2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Formation Control on Lines, Circles and Ellipses: Genericity Results and Morse Theoretic Ideas

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Abstract— **In this work we consider natural potential functions for 1-dimensional formation control problems on the real line, the circle and ellipses. It is shown that generically such functions on the line and the circle are Morse functions, i.e., their critical points are nondegenerate. This property is important in order to establish sharp upper and lower bounds for the number of critical points. For formations of higher dimensional agents it is an open problem to decide whether the Morse property is satisfied for generic choices of desired distances. For the circular case we apply methods from algebraic geometry, such as Bézout's theorem and the Bernstein-Kushnirenko-Khovanski theorem, to provide novel upper bounds on the number of critical points. These upper bounds grow exponentially in the number N of point agents, which indicates the underlying complexity of the problem of characterizing critical formations. Studying formations on an arbitrary curve is much more complicated and may lead to the generic appearance of degenerate critical points. We provide an example of a family of potential functions on an ellipse that is never a Morse function.**

I. INTRODUCTION

In formation control one frequently seeks to steer a group of N agents such that the group assumes a specific shape and/or the group moves as a rigid body with one agent or the centroid following a specified trajectory in an underlying ambient space, $\mathbb{R}^2$ say. The task of developing control laws to achieve such objectives has received much attention in the last years, motivated by applications in the use of multiple agents to shift a single heavy object [1], or in the use of multiple agents such as unmanned airborne vehicles (UAVs) to localise ground targets. Note that in many localization problems, there are optimum positions for the agents arising from Cramer-Rao bound considerations [2], [3] In many of these applications, the agents are regarded as point agents, modelled kinematically by a first order integrator, and this will be our approach in this paper.

One central task is then to construct a control law which ensures that the group of agents assumes a desired shape; this shape is often described by the fixed distances required between the agents. With N agents and M interagent communication links, there are $M \leq (1/2)N(N-1)$ such distances. Thus, especially for large formations, to avoid issues of scalability, advantage is taken of the fact that specification of $O(N)$ inter-agent distances, appropriately chosen, can be enough to uniquely define the formation shape, see e.g. [4]. Again in the interests of scalability one is usually interested in distributed control laws, that is, each agent implements its own control law which takes into account information obtained from other, neighbouring agents but not generally from all agents in the formation (unless the formation is small). Thus the control law is decentralized and reflects communication constraints which are described by an communication graph $\Gamma = (\mathcal{V}, \mathcal{E})$. The N nodes of the graph correspond to the respective agents and the M edges to communication links (which coincide with the agent pairs for which there is a prescribed distance in the desired formation.)

In this paper, we assume that the graph is constant over time, undirected, i.e., communication/control is bidirectional, and has no self-loops. Moreover, it is assumed that each agent has access to the relative positions of its neighbours, and ability to set its velocity. It is conventional also to note that each agent can have its own coordinate basis, but pursuing the analysis at that level in this paper would add an unnecessary complication. Furthermore, we assume throughout this paper that the graph Γ is connected, as the condition is necessary for all results presented here.

If the dynamics of the agents are given by simple first-order integrator dynamics in an ambient space $\mathbb{R}^d$, a commonly used approach is use the gradient descent vector field for a potential function of the form

$$V(x_1, \ldots, x_N) = \sum_{ij \in \mathcal{E}} (\|x_i - x_j\|^2 - d_{ij}^2)^2, \qquad (1)$$

see e.g. [5], [6]. This yields a distributed control law which seeks local minima of the potential function. If the distances d_{ij} are realizable, i.e., there exist formations for which the interagent distances coincide with the specified ones, then formations realizing the distances correspond to stable equilibria of the closed-loop dynamics and a global minimum for V.

This observation raises the question in relation to the convergence to formations with the desired shape as to whether such a gradient law is (almost) globally convergent, or if other local minima exist. A line of work to tackle this question was opened by papers [7], [8] where Morse-theoretic and algebraic-geometric approaches to this problem were proposed (That there exist critical points of V which are not local minima is fairly easy to establish.) In this work we review some of the results from [8] for formations

978-1-4799-7885-4/15/$31.00 ©2015 IEEE

on the real line and from [9] for formations on the unit circle S^1. Furthermore, we provide some new algebraic-geometric bounds for the S^1 case and discuss briefly the extension to formations on ellipses. These approaches cannot straightforwardly extended to higher dimensional spaces and we leave this topic for future research.

