2015 IEEE 54th Annual Conference on Decision and Control (CDC)
December 15-18, 2015. Osaka, Japan

Convergence Rate of Optimal Periodic Gossiping on Ring Graphs

S. Mou A. S. Morse B. D. O. Anderson

***Abstract*— In an n-node connected graph $\mathbb{A}$, each node i is with a real-valued state $x_i(t)$, $i = 1, 2, ..., n$, and is able to communicate with certain other nodes. A periodic gossip sequence is able to drive all $x_i(t)$ to converge to $\frac{1}{n}\sum_{i=1}^{n} x_i(0)$ exponentially fast. Different sequences are usually associated with different convergence rates for graphs with cycles. This paper mainly focuses on a type of optimal periodic gossip sequences for ring graphs. Explicit formulas to compute their convergence rates are given, which are determined by the adjacency matrix of the $\frac{n}{2}$-node ring graph when n is even and Chebychev polynomials of the second kind when n is odd.**

I. INTRODUCTION

Increasing interests have been given to sensor networks, which have found extensive applications in surveillance, target tracking, etc [1]–[3]. Each node in a sensor network usually controls a state initialized by a real-valued measurement and is able to communicate with certain other sensor nodes called its "*neighbors*". Since these measurements in practice are usually with zero-mean noise, it is meaningful to develop methods for each node to achieve the *global average*, which is the average of initial state values of all nodes across the network, by only communications among neighbors. Such methods are often referred as to *distributed averaging algorithms* [4]–[8], which are special types of distributed consensus algorithms [9]–[17].

One type of distributed averaging algorithms is the Metropolis Algorithm [18], in which each node update its state value to be a weighted average of states of all its neighbors at each iteration. Implementation of this algorithm usually requires each node to be able to broadcast and at least know an upper bound of degrees of all its neighbors. Another popular type of distributed algorithms are gossiping algorithms [19]–[21], in which one pair of nodes *gossip* at each iteration by updating their state values to the average of their previous ones. Thus in a gossiping algorithm each node is only required to communicate with one node instead of broadcasting. To enable more than one disjoint pairs of nodes to gossip at each time with the goal of potentially speeding up the convergence, periodic gossiping algorithms are recently studied in [22], which involve a periodic gossip sequence such that one pair of nodes gossip just once over one period. Convergence rate of such a periodic gossiping is determined by the magnitude of the second largest eigenvalue of a matrix associated with the gossip sequence occurring over one period. When the underlying graph is a tree, the convergence rate does not change with respect to the order of gossips [23], [24], which is usually not true when it comes to graphs with cycles.

The research of A. S. Morse is supported by the US Air Force Office of Scientific Research and the National Science Foundation. The work of B. D. O. Anderson was supported by the Australian Research Council's Discovery Projects DP-0877562 and DP-110100538 and by NICTA (National ICT Australia). Corresponding author: S. Mou.

S. Mou is with the School of Aeronautics and Astronautics, Purdue University. A. S. Morse is with the Department of Electrical Engineering, Yale Univiserity. B. D. O. Anderson is with the College of Engineering and Computer Science, Australian National University. Email: mous@purdue.edu, as.morse@yale.edu, brian.anderson@anu.edu.au

In this paper we study periodic gossip sequences which allow gossips of disjoint node pairs to take place simultaneously in one step inspired by [25]. If the period of a periodic gossip sequence is equal to $\chi'(\mathbb{A})$, the chromatic index of the network $\mathbb{A}$, we call it an *optimal* periodic gossip sequence. By *optimal* is meant that one of its period requires the minimal number of steps for implementation. Given the network is connected, all optimal periodic gossip sequences drive each node's state to converge to the global average as fast as ρ^t converges to 0, where ρ is called the convergence rate and is related to the second largest eigenvalue in magnitude of a matrix. A major contribution of this paper is to obtain explicit formulas for the convergence rate ρ for optimal periodic gossip sequences when $\mathbb{A}$ is a ring graph. Especially when $\mathbb{A}$ is a n-node ring graph with n even, the convergence rate ρ is determined by the adjacency matrix of the $\frac{n}{2}$-node ring graph, which implies that ρ is fixed as long as the number of nodes in a ring graph is fixed. When $\mathbb{A}$ is a n-node ring graph with n odd, we find that ρ is related to Chebyshev polynomials of the second kind.

The remainder of this paper is organized as follows: We formulate the problem of solving distributed averaging problem using optimal periodic gossiping and present our main result in Section II. Proof of all theorems will be given in Section III. Then we conclude in Section IV.

II. PROBLEM FORMULATION AND MAIN RESULT

Consider a network of n sensor nodes, in which each node i controls a real-valued state $x_i(t) \in \mathbb{R}$ and is able to communicate with certain other nodes called its neighbors. Suppose the neighbor relationship of the network can be characterized by a connected undirected graph $\mathbb{A}$ such that there is an edge $(i, j) \in \mathcal{E}$ if and only if node i and j are neighbors, where $\mathcal{E}$ denotes the edge set of $\mathbb{A}$.

