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Rigid formation shape control in general dimensions: an invariance principle and open problems

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Abstract— **The control systems resulting from seeking to control the shape of a rigid formation are frequently described by gradient flows which are designed to minimize some predefined and relevant potential functions linked to the desired formation shape. This paper discusses and establishes a rank-preserving property of such formation shape stabilization systems. We further show some properties of the degenerate critical formations that live in a lower dimensional space, and prove that they are unstable. The implication of these results is that if all the agents start with generic initial positions, then their trajectories will be strictly bounded away from the set of degenerate formations at any finite or infinite time. By establishing this invariance principle, some previous results on equilibrium analysis for rigid formation systems can be greatly simplified. The results also have applications in other fields such as the multidimensional scaling study. The results are obtained from a joint analysis of rank-preserving flow theory, graph rigidity theory and invariant manifold theory.**

I. Introduction

Rigid formation shape control for a collection of N point agents in Euclidean space is concerned with designing distributed control laws, by employing tools from graph theory, for individual agents so that the formation can converge to a prescribed rigid shape specified by a certain set of desired interagent distances [1], [2]. In this distance-based formation control framework, the global property for the convergence analysis of formation control systems with general shapes has been an open problem.

The stability analysis of the formation control system usually involves an interplay of graph topology concepts and nonlinear analysis tools. On the one hand, there have been several papers focusing on the role of topology aspect in formation control system. In [3] the authors discussed the convergence property for the distance-based formation control system when the underlying graph is a tree, and showed that with this assumption the commonly-used descent gradient control derived from a potential function can stabilize the formation system to the desired relative distances. However in this tree-graph case the formation is not a rigid one and thus the shape is not well defined. Recently a vast number of literature has been emerged for analyzing the formation control system with certain simple and specific shapes, such as the 2-D triangular formation [4], [5], the 2-D four-agent rectangular formation [6] and the 3-D tetrahedral formation [7]. For formation systems with general shapes, however, a full understanding of the convergence property of different equilibria has not yet been explored.

On the other hand, some other approaches have been reported recently to gain a further understanding of the bounds and properties of critical points of *nonlinear* formation systems, including the semidefinite programming viewpoints for solving semialgebraic problems in [8] and the Morse theory method [9], [10], [2]. The application of Morse theory to robot navigation control can be tracked back to an early paper [11]. It was recently shown in [9] by using this theory that multiple equilibria, including incorrect equilibria, are a consequence of any formation shape control algorithm which evolves in a steepest descent direction of a smooth cost function that is invariant under translations and rotations. The existence of multiple equilibria of the potential function adds considerable complexity to the convergence analysis of formation control algorithms. We refer the readers to the recent thesis [12] for more discussions on applying Morse theory to the rigid formation control problem.

This paper aims to explore more invariance properties in formation control system and to provide additional insights on the stability analysis of critical formations. We will show a rank-preserving property for formation systems described by a gradient flow, which means that the dimension of the smallest affine space that contains agents' positions is an invariant under the gradient flow. Furthermore, this rank-preserving property also indicates certain invariance properties for 2-D and 3-D formation systems.

In the second main part of this paper, we will show that any incorrect equilibria which lie in a degenerate space (definitions will become clear later) are unstable. This is a consequence of the rank-preserving property of the formation gradient system and an eigenvalue result relating to the Hessian matrix of the potential function. The results hold for formation systems in any higher dimensions, which also have applications in other fields such as the multidimensional scaling study [13]. Actually, the optimization problem involving minimizing certain stress functions in multidimensional scal-

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ing field has a similar problem description as in the formation control field with similar forms of cost functions.

The rest of this paper is organized as follows. Section II briefly reviews some background on the rank-preserving flow. Section III presents formation equations and problem setup. The rank-preserving property for formation systems in arbitrary dimensions is proved in Section IV. In Section V, we discuss the stability property of degenerate equilibria (definitions will be clear in that section) and show they are unstable. Conclusions and some open problems are provided in Section VI which close this paper.

II. Background on rank-preserving matrix flow

In this section we will briefly review some background on the rank-preserving flow theory [14, Chapter 5].

For integers $1 \leq n \leq \min(M, N)$, let

$$\mathbb{M}(n, M \times N) = \{X \in \mathbb{R}^{M \times N} | \text{rank}(X) = n\}$$

denote the set of real $M \times N$ matrices of fixed rank n. The following results will be useful in later analysis.