II. Formations on the real line

In this section we discuss the case of point agents on the real line and review the results from [8]. The dynamics of the one-dimensional agents are given by

$$\dot{x}_j = u_j$$

with u_j a control input. The potential function takes the form

$$V(\mathbf{x}) = \frac{1}{4}\sum_{ij\in\mathcal{E}}((x_i - x_j)^2 - d_{ij}^2)^2. \tag{2}$$

As the d_{ij} are the desired distances between the agents, one can view V as an error measure for the difference between the formation described by $\mathbf{x}$ and the desired formation.

The gradient descent law

$$\dot{x} = -\nabla V(\mathbf{x})$$

yields for the ith agent the dynamics

$$\dot{x}_i = -\sum_{ij\in\mathcal{E}}((x_i - x_j)^2 - d_{ij}^2)(x_i - x_j). \tag{3}$$

The potential function V is invariant under translation of the group of agents. Hence, one can define a *reduced potential*

$$V_r(x_1,\ldots,x_{N-1}) = V(x_1,\ldots,x_{N-1},0)$$

which will be used henceforth. A critical point of V_r corresponds to a manifold of critical points of V; these manifolds are 1-dimensional affine subspaces in $\mathbb{R}^N$. Hence a bound on the number of critical points of V_r yields a bound on the number of lines of critical points of V.

A Morse function [10] on a manifold is a real-valued, smooth function with compact sublevel sets, such that all critical points are non-degenerate, i.e., the Hessian of the function at each critical point has full rank. Morse theory provides lower bounds on the number of critical points of Morse functions via formulas involving topological characteristics of the manifold. It was shown in [8], Thm 11, that for generic d_{ij} the reduced potential is a Morse function and lower bounds on the number of critical points were obtained.

Theorem 1 ([8]): There is an open, dense set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ such that V_r is a Morse function. The number of critical points is finite and bounded below by $2N-1$. Furthermore, there are at least $2N-4$ saddle points.

Note that it is implicitly assumed that the $(d_{ij})_{ij\in\mathcal{E}}$ are free, i.e., not subject to any algebraic constraint on the paramters d_{ij}. However, it was also shown in [8] that if Γ is a spanning subgraph of a minimally rigid graph for formations in $\mathbb{R}^d$ then for an open, dense set of reference formations $(x_1^*,\ldots,x_N^*)\in\mathbb{R}^{d\times N}$, i.e., for reference formations in an d-dimensional Euclidean space, the potential function (2) with prescribed distances $d_{ij} := \|x_i^* - x_j^*\|$ is Morse.

For upper bounds on the number of critical points of V_r one has to resort to tools from complex algebraic geometry. Observe that the equation

$$0 = \nabla V_r(x_1,\ldots,x_{N-1})$$

can be regarded as $N-1$ polynomial equations ($i = 1,\ldots,N-1$)

$$0 = p_i(x_1,\ldots,x_{N-1}) \tag{4}$$

in $\mathbb{C}^{N-1}$. The critical points of V_r can be identified with the real solutions of these equations. Complex algebraic geometry provides methods to bound the number of solutions (but without distinguishing real from complex solutions).

The first approach used in [8] was Bézout's theorem, which states that the number of isolated solutions is bounded above by the product of the degrees of the polynomials p_i. This yields the following bound.

Theorem 2 ([8]): There is an open, dense set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ such that V_r has at most 3^{N-1} critical points.

The second approach from [8] is the Bernstein-Kushnirenko-Khovanski (BKK), [11], [12], [13], bound. For this bound one has first to introduce the mixed volume of k polytopes $\mathcal{N}_1,\ldots,\mathcal{N}_k$ in $\mathbb{R}^k$ as

$$\begin{aligned} MV(\mathcal{N}_1,\ldots,\mathcal{N}_k) &= (-1)^{k-1}\sum_{j=1}^{k}\mathrm{vol}(\mathcal{N}_j) \\ &\quad + (-1)^{k-2}\sum_{1\le i<j\le k}\mathrm{vol}(\mathcal{N}_j+\mathcal{N}_i) \\ &\quad + \cdots + \mathrm{vol}(\mathcal{N}_1+\cdots+\mathcal{N}_k). \end{aligned}$$

where vol denotes the k-dimensional volume of a bounded set in $\mathbb{R}^k$. For a Laurent polynomial[1] $p \in \mathbb{C}[x_1, x_1^{-1},\ldots,x_k,x_k^{-1}]$, $p(\mathbf{x}) = \sum_{\alpha\in\mathbb{Z}^k} a_\alpha \mathbf{x}^\alpha$ one defines successively the support

$$\mathrm{supp}\, p = \{\alpha\in\mathbb{Z}^k \mid a_\alpha \neq 0\},$$

the Newton polytope

$$\mathcal{N}_p = \mathrm{conv}\,\mathrm{supp}\, p$$

and the extended Newton polytope

$$\hat{\mathcal{N}_p} = \mathrm{conv}(\{0\}\cup\mathrm{supp}\, p).$$

Here, conv denotes the convex hull of a subset in $\mathbb{R}^k$. The BKK bound states that the number of isolated common zeros of k Laurent polynomials in $(\mathbb{C}\setminus\{0\})^k$ is bounded by the mixed volume of the Newton polytopes of the polynomials [11], [12], [13].