The goal of gossiping algorithms is to enable each node to achieve the global average $\frac{1}{n}\sum_{i=1}^{n} x_i(0)$ by letting one pair of neighbor nodes *gossip* at each time step, namely, updating their state values to be the average of their previous ones while all other agents' states remain unchanged. We use a *gossip sequence* $\mathbf{E} = e_1, e_2, ..., e_m$ to denote the neighbor node pairs gossiping in order, where each $e_k = (i_k, j_k) \in \mathcal{E}$.

In this paper, we only consider the infinite periodic gossip sequence $\mathbf{E}, \mathbf{E}, \mathbf{E}, ...$ Its *period* is denoted by T, which is the number of steps needed to implement its one-period subsequence $\mathbf{E}$. Suppose each edge in $\mathbb{A}$ only appears once in one period $\mathbf{E}$. If only one gossip is allowed at one time step, T is equal to the number of edges in $\mathbf{E}$. When a subset of edges are such that no two edges are adjacent to the same node, these edges are said to be disjoint and the gossips on these edges can be performed simultaneously in one time step, which is called *multi-gossip*. In this case the minimum value of T is related to an edge-coloring problem on $\mathbb{A}$. In graph theory, edge coloring is an assignment of different colors to the edges of a graph $\mathbb{A}$ such that no two edges sharing a common node have the same color. The minimum number of colors needed in an edge coloring problem is called the *chromatic index* denoted by $\chi'(\mathbb{A})$ [26]. Note that $\chi'(\mathbb{A}) = d_{\max}$ or $d_{\max} + 1$, where $d_{\max}$ is the maximum degree of a graph.

When multi-gossip is allowed, a periodic gossip sequence $\mathbf{E}, \mathbf{E}, \mathbf{E}, ...$ with $T = \chi'(\mathbb{A})$ is called an *optimal periodic gossip sequence* in the sense that one period of such sequence requires a minimal number to implement. It can be easily shown that optimal periodic gossip sequence drive all $x_i(t)$ to converge to $\frac{1}{n}\sum_{i=1}^{n} x_i(0)$ as fast as $\rho^{\frac{t}{T}}$ converges to 0. Here, ρ is called the *convergence rate* of periodic gossip sequences, which is equal to the second largest eigenvalue in magnitude of a matrix as shown later. When $\mathbb{A}$ is a tree graph, all optimal periodic gossip sequences share the same convergence rate by results in [22]–[24], which however are usually not true for graphs with cycles. To explore more about convergence rate invariance and obtain explicit formulas for computing ρ, we focus on the case when $\mathbb{A}$ is a ring graph with the node set $\{1, 2, ..., n\}$ and the edge set $\{(1, 2), (2, 3), ..., (n-1, n), (n, 1)\}$.

When n is even, the chromatic index $\chi'(\mathbb{A}) = 2$ and $\mathbb{A}$ is uniquely 2-colorable graph, that is, $\mathcal{E}$ can be uniquely divided into two disjoint sets $\mathcal{E}_1 = \{(1, 2), (3, 4), ..., (n-1, n)\}$ and $\mathcal{E}_2 = \{(2, 3), (4, 5), ..., (n-2, n-1), (n, 1)\}$. In this case we have the following result:

Theorem 1: When $\mathbb{A}$ is an n-node ring graph with n even and $n \geq 4$, all optimal periodic gossip sequences share the same convergence rate

$$\rho = \cos^2 \frac{2\pi}{n} \tag{1}$$

In the case that $\mathbb{A}$ is an n-node ring graph with n an odd integer, the chromatic index $\chi'(\mathbb{A}) = 3$. Note that $\mathbb{A}$ is *not* uniquely 3-colorable in this case. Take the 9-node ring graph for example, the edge set $\mathcal{E}$ can be the union of three disjoint sets $\mathcal{E}_1 = \{(1, 2), (3, 4), (5, 6), (7, 8)\}$, $\mathcal{E}_2 = \{(9, 1)\}$ and $\mathcal{E}_3 = \{(2, 3), (4, 5), (6, 7), (8, 9)\}$, or the union of $\bar{\mathcal{E}}_1 = \{(1, 2), (4, 5), (7, 8)\}$, $\bar{\mathcal{E}}_2 = \{(2, 3), (5, 6), (8, 9)\}$ and $\bar{\mathcal{E}}_3 = \{(3, 4), (6, 7), (9, 1)\}$. Optimal periodic gossip sequences corresponding to these two types of edge coloring usually do not share the same convergence rate. To get similar result to Theorem 1, we limit ourselves to a special type of optimal periodic gossip sequences corresponding to the edge coloring such that the edge set of $\mathbb{A}$ is the union of three disjoint sets $\mathcal{E}_1 = \{(2, 3), (4, 5), \cdots, (n-1, n)\}$, $\mathcal{E}_2 = \{(n, 1)\}$ and $\mathcal{E}_3 = \{(1, 2), (3, 4), \cdots, (n-2, n-1)\}$. We call these sequences *optimal periodic gossip sequences of type 1*. Then we have the following result:

Theorem 2: When $\mathbb{A}$ is an n-node ring graph with n an odd integer and $n \geq 3$, all optimal periodic gossip sequences of type 1 share the same convergence rate ρ, which is the 2nd largest of $\{\lambda : \lambda = \frac{q+1}{2},\quad 2(q+1)(U_{\frac{n-1}{2}} - U_{\frac{n-3}{2}}) - q - 3 = 0\}$, where $U_k(q)$ are Chebychev polynomials of the second kind defined as

$$\begin{aligned} U_0(q) &= 1 \\ U_1(q) &= 2q \\ U_k(q) &= 2qU_{k-1}(q) - U_{k-2}(q), \quad k \geq 2 \end{aligned}$$

Remark 1: Note that Chebyshev polynomials play a significant role in almost every area of numerical analysis including polynomial approximation, numerical integration, spectral methods for partial differential equations [27]. The finding in this paper may give us a clue to bridge the connection between convergence rate of optimal periodic gossip sequences and Chebyshev polynomials for more general graphs.