***Lemma** 1:* $\mathbb{M}(n, M \times N)$ is a smooth and connected manifold of dimension $n(M + N - n)$, if $\max(M, N) > 1$. The tangent space of $\mathbb{M}(n, M \times N)$ at an element X is

$$T_X\mathbb{M}(n, M \times N) = \{\Delta_1 X + X\Delta_2 | \Delta_1 \in \mathbb{R}^{M\times M}, \Delta_2 \in \mathbb{R}^{N \times N}\} \tag{1}$$

A matrix differential equation $\dot{X} = F(t, X)$ evolving on the matrix space $\mathbb{R}^{M\times N}$ is said to be *rank-preserving* if the rank of every solution $X(t)$ is constant as a function of t, that is, $\text{rank}(X(t)) = \text{rank}(X(0))$ for all $t \geq 0$. The following lemma characterizes such rank-preserving flows (cf. Lemma 1.22 in Chapter 5 of [14]).

***Lemma** 2:* Let $I \subset \mathbb{R}$ be an interval and let $A(t) \in \mathbb{R}^{M\times M}$, $B(t) \in \mathbb{R}^{N\times N}$, $t \in I$, be a continuous time-varying family of matrices. Then

$$\dot{X}(t) = A(t)X(t) + X(t)B(t), \;\; X(0) \in \mathbb{R}^{M\times N} \tag{2}$$

is rank-preserving. Conversely, every rank-preserving differential equation on $\mathbb{R}^{M\times N}$ is of the form (2) for matrices $A(t)$ and $B(t)$.

The proof of Lemma 2 is based on the fact that (2) defines a time varying vector field on the subset of the tangent space of $\mathbb{M}(n, M \times N)$ described by (1). The full proof can be found in [14, Page 139].

***Remark** 1:* Note that the above lemma on rank-preserving flows also implies that the limit value $X(\infty)$ has rank *less than or equal to* $\text{rank}(X(0))$.

III. Problem setup and motion equations

Since formations of N mobile agents are best described in terms of graph theory, we give a brief descriptions of some of the basic definitions and facts needed. Consider an undirected and connected graph with M edges and N vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, N\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The matrix relating the nodes to the edges is called the incidence matrix $H = \{h_{ki}\} \in \mathbb{R}^{M\times N}$, whose entries are defined as (with arbitrary edge orientations for the *undirected* formations considered here)

$$h_{ki} = \begin{cases} 1, & \text{the } k\text{-th edge sinks at node } i \\ -1, & \text{the } k\text{-th edge leaves node } i \\ 0, & \text{otherwise} \end{cases}$$

For a connected and undirected graph, one has $\text{rank}(L) = N - 1$ and $\ker(L) = \ker(H) = \text{span}\{\mathbf{1}_N\}$.

Given a vertex element $i \in \mathcal{V}$ we associate to it a point p_i of Euclidean space $\mathbb{R}^d$. The column vector $p = \text{col}(p_1, \dots, p_N)$ thus describes a *formation* $(\mathcal{G}, p)$ of N agents, labelled by the set of vertices of $\mathcal{G}$. For any edge $k \in \mathcal{E}$ with head j and tail i which is consistent with the construction of the matrix H, consider the associated relative position vector defined as $z_k = p_j - p_i$. Let

$$z = \text{col}(z_1, z_2, \cdots, z_M) := [z_1^\top, z_2^\top, \cdots, z_M^\top]^\top \in \mathbb{R}^{dM}$$
$$D(z) = \text{diag}(z_1, z_2, \cdots, z_M) \in \mathbb{R}^{dM\times M}$$

denote the associated column vector and block diagonal matrix, respectively. Note that there holds $z = (H \otimes I_d)p$. With this notation at hand, we consider the smooth distance map

$$\mathcal{D} : \mathbb{R}^{dN} \longrightarrow \mathbb{R}^M, \mathcal{D}(p) = (\|p_i - p_j\|^2)_{(i,j)\in\mathcal{E}} = D(z)^\top z.$$

The **rigidity matrix** then is defined as the Jacobian matrix $R(z) = \frac{1}{2}\partial\mathcal{D}(p)/\partial(p)$. By inspection, $R(z)$ is the $M \times dN$ matrix given as

$$R(z) = D(z)^\top (H \otimes I_d) \tag{3}$$

Let $d_{k_{ij}}$ denotes the desired distance of edge k which links agent i and j. In the following, the set of all frameworks $(\mathcal{G}, p)$ which satisfies the distance constraints is referred to as the set of *target formations*. We further define $e_{k_{ij}} = \|p_i - p_j\|^2 - (d_{k_{ij}})^2$ to denote the squared distance error for edge k. Note we may also use e_k and d_k occasionally for notational convenience in the sequel if no confusion is expected. The squared distance error vector is denoted by $e = [e_1, e_2, \cdots, e_M]^\top$.