Since 0 is a valid value for any agent position, in [8] the authors resorted to using a slightly more general result of

[1] A complex polynomial $p\in\mathbb{C}[x_1,\ldots,x_k]$ is a special case of a Laurent polynomial. We detail the bound for Laurent polynomials as these appear in the circular case later on.

Li and Wang [14] which bounds the number of common roots of k polynomials in $\mathbb{C}^k$ by the mixed volume of their extended Newton polytopes. This next result refines Thm 9 in [8].

Theorem 3: Let e_i denoting the i-th standard basis vector in $\mathbb{R}^{N-1}$ and a_{ij} be the ij-entry of the adjacency matrix of the connected graph Γ. There is an open, dense set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ such that V_r has at most $MV(\mathcal{N}_1,\ldots,\mathcal{N}_{N-1})$ critical points with the polytope $\mathcal{N}_i$ given by

$$\mathrm{conv}_{1\le j\le N-1}\{0,3e_i,a_{ij}3e_j,a_{ij}(2e_i+e_j),a_{ij}(2e_j+e_i)\},$$

The mixed volume $MV(\mathcal{N}_1,\ldots,\mathcal{N}_{N-1})$ is equal to 3^{N-1}.

Proof: For the first claim we refer to [8]. To compute the mixed volume, let $B_i := \mathrm{conv}\{0,3e_i\}$ for $i=1,\ldots,N-1$. Then $B_i\subset\mathcal{N}_i$. Since the Minkowski sums $B_{i_1}+\ldots+B_{i_k}$, $1\le i_1<\ldots<i_k\le N-1$ have $N-1$ dimensional volume 0 unless $k=N-1$, we obtain

$$\begin{aligned}MV(B_1,\ldots,B_{N-1}) &= \mathrm{vol}(B_1+\ldots+B_{N-1})\\ &= \mathrm{vol}([0,3]^{N-1})=3^{N-1}.\end{aligned}$$

Since the mixed volume in monotonically increasing in each of its arguments, we obtain

$$MV(B_1,\ldots,B_{N-1})\le MV(\mathcal{N}_1,\ldots,\mathcal{N}_{N-1}).$$

As the upper bound 3^{N-1} for the mixed volume $MV(\mathcal{N}_1,\ldots,\mathcal{N}_{N-1})$ was already established in [8], our claim follows. ■

Theorem 3 implies that the BKK bound coincides with the Bézout bound, for arbitrary connected graphs Γ. However, the Bézout bound can be further improved for specific graphs by taking into account the zeros at infinity of the system of polynomial equations (4). This was exploited in [8] for the analysis of few-agent cases ($N=3$ and $N=4$).

For small number of agents, i.e., for $N=3$ and $N=4$, and a complete graph a more detailed analysis of the critical points of V_r was given in [8] (Thm. 12 and Thm. 13):

- $N=3$. There are exactly 5 critical points, in contrast to the 9 critical points, as predicted by the BKK bound: two global minima (x_1^*,x_2^*), two saddle points $(\hat{x}_1,\hat{x}_2)$ and a local maximum at $(0,0)$.
- $N=4$. There are at least 7 critical points - two global minima, 4 saddle points and a local maximum. Furthermore, there are exactly 27 complex critical formations. The coordinates of no two critical agents coincide, except when all agents are collocated.

Remarkably, for the case that Γ is complete, in the recent work [15], Theorem 3.8, the following formula for the precise number for complex solutions of (4) was determined. This was the first instance of a general formula for the exact number of complex solutions, albeit only for complete graphs.

Theorem 4 ([15]): Let Γ be complete. There is an open, dense set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ such that

TABLE I
BOUNDS FOR CRITICAL FORMATIONS ON $\mathbb{R}$; Γ COMPLETE

N	Morse lower	BKK/Bézout upper
3	5	9
4	7	27
5	9	81
6	11	243
7	13	729

1) if $N\equiv 1,2 \mod 3$ then (4) has exactly 3^{N-1} complex zeros,
2) if $N\equiv 0 \mod 3$ then (4) has exactly $3^{N-1}-\frac{2N!}{3((N/3)!)^3}$ complex zeros.