III. Analysis

To prove Theorem 1-2, we will first derive an evolution model in state space. Toward to this end, we define a *primitive gossip matrix* $P_{ij} = [p_{kl}]_{n\times n}$ for each gossip step (i, j) with entries defined as

$$p_{kl} = \begin{cases} \frac{1}{2}, & (k, l) \in \{(i, i), (i, j), (j, i), (j, j)\}; \\ 1, & k = l, k \neq i, k \neq j; \\ 0, & \text{otherwise}. \end{cases}$$

For a gossip sequence $(i_1, j_1), (i_2, j_2), \cdots, (i_k, j_k)$, we define the corresponding *gossip matrix* as

$$P_{i_k j_k} \cdots P_{i_2 j_2} P_{i_1 j_1}$$

Let $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix}'$. If i and j gossip at time $t + 1$, one has

$$x(t + 1) = P_{ij} x(t), \quad t \geq 0 \tag{2}$$

For a periodic gossip sequence $\mathbf{E}, \mathbf{E}, ..., \mathbf{E}$ with period T, we let W be the gossip matrix corresponding to $\mathbf{E}$. Then

$$x((k + 1)T) = W x(kT), \quad k = 0, 1, 2, ...$$

It follows that

$$x(kT) = W^k x(0), \quad k = 0, 1, 2, ... \tag{3}$$

By (3) one has

$$\begin{aligned} x(kT) - \frac{1}{n}\sum_{i=1}^{n} x_i(0)\mathbf{1} &= W^k x(0) - \frac{\mathbf{11}'}{n}x(0) \\ &= W^k(I - \frac{\mathbf{11}'}{n})x(0) \\ &= W^k(I - \frac{\mathbf{11}'}{n})^k x(0) \\ &= (W - \frac{\mathbf{11}'}{n})^k x(0) \end{aligned}$$

where I is the $n \times n$ identity matrix and $\mathbf{1}$ is the $n \times 1$ vector with all elements equal to 1s. By results in [22], one has $|\lambda_2(W)| < 1$ and $\lambda_1(W - \frac{\mathbf{11}'}{n}) = \lambda_2(W)$, where $\lambda_i(W)$ denote the ith largest eigenvalue in magnitude of matrix W. Thus $||x(kT) - \frac{1}{n}\sum_{i=1}^{n} x_i(0)\mathbf{1}||$ converges to 0 as fast as ρ^k converges to 0, where $\rho = |\lambda_2(W)|$. Then all $x_i(t)$ converge to $\frac{1}{n}\sum_{i=1}^{n} x_i(0)$ as fast as $\rho^{\frac{t}{T}}$ converges to 0.

Proof of Theorem 1: Let W be the gossip matrix corresponding to one-period subsequence of any optimal periodic gossip sequence on $\mathbb{A}$. Let W_0 be the gossip matrix corresponding to one specific such sequence

$$\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2$$

where $\mathbf{E}_1 = (2,3), ..., (n-2, n-1), (n,1)$ and $\mathbf{E}_2 = (1,2), (3,4), ..., (n-1, n)$. To prove all optimal periodic gossip sequences share the same convergence rate, it is sufficient to show that

$$cp(W) = cp(W_0) \tag{4}$$

where $cp(\cdot)$ denotes the characteristic polynomial of a matrix. Let S_i denote the gossip matrix corresponding to $\mathbf{E}_i$, $i = 1, 2$. Then

$$W_0 = S_2 S_1$$

Since $\mathbb{A}$ is uniquely 2-colorable, one period of any optimal periodic gossip sequence must be either $\pi_1(\mathbf{E}_1), \pi_2(\mathbf{E}_2)$ or $\pi_2(\mathbf{E}_2), \pi_1(\mathbf{E}_1)$, where $\pi_i(\mathbf{E}_i)$ is a permutation of the gossip sequence $\mathbf{E}_i$, $i = 1, 2$. Let $\bar{S}_i$ denote the gossip matrix corresponding to the gossip sequence $\pi_i(\mathbf{E}_i)$. Then

$$W = \bar{S}_1 \bar{S}_2$$

or

$$W = \bar{S}_2 \bar{S}_1$$

Note that all edges in $\mathbf{E}_i$ are disjoint, then

$$S_i = \bar{S}_i, \;\; S_i' = S_i, \;\; i = 1, 2$$

Then

$$cp(\bar{S}_1\bar{S}_2) = cp(\bar{S}_2\bar{S}_1) = cp(S_2 S_1)$$

Thus (4) is true.

Next we show the eigenvalues of W_0 are those in [1]

$$0, \cos^2\frac{2k\pi}{n}, k = 0, 1, 2, ..., \frac{n}{2} - 1 \tag{5}$$

[1]Key idea of the analytical proof for (5) was initially from discussions with Steve Butler at Iowa State University.

