(t) = \sum_{i=1}^N \mathcal{P}_{\mathcal{A}}\big(\mathbf{x}_i(0)\big)/N,\ \forall i \in \mathrm{V}.$$

(ii) Suppose $\mathrm{G}_{\sigma(t)} \equiv \mathrm{G}^\S$ for some fixed, strongly connected, digraph $\mathrm{G}^\S$ and for any $i, j \in \mathrm{V}$, $a_{ij}(t) \equiv a_{ij}^\S$ for some constant $a_{ij}^\S$. Let $\mathbf{w} := (w_1 \dots w_N)^{\mathrm{T}}$ with $\sum_{i=1}^N w_i = 1$ be the left eigenvector corresponding to the simple eigenvalue zero of the Laplacian[2] $\mathbf{L}^\S$ of the

[2] The Laplacian matrix $\mathbf{L}^\S$ associated with the graph $\mathrm{G}^\S$ under the given arc weights is defined as $\mathbf{L}^\S = \mathbf{D}^\S - \mathbf{A}^\S$ where $\mathbf{A}^\S = [\mathbb{I}_{(j,i)\in\mathrm{E}^\S} a_{ij}^\S]$ and $\mathbf{D}^\S = \mathrm{diag}\big(\sum_{j=1}^N \mathbb{I}_{(j,1)\in\mathrm{E}^\S} a_{1j}^\S, \dots, \sum_{j=1}^N \mathbb{I}_{(j,N)\in\mathrm{E}^\S} a_{Nj}^\S\big)$. In fact, we have $w_i > 0$ for all $i \in \mathrm{V}$ if $\mathrm{G}^\S$ is strongly connected [6].

digraph $\mathrm{G}^{\S}$. Then we have

$$\lim_{t\to\infty}\mathbf{x}_i(t)=\sum_{i=1}^{N}w_i\mathcal{P}_{\mathcal{A}}\big(\mathbf{x}_i(0)\big),\ \forall i\in\mathrm{V}.$$

Note that $\mathrm{G}_{\sigma(t)}$ is balanced if $\mathrm{G}_{\sigma(t)}$ is undirected with the $a_{ij}(t)$ being symmetric. We remark that due to the linear nature of the systems (3) and (4), the convergence stated in Theorems 1, 2, 3 and 4 is exponential if $\mathrm{G}_{\sigma(t)}$ is periodic and δ-UJSC.

IV. Least Squares Solutions

In this section, we turn to Case (III) and consider that (1) admits a unique least squares solution $\mathbf{y}^{\star}$. Evidently, neither of the two continuous-time distributed flows (3) and (4) in general can yield exact convergence to the least squares solution of (1) since, even for a fixed interaction graph, $\mathbf{y}^{\star}$ is not an equilibrium of the two network flows.

It is indeed possible to find the least squares solution using double-integrator node dynamics [19], [20]. However, the use of double integrator dynamics was restricted to networks with fixed and undirected (or balanced) communication graphs [19], [20]. On the other hand, one can also borrow the idea of the use of square-summable step-size sequences with infinite sums in discrete-time algorithms [21] and build the following flow

$$\dot{\mathbf{x}}_i=K\Big(\sum_{j\in\mathrm{N}_i(t)}a_{ij}(t)\big(\mathbf{x}_j-\mathbf{x}_i\big)\Big)+\frac{1}{t}\Big(\mathcal{P}_{\mathcal{A}_i}(\mathbf{x}_i)-\mathbf{x}_i\Big), \tag{7}$$

for $i\in\mathrm{V}$. The least squares case can then be solved under graph conditions of connectedness and balance [21], but the convergence rate is at best $O(1/t)$. This means (7) will be fragile against noises.

For the "projection+consensus" flow, we can show that under fixed and connected bidirectional interaction graphs, with a sufficiently large K, the node states will converge to a ball around the least squares solution whose radius can be made arbitrarily small. This approximation is global in the sense that the required K only depends on the accuracy between the node state limits and the $\mathbf{y}^{\star}$.

Theorem 6: Let (III) hold with $\mathrm{rank}(\mathbf{H})=m$ and denote the unique least squares solution of (1) as $\mathbf{y}^{\star}$. Suppose $\mathrm{G}_{\sigma(t)}=\mathrm{G}^{\S}$ for some bidirectional, connected graph $\mathrm{G}^{\S}$ and for any $i,j\in\mathrm{V}$, $a_{ij}(t)=a_{ji}(t)\equiv a_{ij}^{\S}$ for some constant $a_{ij}^{\S}$. Then along the flow (3), for any $\epsilon>0$, there exists $K_*(\epsilon)>0$ such that $\mathbf{x}(\infty):=\lim_{t\to\infty}\mathbf{x}(t)$ exists and

$$\big\|\mathbf{x}_i(\infty)-\mathbf{y}^{\star}\big\|\leq\epsilon,\ i\in\mathrm{V}$$

for any initial value $\mathbf{x}(0)$ if $K\geq K_*(\epsilon)$.