From a control-engineering point of view, it would be very helpful to understand in more details how many stable equilibria there are, and how these might be characterized. However, besides the aforementioned results for $N=3$ and $N=4$, a detailed analysis is still missing.

III. FORMATIONS ON THE UNIT CIRCLE

We next discuss the case where the agents move on the unit circle. We are motivated to consider this case by the large amount of work regarding synchronization phenomena on S^1, [16], [17] and in particular their relevance to engineering applications like synchronization in power networks [18]. Agents positioning on a planar, convex curve by distributed control algorithms has been considered in [19] for approximation of a convex contour by a polygon and in [20] for optimal sensor placement for tracking applications. To begin, we first review the Morse theoretic bounds from [9]. From Proposition 1 on, we provide some new, general algebraic bounds.

For a more terse notation we assume that the unit circle is embedded in the complex plane, i.e.,

$$S^1=\{z\in\mathbb{C}\mid |z|=1\}.$$

The dynamics of simple, first order integrator agents on S^1 is given by

$$z_j=\mathbf{i}z_ju_j$$

with u_j a real-valued control input and $\mathbf{i}=\sqrt{-1}$. The collective state of the agents is a point in the N-torus $\mathbb{T}^N=S^1\times\cdots\times S^1$.

For formations on S^1 we consider the potential function

$$V(\mathbf{z})=\frac{1}{4}\sum_{ij\in\mathcal{E}}(|z_i-z_j|^2-d_{ij}^2)^2.$$

To formulate a gradient descent law, we need a Riemannian metric on $\mathbb{T}^N$. To this end we identify $\mathbb{C}$ with the Euclidean space $\mathbb{R}^2$ and equip S^1 with the induced Riemannian metric, which is the restriction of the Euclidean scalar product to the tangent spaces of S^1. This yields a Riemannian metric on $\mathbb{T}^N$. The gradient of V with respect to this metric is easily computed and the gradient law

$$\dot{\mathbf{z}}=-\,\mathrm{grad}\,V(\mathbf{z})$$

yields the individual agent dynamics

$$\dot{z}_i = -\frac{1}{2} z_i \sum_{\substack{j=1,\ldots,N\\ ij\in\mathcal{E}}} (|z_i - z_j|^2 - d_{ij}^2)(z_i\overline{z}_j - \overline{z}_i z_j).$$

The potential V is invariant under rotations of the formations along the unit circle; these can be realized by multiplying the entries of the N-tuple $\mathbf{z}$ with a fixed element $a \in S^1$. Therefore, one considers again a reduced potential

$$V_r(z_1, \ldots, z_{N-1}) = V(z_1, \ldots, z_{N-1}, 1).$$

A critical point of V_r corresponds to a 1-dimensional manifold of critical points of V of the form $\{(az_1, \ldots, az_{N-1}, a) \mid a \in S^1\}$.

In [9] it was shown in an actually more general setting for the potential function that generically V_r is a Morse function and bounds on the number of critical points were obtained (cf. [9], Thm 1).

Theorem 5: There is an open, dense set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ such that V_r is a Morse function and has at least 2^{N-1} critical points.

Note that, once again, the d_{ij} are free and not subject to the constraint of being realizable. Again, this raises the question as to whether the genericity result can be extended to the case where the d_{ij} are replaced with distances in reference formations $(z_1^*, \ldots, z_N^*) \in \mathbb{T}^N$. Note, that the combination of Γ connected and Γ is a subgraph of a minimally rigid graph for formations in $\mathbb{R}^1$ with the same vertices, is equivalent to Γ being a tree. Therefore, we have to establish the result for trees. Fortunately, in this case it is simply a corollary for a genericity result established for trees in [9].

Corollary 1: Let Γ be a tree. For an open, dense set of reference formations $(z_1^*, \ldots, z_N^*) \in \mathbb{T}^N$ the reduced potential function

$$\frac{1}{4} \sum_{ij\in\mathcal{E}} (|z_i - z_j|^2 - |z_i^* - z_j^*|^2)^2; \quad z_N = 1$$

is a Morse function.

Proof: By Thm 2 in [9], V_r is a Morse function for Γ a tree and all $d_{ij}^2 \in (0,4)$. Since the map $\mathbb{T}^N \to [0,4]^{N-1}$, $\mathbf{z} \mapsto (\ldots, |z_i - z_j|^2, \ldots)$, is a surjective, real-algebraic map, the preimage of $(0,4)^{N-1}$ is an open, dense subset of $\mathbb{T}^N$. ∎

Note that for Γ a tree and all $d_{ij} \in (0,2)$ the reduced potential has exactly 4^{N-1} critical points and the only local minima correspond to the formations realizing the given distance.