We first check the case when $n = 4$, in which one has

$$W_0 = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

the eigenvalues of which are $\{0, 1\}$. Then (5) is true.

When n is even and $n \geq 6$, we let e_k denote the kth column of the $n \times n$ identity matrix, $k = 1, 2, ..., n$. One has $e_i' e_j = 1$ if $i = j$ and $e_i' e_j = 0$ otherwise. Then

$$\begin{aligned} W_0 &= (P_{n(n-1)} \cdots P_{43} P_{21})(P_{1n} P_{(n-1)(n-2)} \cdots P_{32}) \\ &= (\frac{1}{2}\begin{bmatrix} e_1' + e_2' \\ e_1' + e_2' \\ e_3' + e_4' \\ e_3' + e_4' \\ \vdots \\ e_{n-1}' + e_n' \\ e_{n-1}' + e_n' \end{bmatrix}_{n\times n})(\frac{1}{2}\begin{bmatrix} e_1' + e_n' \\ e_2' + e_3' \\ e_2' + e_3' \\ \vdots \\ e_{n-2}' + e_{n-1}' \\ e_{n-2}' + e_{n-1}' \\ e_1' + e_n' \end{bmatrix}_{n\times n}') \\ &= \frac{1}{4}\begin{bmatrix} e_1' + e_2' + e_3' + e_n' \\ e_1' + e_2' + e_3' + e_n' \\ e_2' + e_3' + e_4' + e_5' \\ e_2' + e_3' + e_4' + e_5' \\ \vdots \\ e_1' + e_{n-2}' + e_{n-1}' + e_n' \\ e_1' + e_{n-2}' + e_{n-1}' + e_n' \end{bmatrix}_{n\times n} \end{aligned}$$

Note that 0 is an eigenvalue of W_0 with $\frac{n}{2}$ left eigenvectors

$$e_1' - e_2', e_3' - e_4', ..., e_{n-1}' - e_n' \tag{6}$$

Let λ denote any non-zero eigenvalues of W_0 and let v denote any right eigenvector of W_0 associated with the eigenvalue λ. Then v is perpendicular to all vectors defined in (6). Let $r = \frac{n}{2}$. Then v must be in the form

$$v = \begin{bmatrix} v_1 & v_1 & v_2 & v_2 & \cdots & v_r & v_r \end{bmatrix}'$$

where $v_j, j = 1, ..., r$ are scalars. Thus

$$W_0 v = \frac{1}{4}\begin{bmatrix} v_r + 2v_1 + v_2 \\ v_r + 2v_1 + v_2 \\ v_1 + 2v_2 + v_3 \\ v_1 + 2v_2 + v_3 \\ \vdots \\ v_{r-1} + 2v_r + v_1 \\ v_{r-1} + 2v_r + v_1 \end{bmatrix} = \lambda v \tag{7}$$

Let $\bar{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}'$. Let

$$\bar{W}_0 = \frac{1}{4}(2I + A_r) \tag{8}$$

where A_r denotes the adjacency matrix of the r-node ring graph with edges $\{(1,2), (2,3), ..., (r-1, r), (r, 1)\}$. By (7) and (8) one has

$$\bar{W}_0 \bar{v} = \lambda \bar{v}$$

All non-zero eigenvalues of W_0 are eigenvalues of $\bar{W}_0$. Note that $\bar{W}_0$ is a circulant matrix with first column $c_0 =$

$\frac{1}{2}, c_1 = \frac{1}{4}, c_2 = 0, c_3 = 0, ..., c_{r-2} = 0, c_{r-1} = \frac{1}{4}$. Let $\omega_k = e^{\frac{2\pi k i}{r}}, i = \sqrt{-1}, k = 0, 1, 2, ..., r-1$. The eigenvalues of $\bar{W}_0$ are

$$\begin{aligned} \lambda_k &= c_0 + c_{r-1}\omega_k + c_{r-2}\omega_k^2 + ... + c_1\omega_k^{r-1} \\ &= \frac{1}{2} + \frac{1}{4}e^{\frac{2\pi k i}{r}} + \frac{1}{4}e^{\frac{2(r-1)\pi k i}{r}} \\ &= \frac{1}{2}(1 + \cos\frac{2k\pi}{r}) \\ &= \cos^2\frac{k\pi}{r} \end{aligned}$$

Note that $r = \frac{n}{2}$. Thus eigenvalues of W_0 are $\{0, \cos^2\frac{2k\pi}{n}, k = 0, 1, 2, ..., \frac{n}{2} - 1\}$. ■

Proof of Theorem 2: Let W be the gossip matrix corresponding to one-period subsequence of any optimal periodic gossip sequence on $\mathbb{A}$. Let W_0 be the gossip matrix corresponding to one specific such sequence

$$\mathbf{E} = \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$$

where $\mathbf{E}_1 = \{(2,3), (4,5), \cdots, (n-1, n)\}$, $\mathbf{E}_2 = (n, 1)$ and $\mathbf{E}_3 = (1,2), (3,4), \cdots, (n-2, n-1)$. To prove all optimal periodic gossip sequences of type 1 share the same convergence rate, it is sufficient to show that

$$cp(W) = cp(W_0) \tag{9}$$

Note that

$$W_0 = S_3 S_2 S_1 \tag{10}$$

where

$$\begin{aligned} S_1 &= P_{n(n-1)} \cdots P_{32} \\ S_2 &= P_{n1} \\ S_3 &= P_{(n-1)(n-2)} \cdots P_{43} P_{21} \end{aligned}$$