We also manage to establish the following semi-global result for switching but balanced graphs.

Theorem 7: Let (III) hold with $\mathrm{rank}(\mathbf{H})=m$ and denote the unique least squares solution of (1) as $\mathbf{y}^{\star}$. Suppose $\mathrm{G}_{\sigma(t)}$ is balanced for all $t\in\mathbb{R}^+$ and δ-UJSC with respect to $T>0$. Let the following assumptions hold:

[A1] The set $\mathcal{W}(\mathbf{y}):=\big\{\mathcal{P}_{i_J}\cdots\mathcal{P}_{i_1}(\mathbf{y}):\ \ i_1,\dots,i_J\in\mathrm{V},J\geq1\big\}$ is a bounded set;

[A2] $\sum_{i=1}^{N}\mathcal{P}_{\mathcal{A}_i}(0)=0$.

Then along the flow (3), for any $\epsilon>0$ and any initial value $\mathbf{x}(0)\in\mathcal{A}_1\times\cdots\mathcal{A}_N$, there exist $K_*(\epsilon,\mathbf{x}(0))>0$ and $T_*(\epsilon,\mathbf{x}(0))$ such that

$$\limsup_{t\to\infty}\big\|\mathbf{x}_i(t)-\mathbf{y}^{\star}\big\|\leq\epsilon,\ i\in\mathrm{V}$$

if $K\geq K_*(\epsilon)$ and $T\leq T_*(\epsilon)$.

Remark 2: The two assumptions, [A1] and [A2] are indeed rather strong assumptions. In general, $[A1]$ holds if $\bigcap_{i=1}^{N}\mathcal{A}_i$ is a nonempty bounded set [28], which is exactly opposite to the least squares solution case under consideration. We conjecture that at least for $m=2$ case with $\mathbf{h}_i$ being pairwise distinct, $[A1]$ should hold. The assumption $[A2]$ requires a strong symmetry in the affine spaces $\mathcal{A}_i$, which turns out to be essential for the result to stand.

For the "projection consensus" flow, we present the following result.

Theorem 8: Let (III) hold with $\mathrm{rank}(\mathbf{H})=m$ and denote the unique least squares solution of (1) as $\mathbf{y}^{\star}$. Suppose $\mathrm{G}_{\sigma(t)}=\mathrm{G}^{\S}$ is fixed, complete, and $a_{ij}(t)\equiv a^{\S}>0$ for all $i,j\in\mathrm{V}$. Then along the flow (4), for any initial value $\mathbf{x}(0)\in\mathcal{A}_1\times\cdots\mathcal{A}_N$, there holds

$$\lim_{t\to\infty}\frac{\sum_{i=1}^{N}\mathbf{x}_i(t)}{N}=\mathbf{y}^{\star}. \tag{8}$$

Based on simulation experiments (which will be presented later), we conjecture that the requirement that $\mathrm{G}_{\sigma(t)}=\mathrm{G}^{\S}$ does not have to be complete for the Theorem 8 to be valid.

V. Conclusions

Two distributed network flows were studied as distributed solvers for a linear algebraic equation $\mathbf{z}=\mathbf{H}\mathbf{y}$, where a node i holds a row $\mathbf{h}_i^{\mathrm{T}}$ of the matrix $\mathbf{H}$ and the entry z_i in the vector $\mathbf{z}$. A "consensus + projection" flow consists of two terms, one from standard consensus dynamics and the other as projection onto each affine subspace specified by the $\mathbf{h}_i$ and z_i. Another "projection consensus" flow simply replaces the relative state feedback in consensus dynamics with projected relative state feedback. Under mild graph conditions, it was shown that that all node states converge to a common solution of the linear algebraic equation, if there is any. The convergence is global for the "consensus + projection" flow while local for the "projection consensus" flow. When the linear equation has no exact solutions, it was proved that the node states can converge to somewhere arbitrarily near the least squares solution as long as the gain of the consensus dynamics is sufficient large for "consensus + projection" flow under fixed and undirected graphs. It was also shown that the "projection consensus" flow drives the average of the node states to the least squares solution if the communication graph is complete.

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