As in the case of the real line, we can use algebraic geometric tools to obtain upper bounds for the number of critical points. To this end, we extend the $N-1$ equations

$$0 = -\frac{1}{2} z_i \sum_{\substack{j=1,\ldots,N\\ ij\in\mathcal{E}}} (|z_i - z_j|^2 - d_{ij}^2)(z_i\overline{z}_j - \overline{z}_i z_j)$$

on $\mathbb{T}^{N-1}$ with $z_N = 1$ to $(\mathbb{C}\setminus\{0\})^{N-1}$ by replacing $\overline{z}_j$ with z_j^{-1}. Further we multiply the equation by z_i^{-1} — as we are only interested in roots in $(\mathbb{C}\setminus\{0\})^{N-1}$ this does not change anything. We obtain a system of $N-1$ equations

$$0 = \sum_{\substack{j=1,\ldots,N\\ ij\in\mathcal{E}}} (2 - z_i z_j^{-1} - z_i^{-1} z_j - d_{ij}^2)(z_j z_i^{-1} - z_i z_j^{-1}) \quad (5)$$

with $z_N = 1$.

Since we are now interested in the common zeros of a set of Laurent polynomials in $(\mathbb{C} \setminus \{0\})^{N-1}$, we will directly apply the BKK bound. However, we first have to establish that generically the common zeros of (5) are non-degenerate, i.e., the Jacobian of the map $\mathbb{C}^{N-1} \to \mathbb{C}^{N-1}$ given by the polynomials on the right hand side of (5) is non-degenerate at the zeros. The parametric transversality theorem, [10], is a crucial tool in establishing that for generic d_{ij} the critical points of V_r are non-degenerate. To prove that the zeros of (5) we need the following result.

Proposition 1: Let $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$ be Zariski-open. Consider a regular function $F\colon X \times Y \to \mathbb{C}^n$. Set $\Delta = \{(z,u) \in X \times Y \mid F(z,u) = 0\}$. If for all $(z,u) \in \Delta$ $\operatorname{rank} DF(z,u) = n$, then there is a Zariski-open, dense subset Q of Y such that for all $(z,u) \in \Delta$ with $u \in Q$ we obtain $\operatorname{rank} D_z F(z,u) = n$.

Due to space restrictions, we omit the proof of Proposition 1. It is similar to the proof of the well-known parametric transversality theorem, [10], using Bertini's theorem, [21], instead of Sard's theorem.

Theorem 6: For a generic, semialgebraic set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ the roots of (5) in $(\mathbb{C} \setminus \{0\})^{N-1}$ are non-degenerate and isolated.

Proof: As a detailed proof is rather lengthy and technical, we just provide a sketch of the argument. We write the right-hand side of (5) as a function $Q\colon (\mathbb{C}\setminus\{0\})^{N-1} \times \mathbb{C}^M \to \mathbb{C}^{N-1}$, $(\mathbf{z}, \mathbf{d}) \to Q(\mathbf{z}, \mathbf{d})$ in the vector $\mathbf{z}$ of the z_i and the vector $\mathbf{d}$ of the d_{ij}. With some technical arguments we can show that $\operatorname{rank} DQ(\mathbf{z}^*, \mathbf{d}^*) = N-1$ at a zero $(\mathbf{z}^*, \mathbf{d}^*)$ of Q. By Proposition 1 there is an Zariski-open, dense subset Ω_1 of $\mathbb{C}^M$ such that for all $\mathbf{d}^* \in \Omega_1$ the map $\mathbf{z} \mapsto Q(\mathbf{z}, \mathbf{d}^*)$ is transversal to $\{0\}$, i.e., the zeros of this map are non-degenerate. Consider $\Omega_2 = \Omega_1 \cap \mathbb{R}^M$. This set is Zariski-open in $\mathbb{R}^M$ (w.r.t. the Zariski-topology on $\mathbb{R}^M$). If this set is empty, $\mathbb{C}^M \setminus \Omega_1$ contains $\mathbb{R}^M$. However, this implies that Ω_1 is empty. Hence Ω_2 is non-open and therefore dense in $\mathbb{R}^M$. ∎

Applying the BKK bound in the generic case we obtain the following bound.