Let $\bar{S}_i$ denote the gossip matrix corresponding to the gossip sequence $\pi_i(\mathbf{E}_i)$, where $\pi_i(\mathbf{E}_i)$ is a permutation of $\mathbf{E}_i$, $i = 1, 2, 3$. Then

$$W = \bar{S}_{k_1}\bar{S}_{k_2}\bar{S}_{k_3} \tag{11}$$

where k_1, k_2, k_3 is a permutation of $1, 2, 3$. Note that all edges in $\mathbf{E}_i$ are disjoint, then

$$S_i = \bar{S}_i, \quad S_i' = S_i, \quad i = 1, 2, 3 \tag{12}$$

By (11) and (12), one has

$$cp(W) = cp(S_{k_1}S_{k_2}S_{k_3}) \tag{13}$$

Note that S_1, S_2, S_3 are symmetric matrices. Then

$$cp(S_1S_2S_3) = cp(S_3S_2S_1) \tag{14}$$

In addition note that,

$$\begin{aligned} cp(S_1S_2S_3) = cp(S_2S_3S_1) = cp(S_3S_1S_2) \\ cp(S_3S_2S_1) = cp(S_2S_1S_3) = cp(S_1S_3S_2) \end{aligned}$$

Then

$$cp(S_1S_2S_3) = cp(S_{k_1}S_{k_2}S_{k_3})$$

Thus $cp(W) = cp(W_0)$.

Next we show the eigenvalues of W_0 are determined by Chebychev polynomials of second kind. For $n = 3$, one has $\mathbf{E} = (2,3), (3,1), (1,2)$ and

$$W_0 = \frac{1}{8}\begin{bmatrix} 2 & 3 & 3 \\ 2 & 3 & 3 \\ 4 & 2 & 2 \end{bmatrix}$$

Non-zero eigenvalues of W_0 are $\{1, -\frac{1}{8}\}$. Note that $2(q+1)[U_1(q) - U_0(q)] - q - 3 = 4q^2 + q - 5 = (4q+5)(q-1)$. Thus the set of all non-zero eigenvalues of W_0 is $\{\lambda : \lambda = \frac{q+1}{2}, \quad 2(q+1)[U_1(q) - U_0(q)] - q - 3 = 0\} = \{1, -\frac{1}{8}\}$. Theorem 2 is true for $n = 3$.

For $n = 5$, one has $\mathbf{E} = (2,3), (4,5), (5,1), (1,2), (3,4)$ and

$$W_0 = \frac{1}{8}\begin{bmatrix} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & 2 \\ 4 & 0 & 0 & 2 & 2 \end{bmatrix}$$

Non-zero eigenvalues of W_0 are $\{1, \frac{1}{8} \pm \frac{1}{8}i\}$. Theorem 2 is true for $n = 5$.

For n is odd and $n \geq 7$, one has $\mathbf{E} = (2,3), (4,5), ..., (n-1, n), (n, 1), (1, 2), (3, 4), ..., (n-2, n-1)$. Let e_k denote the $k$the column of the $n \times n$ identity matrix, $k = 1, 2, ..., n$. Then the complete gossip matrix is

$$\begin{aligned} W_0 &= (\frac{1}{2}\begin{bmatrix} e_1' + e_2' \\ e_1' + e_2' \\ e_3' + e_4' \\ e_3' + e_4' \\ \vdots \\ e_{n-2}' + e_{n-1}' \\ e_{n-2}' + e_{n-1}' \\ 2e_n' \end{bmatrix}_{n\times n}) P_{n1} (\frac{1}{2}\begin{bmatrix} 2e_1' \\ e_2' + e_3' \\ e_2' + e_3' \\ e_4' + e_5' \\ e_4' + e_5' \\ \vdots \\ e_{n-1}' + e_n' \\ e_{n-1}' + e_n' \end{bmatrix}_{n\times n}') \\ &= (\frac{1}{4}\begin{bmatrix} e_1' + 2e_2' + e_n' \\ e_1' + 2e_2' + e_n' \\ 2e_3' + 2e_4' \\ 2e_3' + 2e_4' \\ \vdots \\ 2e_{n-2}' + 2e_{n-1}' \\ 2e_{n-2}' + 2e_{n-1}' \\ 2e_1' + 2e_n' \end{bmatrix}_{n\times n}) (\frac{1}{2}\begin{bmatrix} 2e_1' \\ e_2' + e_3' \\ e_2' + e_3' \\ e_4' + e_5' \\ e_4' + e_5' \\ \vdots \\ e_{n-1}' + e_n' \\ e_{n-1}' + e_n' \end{bmatrix}_{n\times n}') \\ &= \frac{1}{8}\begin{bmatrix} 2(e_1' + e_2' + e_3') + e_{n-1}' + e_n' \\ 2(e_1' + e_2' + e_3') + e_{n-1}' + e_n' \\ 2(e_2' + e_3' + e_4' + e_5') \\ 2(e_2' + e_3' + e_4' + e_5') \\ 2(e_4' + e_5' + e_6' + e_7') \\ 2(e_4' + e_5' + e_6' + e_7') \\ \vdots \\ 2(e_{n-3}' + e_{n-2}' + e_{n-1}' + e_n') \\ 2(e_{n-3}' + e_{n-2}' + e_{n-1}' + e_n') \\ 4e_1' + 2(e_{n-1}' + e_n') \end{bmatrix}_{n\times n} \end{aligned}$$

Then 0 is an eigenvalue of W_0 with $\frac{n-1}{2}$ left eigenvectors

$$e_1' - e_2', e_3' - e_4', ..., e_{n-2}' - e_{n-1}' \tag{15}$$

Let λ denote any non-zero eigenvalues of W_0 and let v denote any right eigenvector of W_0 associated with the eigenvalue λ. Then v is perpendicular to all vectors defined in (15). Let $r = \frac{n-1}{2}$. Then v must be in the form of

$$v = \begin{bmatrix} v_1 & v_1 & v_2 & v_2 & \cdots & v_r & v_r & v_{r+1} \end{bmatrix}'$$

Then

$$W_0 v = \frac{1}{8} \begin{bmatrix} 4v_1 + 2v_2 + v_r + v_{r+1} \\ 4v_1 + 2v_2 + v_r + v_{r+1} \\ 2v_1 + 4v_2 + 2v_3 \\ 2v_1 + 4v_2 + 2v_3 \\ 2v_2 + 4v_3 + 2v_4 \\ 2v_2 + 4v_3 + 2v_4 \\ \vdots \\ 2v_{r-1} + 4v_r + 2v_{r+1} \\ 2v_{r-1} + 4v_r + 2v_{r+1} \\ 4v_1 + 2v_r + 2v_{r+1} \end{bmatrix} = \lambda v$$

where $v_i, \quad i = 1, ..., r+1$ are scalars. Let $\bar{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_{r+1} \end{bmatrix}'$. Let $\bar{e}_i$ denote the ith column of the $r \times r$ identity matrix, $i = 1, 2, .., r$. Let A_k denote the adjacency matrix of the k-node path graph with edges $\{(1,2), (2,3), ..., (k-1,k)\}$, $k = 2, 3, .., r$. Let

$$\bar{W}_0 = \begin{bmatrix} 2A_r + \bar{e}_1 \bar{e}_r' & \bar{e}_1 + 2\bar{e}_r \\ 4\bar{e}_1 + 2\bar{e}_r' & -2 \end{bmatrix}_{(r+1)\times(r+1)}$$

Then

$$\frac{1}{8}(\bar{W}_0 + 4)\bar{v} = \lambda \bar{v} \tag{16}$$

Note that all non-zero eigenvalues of W_0 are obtained by shifting and scaling all eigenvalues of $\bar{W}_0$. In the following we'll try to compute eigenvalues of $\bar{W}_0$. Toward to this end, we let $B_k = pI - 2A_k$, $2 \le k \le r$. Let $D = B_r - \bar{e}_1 \bar{e}_r'$. Let p denote the eigenvalue of $\bar{W}_0$. Let $cp(\bar{W}_0)$ denote the characteristic polynomial of $\bar{W}_0$. Then one has

$$cp(\bar{W}_0) = \det(pI - \bar{W}_0) = \det(\begin{bmatrix} D & b \\ c & a \end{bmatrix})$$

where

$$a = p + 2, \quad b = -\bar{e}_1 - 2\bar{e}_r, \quad c = -4\bar{e}_1' - 2\bar{e}_r'.$$

By using Schur complement one has

$$cp(\bar{W}_0) = \det(D)(a - cD^{-1}b) \tag{17}$$

To compute (17) we note that

$$\begin{aligned} cD^{-1}b &= 4D^{-1}(1,1) + 2D^{-1}(r,1) \\ &\quad + 8D^{-1}(1,r) + 4D^{-1}(r,r) \end{aligned} \tag{18}$$

By rank-one permutation, one has

$$\begin{aligned} D^{-1} &= (B_r - \bar{e}_1 \bar{e}_r')^{-1} \\ &= (I - B_r^{-1} \bar{e}_1 \bar{e}_r')^{-1} B_r^{-1} \\ &= (I + \frac{B_r^{-1} \bar{e}_1 \bar{e}_r'}{1 - \bar{e}_r' B_r^{-1} \bar{e}_1}) B_r^{-1} \\ &= B_r^{-1} + \frac{B_r^{-1} \bar{e}_1 \bar{e}_r' B_r^{-1}}{1 - \bar{e}_r' B_r^{-1} \bar{e}_1}, \end{aligned} \tag{19}$$

Then

$$D^{-1}(1,1) = B_r^{-1}(1,1) + \frac{B_r^{-1}(1,1) B_r^{-1}(r,1)}{1 - B_r^{-1}(r,1)} \tag{20}$$

$$D^{-1}(1,r) = B_r^{-1}(1,r) + \frac{B_r^{-1}(1,1) B_r^{-1}(r,r)}{1 - B_r^{-1}(r,1)} \tag{21}$$

$$D^{-1}(r,1) = B_r^{-1}(r,1) + \frac{B_r^{-1}(r,1)^2}{1 - B_r^{-1}(r,1)} \tag{22}$$

$$D^{-1}(r,r) = B_r^{-1}(r,r) + \frac{B_r^{-1}(r,1) B_r^{-1}(r,r)}{1 - B_r^{-1}(r,1)} \tag{23}$$