Theorem 7: For a generic, semialgebraic set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ V_r has at most $MV(\mathcal{N}_1, \ldots, \mathcal{N}_{N-1})$ critical points with $\mathcal{N}_i$ given as

$$\operatorname{conv}(\{-2e_i + 2e_j, 2e_i - 2e_j \mid ij \in \mathcal{E}, 1 \le j < N\} \cup Q_i)$$

and

$$Q_i = \begin{cases} \emptyset & \text{for } iN \notin \mathcal{E} \\ \{-2e_i, 2e_i\} & \text{for } iN \in \mathcal{E} \end{cases}.$$

Proof: By Theorem 6 there is a generic, semialgebraic set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ such that the roots of (5) in $(\mathbb{C}\setminus\{0\})^{N-1}$ are non-degenerate and isolated. Thus we can for these $(d_{ij})_{ij\in\mathcal{E}}$ the BKK bound. The system of equations (5) consists of Laurent polynomials containing the monomials

$$z_i z_j^{-1}, z_j z_i^{-1}, z_i^2 z_j^{-2}, z_i^{-2} z_j^2$$

if and only if $ij\in\mathcal{E}$. This gives the Newton polytopes $\mathcal{N}_i$ as stated above for the right hand sides of (5). Applying the BKK bound yields the result. ∎

To use the Bézout bound on (5), one would have to multiply the equations with appropriate powers of the relevant z_i to get rid of the inverse powers. However, applying e.g. the affine Bézout bound, [22], to these equations leads to the bound $(2N)^{N-1}$. To obtain a better bound we replace the inverse z_i^{-1} by w_i and provide additional constraint equations $z_i w_i - 1 = 0$. This gives the system of $2N-2$ equations of the form

$$0 = \sum_{\substack{j=1,\dots,N\\ ij\in\mathcal{E}}} (2 - z_i w_j - w_i z_j - d_{ij}^2)(z_j w_i - z_i w_j) \quad (6)$$

$$0 = z_i w_i - 1 \quad (7)$$

For these equations, we can provide a similar genericity result.

Theorem 8: For a generic, semialgebraic set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ the roots of the system of $2N-2$ equations (6),(7) in $\mathbb{C}^{2N-2}$ are non-degenerate and isolated.

Proof: The proof is similar to the proof of Theorem 6. We define the function $Q\colon \mathbb{C}^{N-1}\times\mathbb{C}^{N-1}\times\mathbb{C}^M \to \mathbb{C}^{N-1}\times\mathbb{C}^{N-1}$, $(\mathbf{z},\mathbf{w},\mathbf{d}) \mapsto Q(\mathbf{z},\mathbf{w},\mathbf{d})$ as the right-hand side of (6), (7). Some technical arguments yield that $\operatorname{rank} DQ(\mathbf{z}^*,\mathbf{w}^*,\mathbf{d}^*) = 2N-2$ at a zero $(\mathbf{z}^*,\mathbf{w}^*,\mathbf{d}^*)$ of Q . We can apply again Theorem 1 and remaining argument follows the proof of Theorem 6. ∎

An application of the affine Bézout theorem to (6), (7) gives the following bound.

Theorem 9: For a generic, semialgebraic set of $(d_{ij})_{ij\in\mathcal{E}}$ in $\mathbb{R}_+^M$ the reduced cost function V_r has at most 8^{N-1} critical points.

Proof: By Theorem 8 the zeros of the system of equations (6), (7) are isolated and non-degenerate. Since the polynomials in (6) have degree 4 and the polynomials in 7 have degree 2 we obtain from the affine Bézout theorem, see [22], the bound

$$4^{N-1}2^{N-1}.$$

∎

In Table II we provide the bounds for a complete graph of $3,\dots,7$ agents. Note, that for formations on S^1 the BKK and Bézout bounds do not coincide, even if the graph Γ is complete. The mixed volume for the bound from Theorem 7 was computed with the software package MixedVol from [23].

TABLE II

BOUNDS FOR CRITICAL FORMATIONS ON S^1, Γ COMPLETE

N	Morse lower	BKK upper	Bézout upper
3	4	24	64
4	8	160	512
5	16	1120	4096
6	32	8064	32768
7	64	59136	262144

IV. Two agents on an ellipse

It is a natural question to ask whether the results and methods employed for formations on the real line and the unit circle can be extended to other curves. As a simple example we consider the ellipse

$$\mathcal{C} = \{z\in\mathbb{C} \mid \frac{(\Re z)^2}{a^2} + \frac{(\Im z)^2}{b^2} = 1\}, \quad a > b > 0.$$

Extending the analysis to more general situations is a topic for future research.

We will show that the following assertions hold for an ellipse, but not for a circle:

- It is no longer adequate to consider the question of whether or not non-degenerate critical points occur. In fact, there exist sets of positive Lebesgue measure of distances for which they do, and others for which they don't.
- Since ellipses are in a sense arbitrarily close to circles, this suggests a lack of robustness about the results for the circle.