Let

$$T_k = \det(B_k) = \det(p - 2A_k), \quad k = 1, 2, ..., r$$

Then

$$B_r^{-1}(1,1) = B^{-1}(r,r) = \frac{T_{r-1}}{T_r}, \tag{24}$$

$$B_r^{-1}(1,r) = B^{-1}(r,1) = \frac{2^{r-1}}{T_r}. \tag{25}$$

Substitute (24) and (25) into (20)-(23), one has Then

$$D^{-1}(1,1) = \frac{T_{r-1}}{T_r - 2^{r-1}} \tag{26}$$

$$D^{-1}(1,r) = \frac{T_{r-1}^2 - 2^{2r-2} + 2^{r-1} T_r}{T_r (T_r - 2^{r-1})} \tag{27}$$

$$D^{-1}(r,1) = \frac{2^{r-1}}{T_r - 2^{r-1}} \tag{28}$$

$$D^{-1}(r,r) = \frac{T_{r-1}}{T_r - 2^{r-1}} \tag{29}$$

Substituting (26)-(29) into (18) one has

$$cD^{-1}b = \frac{1}{T_r - 2^{r-1}} [8T_{r-1} + 10 \times 2^{r-1} + \frac{8T_{r-1}^2 - 2^{2r+1}}{T_r}]$$

which and $\det(D) = T_r - 2^{r-1}$, $a = p + 2$ and (17) implies

$$cp(\bar{W}_0) = (p+2)(T_r - 2^{r-1}) - (8T_{r-1} + 5 \times 2^r + \frac{8T_{r-1}^2 - 2^{2r+1}}{T_r}). \tag{30}$$

To further simplify (30), we introduce Chebychev polynomials of the second kind as follows

$$U_k(p) = 2pU_{k-1} - U_{k-2}, \quad k = 3, 4, ..., r \tag{31}$$

with $U_1(p) = 2p$ and $U_2(p) = 4p^2 - 1$. Note that there is a special relation between $U_k(p)$ and the determinant of special matrix, which is

$$U_k(p) = \det(2p - A_k)$$

where A_k is the adjacency matrix of the k-node path graph with edges $\{(1,2), (2,3), ..., (k-1,k)\}$. Then

$$\begin{aligned} U_k(\frac{p}{4}) &= \det(\frac{1}{2}(p - 2A_k)) \\ &= 2^{-k} \det(p - 2A_k) \\ &= 2^{-k} T_k(p) \end{aligned}$$

Substitute this into (30) one has

$$cp(\bar{W}_0) = (p+2)[U_r(\frac{p}{4}) - \frac{1}{2}] - [4U_{r-1}(\frac{p}{4}) + 5 + 2\frac{U_{r-1}^2(\frac{p}{4}) - 1}{U_r(\frac{p}{4})}]. \tag{32}$$

Let $q = \frac{p}{4}$. Then q is root of

$$(2q+1)(2U_r - 1) - (4U_{r-1} + 5 + 2\frac{U_{r-1}^2 - 1}{U_r}) = 0. \quad (33)$$

Note that for Chebychev polynomials of the second kind,

$$U_{r-1}^2 - 1 = U_r U_{r-2}.$$

Substituting this into (33) leads to

$$(2q+1)U_r - 2U_{r-1} - U_{r-2} - (q+3) = 0$$

which is

$$2(q+1)U_r - 2U_{r-1} - (U_r + U_{r-2}) - (q+3) = 0 \quad (34)$$

By the definition of Chebychev polynomials of the second kind one has

$$U_r + U_{r-2} = 2qU_{r-1}$$

Then (34) is simplified to

$$2(q+1)(U_r - U_{r-1}) - q - 3 = 0 \quad (35)$$

From $\lambda = \frac{1}{8}(p+4)$ and $p = 4q$, one has $\lambda = \frac{1}{2}(q+1)$. Thus non-zero eigenvalues of W_0 are

$$\{\lambda : \lambda = \frac{1}{2}(q+1), \;\; 2(q+1)(U_r - U_{r-1}) - q - 3 = 0\}$$

with $r = \frac{n-1}{2}$. We complete the proof. ■

IV. Conclusion

In this paper we have studied a type of optimal periodic gossip sequences on ring graphs. Explicit formulas have been given to compute their convergence rates. Simulations suggest that these sequences are with the fastest convergence rate among all periodic gossip sequences on ring graphs though finding analytical proof for this observation is still an open problem. Our future work includes investigating challenging communication issues of implementing periodic gossiping algorithms that may arise from dynamic network topology and unpredictable erasures [28]. We will also consider using the idea of periodic gossiping to develop distributed algorithms for solving linear equations [29]–[31].

V. Acknowledgement

The authors would like to thank Fan Chung at University of California, San Diego and Steve Butler at Iowa State University for helpful discussions.

References

[1] C. Y. Chong and S. P. Kumar. Sensor networks: evolution, opportunities, and challenges. *Proceedings of IEEE*, 91(8):1247–1256, 2003.

[2] K. Chakrabarty, S. S. Lyengar, H. Qi, and E. Cho. Grid coverage for surveillance and target location in distributed sensor networks. *IEEE Transactions on Computers*, 51(12):1448–1453, 2002.

[3] C. Li and N. Elia. Stochastic sensor scheduling via distributed convex optimization. *Automatica*, 58:173–182, 2015.

[4] A. Olshevsky and J. N. Tsitsiklis. Convergence speed in distributed consensus and averaging. *SIAM Journal on Control and Optimization*, 48:33–35, 2009.

[5] A. Nedic, A. Olshevsky, A. Ozdaglar, and J. N. Tsitsiklis. On distributed averaging algorithms and quantization effects. *IEEE Transactions on Automatic Control*, 54:2506–2517, 2009.

[6] T. C. Aysal, M. E. Yildiz, A. D. Sarwate, and A. Scaglione. Broadcast gossip algorithms for consensus. *IEEE Transactions on Signal Processing*, 57:2748–2761, 2009.

[7] S. Mou and M. Cao. Distributed averaging using compensation. *IEEE Communication Letters*, (8):1672–1675, 2013.

[8] S. Mou and A. S. Morse. Finite-time distributed averaging. *Proc. of the 2014 American Control Conference*, pages 5260–5263, 2014.

[9] G. Shi, A. Proutiere, M. Johansson, and K. H. Johansson. Randomized consensus with attractive and repulsive links. *Proc. of the 52nd Conference on Decision and Control*, pages 2599–2604, 2013.

[10] A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, pages 988–1001, june 2003. also in Proc. 41st IEEE CDC, pages 2953 - 2958,2002.

[11] W. Ren and R. W. Beard. Consensus seeking in multiagent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, (5):655–661, 2005.

[12] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing enviornment: a graphical approach. *SIAM Jounal on Control and Optimization*, 47:575–600, 2008.

[13] M. Cao, A. S. Morse, and B. D. O. Anderson. Reaching a consensus in a dynamically changing environment – convergence rates, measurement delays and asynchronous events. *SIAM Jounal on Control and Optimization*, pages 610–623, 2008.

[14] M. Cao, A. S. Morse, and B. D. O. Anderson. Agree asychronously. *IEEE Transactions on Automatic Control*, 53(8):1826–1838, 2008.

[15] Y. Lou, G. Shi, K. H. Johansson, and Y. Hong. Convergence of random sleep algorithms for optimal consensus. *Systems and Control Letters*, 12:1196–1202, 2013.

[16] S. Kia, J. Cortes, and S. Martinez. Dynamic average consensus with distributed event-triggered communication. *Proc. of the 53nd Conference on Decision and Control*, pages 890–895, 2014.

[17] A. Olshevsky. Consensus with ternary messages. *SIAM Journal on Control and Optimization*, 52:987–1009, 2014.

[18] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems and Control Letters*, 53:65–78, 2004.

[19] S. Mou, C. Yu, B. D. O. Anderson, and A. S. Morse. Deterministic gossiping with a periodic protocol. *Proc. of the 49th Conference on Decision and Control*, pages 5787–5791, 2010.

[20] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Transaction on Information Theory*, 52:2508–2530, 2006.

[21] A. G. Dimakis, S. Kar, J. M. F. Moura, M. G. Rabbat, and A. Scaglione. Gossip algorithms for distributed signal processing. *Proceeding of IEEE*, 98:1847–1864, 2010.

[22] J. Liu, S. Mou, A. S. Morse, B. D. O. Anderson, and C. Yu. Deterministic gossiping. *Proceedings of the IEEE*, (9):1505–1524, 2011.

[23] B. D. O. Anderson, C. Yu, and A. S. Morse. Convergence of periodic gossiping algorithms. *Perspectives in Mathematical Systems Theory, Control and Signal Processing*, pages 127–138, 2010.

[24] F. He, A. S. Morse, J. Liu, and S. Mou. Periodic gossiping. *Proc. of the 18th IFAC Congress*, pages 8718–8723, 2011.

[25] C. Yu, B. D. O. Anderson, S. Mou, J. Liu, F. He, and A. S. Morse. Gossiping periodically. In Preparation.

[26] F. R. K. Chung and R. L. Graham. Erdos on graphs: His legacy of unsolved problems. *Taylor and Francis*, 1999.

[27] T. J. Rivlin. Chebyshev polynomials: From approximation theory to algebra and number theory. *New York*, 1990.

[28] C. Li, J. Jose, and X. Wu. Distributed-fountain network code (DFNC) for content delivery in vehicular networks. *Proceeding of the 10th ACM International Workshop on Vehicular inter-networking, systems,and applications*, pages 31–40, 2013.

[29] S. Mou, J. Liu, and A. S. Morse. A distributed algorithm for solving a linear algebraic equation. *IEEE Transactions on Automatic Control*. DOI: 10.1109/TAC.2015.2414771.

[30] S. Mou and A. S. Morse. A fixed-neighbor, distributed algorithm for solving a linear algebraic equation. *European Control Conference*, pages 2269–2273, 2013.

[31] S. Mou, A. S. Morse, Z. Lin, L. Wang, and D. Fullmer. A distributed algorithm for efficiently solving linear equations. *Proc. of the 54rd Conference on Decision and Control*, 2015. To appear.