As a very simple situation let us consider two agents linked by a single communications link. The potential function then is

$$V(z_1,z_2) = \frac{1}{4}(|z_1 - z_2|^2 - d_{12}^2)^2$$

and the gradient feedback law is easily computed. To analyse the critical points of V we use the parametrization of $\mathcal{C}$ as

$$\phi(t) = a\cos(t) + b\sin(t)\mathbf{i}.$$

Define the local diffeomorphism $\phi\colon \mathbb{R}^2 \to \mathcal{C}\times\mathcal{C}$, $\phi(t_1,t_2) = (\phi(t_1),\phi(t_2))$. We can write $V\circ\phi(t_1,t_2) = \frac{1}{4}(f(t_1,t_2) - d_{12}^2)^2$ with $f(t_1,t_2) = |\phi(t_1)-\phi(t_2)|^2$. Then gradient and Hessian are given as

$$\nabla(V\circ\phi)(t_1,t_2) = \frac{1}{2}(f(t_1,t_2) - d_{12}^2)\nabla f(t_1,t_2)$$

and

$$H_{V\circ\phi}(t_1,t_2) = \frac{1}{2}\nabla f(t_1,t_2)\nabla f(t_1,t_2)^\top + \frac{1}{2}(f(t_1,t_2) - d_{12}^2) H_f(t_1,t_2).$$

Thus V has a critical point at $(\phi(t_1),\phi(t_2))$ if and only if f has a critical point at (t_1,t_2) or $(f(t_1,t_2) - d_{12}^2)$ is zero. Furthermore, a non-degenerate critical point of f is a non-degenerate critical point of V, unless $f(t_1,t_2) - d_{12}^2 = 0$.

Simple calculations show that the critical points of f in $[0,2\pi]^2$ are given by

$$\{(0,\pi),(\pi,0),(\tfrac{1}{2}\pi,\tfrac{3}{2}\pi),(\tfrac{3}{2}\pi,\tfrac{1}{2}\pi)\}\cup\{(t,t)\mid t\in[0,2\pi]\}$$

and that

$$\{(0,\pi),(\pi,0),(\tfrac{1}{2}\pi,\tfrac{3}{2}\pi),(\tfrac{3}{2}\pi,\tfrac{1}{2}\pi)\}$$

are non-degenerate, since $a \neq b$.

Consider now V. Unless d_{12}^2 coincides with $4a^2$ or $4b^2$, the non-degenerate critical points of f are also non-degenerate critical points of V. However, if $d_{12}^2 \in (0,4a^2)$ we obtain additional degenerate critical points of V given by

$$\{(z_1,z_2)\in\mathcal{C}^2 \mid |z_1-z_2|^2 = d_{12}^2\}.$$

This set consists of two continua, thus leading to a family of degenerate critical points of V.

Thus for the open set of desired distances with $0 < d_{12} < 2a$, the potential function V has degenerate critical points besides $\{(z,z) \mid z\in\mathcal{C}\}$. Due to the presence of isolated, non-degenerate critical points, a symmetry-based dimension reduction as in the real line and circle case is generically not possible for the cost function V on the ellipse $\mathcal{C}$. Hence, the techniques used for the real line and S^1 are not directly applicable for formations on the ellipse.

V. Conclusions and Open Problems

In the previous sections we discussed the reduced cost functions V_r for 1-dimensional formations - on the real line and on S^1. It is shown that, in both cases, the potential functions V_r are generically Morse functions, thus leading to explicit lower and upper bounds for the number of critical formations.

In the higher dimensional situation nothing much is known on the Morse property of the potential functions. The results in [7] have been established under the assumption that the potential function V is Morse for some selection of desired distances; but this condition has not been verified so far. In [24] bounds on the number of critical orbits of (1), i.e., on the numer of orbits of critical points of (1) under the action of Euclidean transformations, were established under the assumption that the potential is an equivariant Morse function. Again, it is unknown if the potential is generically an equivariant Morse function.

As a concluding statement, we want to point out that proving genericity of Morse potential functions and an analysis of the critical points is relevant far beyond formation control problems. In fact, related questions arise e.g. in the analysis of central configurations for the N-body problem [25].

VI. ACKNOWLEDGMENTS

The work by U. Helmke is supported by DFG Grant HE 1858/13-1 and DAAD Go8 German-Australian Collaboration Project. The work of B.D.O. Anderson was supported by NICTA, which is funded by the Australian Government through the ICT Centre of Excellence program, and by the Australian Research Council under grant DP110100538 and DP130103610. His research stay at the University of Würzburg was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach.

References

[1] S. Y. Nof, Ed., *Handbook of Industrial Robotics*, 2nd ed. Wiley, 1999, vol. 1.

[2] A. N. Bishop, B. Fidan, B. D. O. Anderson, K. Dogancay, and P. N. Pathirana, "Optimal range-difference based localization considering geometrical constraints," *IEEE Journal on Oceanic Engineering*, vol. 33, no. 3, pp. 289–301, 2008.

[3] A. N. Bishop, B. Fidan, B. D. O. Anderson, K. Dogcancay, and P. N. Pathirana, "Optimality analysis of sensor-target localization geometries," *Automatica*, vol. 46, pp. 479–492, 2010.

[4] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations," *IEEE Control Systems Magazine*, vol. 28, no. 6, pp. 48–63, 2008.

[5] B. D. O. Anderson, C. Yu, S. Dasgupta, and A. Morse, "Control of a three-coleader formation in the plane," *Syst. and Control Letters*, vol. 56, pp. 573–578, 2007.

[6] L. Krick, M. E. Broucke, and B. Francis, "Stabilization of infinitesimally rigid formation of multi-robot networks," *Int. J. Control*, vol. 82, pp. 423–439, 2009.

[7] B. Anderson, "Morse theory and formation control," in *Proc. 19th Mediterranean Conference on Control and Automation*, Cofru, Greece, 2011, pp. 656–661.

[8] B. D. O. Anderson and U. Helmke, "Counting critical formations on the line," *SIAM J. Control Optim.*, vol. 52, no. 1, pp. 219–242, 2014.

[9] C. Lageman and U. Helmke, "Critical points of potential functions for circular formation control," in *Proceedings of the 4th IFAC Workshop on Distributed Estimation and Control in Networked Systems (NecSys 2013)*, Koblenz, Germany, 2013, pp. 31–37.

[10] M. W. Hirsch, *Differential Topology*. Berlin: Springer-Verlag, 1976.

[11] D. N. Bernstein, "The number of roots of a system of equations," *Functional Anal. Appl.*, vol. 9, pp. 183–185, 1975.

[12] A. G. Kušnirenko, "Newton polyhedra and Bezout's theorem," *Funkcional. Anal. i Priložen.*, vol. 10, no. 3, 82–83., 1976.

[13] A. G. Khovanskiĭ, "Newton polyhedra, and the genus of complete intersections," *Funktsional. Anal. i Prilozhen.*, vol. 12, no. 1, pp. 51–61, 1978.

[14] T. Y. Li and X. Wang, "The BKK root count in $\mathbb{C}^n$," *Math. Comp.*, vol. 65, pp. 1477–1485, 1996.

[15] J. Draisma, E. Horobeţ, G. Ottaviani, B. Sturmfels, and R. R. Thomas, "The euclidean distance degree of an algebraic variety," *to appear in Foundations of Computational Mathematics*, 2014, axiv:1309.0049.

[16] J. Acebrón, L. Bonilla, C. Vicente, F. Ritort, and R. Spigler, "The Kuramoto model: A simple paradigm for synchronization phenomena," *Rev. Modern Physics*, vol. 77, pp. 137–185, 2005.

[17] F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica*, vol. 50, pp. 1539–1564, 2014.

[18] ——, "Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators," *SIAM Journal on Control and Optimization*, vol. 50, no. 3, pp. 1616–1642, 2012.

[19] S. Susca, F. Bullo, and S. Martinez, "Gradient algorithms for polygonal approximation of convex contours," *Automatica*, vol. 45, no. 2, pp. 510–516, 2009.

[20] S. Martinez and F. Bullo, "Optimal sensor placement and motion coordination for target tracking," *Automatica*, vol. 42, no. 4, pp. 661–668, 2006.

[21] I. R. Shafarevich, *Basic Algebraic Geometry 1*. Berlin: Spinger-Verlag, 1994.

[22] J. Heintz, "Definability and fast quantifier elimination in algebraically closed fields," *Theoretical Computer Science*, vol. 24, pp. 239–277, 1983.

[23] T. Gao, T. Y. Li, and M. Wu, "Algorithm 846: MixedVol: A software package for mixed-volume computation," *ACM Trans. Math. Softw.*, vol. 31, no. 4, pp. 555–560, 2005. [Online]. Available: http://doi.acm.org/10.1145/1114268.1114274

[24] U. Helmke and B. D. O. Anderson, "Equivariant Morse theory and formation control," in *Proc 51st Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, USA, 2013, pp. 1576 – 1583.

[25] F. Pacella, *Arch. Rational Mech. Anal.*, vol. 97, no. 1, pp. 59–74, 1987.