We consider the following formation control system

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)(p_i - p_j), \quad i = 1, \dots, N \tag{4}$$

which defines the steepest descent gradient flow of the potential function

$$V(p) = \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)^2 \tag{5}$$

This potential function (5) for rigid shape stabilization and the associated gradient flow (4) has been extensively studied in the literature (see e.g. [1], [2], [15], [5], [4], [9], [16]). More general forms of the controllers and the associated convergence results which extend the control form of (4) have been discussed in [17]. In this paper we focus on the formation system described by the gradient flow (4) and the potential function (5). More general results can be obtained for other potential functions by using the results in [17].

IV. Rank-preserving property for formation systems

This section aims to show the rank-preserving property of the gradient system (4). The results hold for formation systems in general dimensions.

By the definition of the incidence matrix H and the rigidity matrix R, we can write the position system (4) in a compact form

$$\dot{p}(t) = -R^\top(z)e = -(H^\top \otimes I_d)D(z)e \tag{6}$$

Then the dynamics for the relative position vector z can be written as

$$\begin{aligned}\dot{z}(t) &= (H \otimes I_d)\dot{p} = -(H \otimes I_d)R^\top e \\ &= -\left((HH^\top) \otimes I_d\right) D(z)e\end{aligned} \tag{7}$$

The following centroid-invariant result of the formation control system is well known (see e.g., [1], [17]).

***Lemma** 3:* The formation centroid is invariant under the gradient flow (6).

We will then show more invariance results, beginning with the following *rank-preserving* property of the formation system (6).

A. Proof of the rank-preserving property

We summarize one of the main results in the following theorem.

***Theorem** 1:* Denote $P := [p_1, p_2, \cdots, p_N] \in \mathbb{R}^{d\times N}$ with the i-th column being agent i's position vector p_i, and denote $Z := [z_1, z_2, \cdots, z_M] \in \mathbb{R}^{d\times M}$ with the k-th column being the relative position vector z_k. Then rank(P) and rank(Z) are constant along any finite time solution $p(t)$ of (6).

Proof: Note that $R^\top(z)e = (H^\top \otimes I_d)De$ according to the expression of the rigidity matrix. By defining a diagonal matrix $\bar{E} = \text{diag}(e_1, e_2, \cdots, e_M) \in \mathbb{R}^{M\times M}$, one can show that

$$(\bar{E} \otimes I_d)(H \otimes I_d)p = (\bar{E} \otimes I_d)z = De \tag{8}$$

Hence, there holds

$$\begin{aligned}R^\top(z)e =&(H^\top \otimes I_d)De \\ =&(H^\top \otimes I_d)(\bar{E} \otimes I_d)(H \otimes I_d)p \\ :=&(E \otimes I_d)p\end{aligned} \tag{9}$$

where the matrix E is defined as $E = H^\top \bar{E} H$. Note that E is a symmetric matrix, which has the same structure as the **stress matrix** in graph rigidity theory [18]. In later sections we will see the matrix E plays an important role in the stability analysis of degenerate equilibria.

By doing the above matrix operation, the gradient flow for the formation control system (4) can be rewritten as

$$\dot{p}(t) = -R^\top(p)e(p) = -(E(p) \otimes I_d)p \tag{10}$$

The vector differential equation (10) can be equivalently stated as the following differential flow on the matrix space $\mathbb{R}^{d\times N}$ (without involving the Kronecker product term)

$$\dot{P}(t) = -P(t)E^\top(p(t)) = -P(t)E(p(t)) \tag{11}$$

Since the solution of (4) is well defined and can be extended to $t \to \infty$ which excludes the case of finite escape time, the existence and uniqueness of the solution to (11) is well guaranteed. Actually, the matrix E can be written as $E(P)$, i.e. a smooth matrix-valued function $E(P)$ of the variable P. Then according to Lemma 2, the rank-preserving property of the matrix flow (11) follows by observing $B(t) = -E(p(t))$ and $A(t) = 0$.

Similarly, the relative position z system (7) can be written as

$$\begin{aligned}\dot{z}(t) &= -\left((HH^\top) \otimes I_d\right) De \\ &= -\left((HH^\top) \otimes I_d\right) (\bar{E} \otimes I_d)z \\ &= -\left((HH^\top \bar{E}) \otimes I_d\right) z\end{aligned} \tag{12}$$

The vector differential equation (12) can be equivalently stated as the following differential flow on the matrix space $\mathbb{R}^{d\times M}$ (without involving the Kronecker product term)

$$\dot{Z}(t) = -Z(t)(HH^\top \bar{E}(t))^\top = -Z(t)\left(\bar{E}(t)HH^\top\right) \tag{13}$$

The existence and uniqueness of the solution to (13) is guaranteed by the solution property of (7). Then according to Lemma 2, the rank-preserving property relating to rank$(Z(t))$ is proved by observing $B(t) = -\bar{E}(t)HH^\top$ and $A(t) = 0$. ■

B. Consequences of the rank-preserving property

We first define *non-degenerate* formations of $\mathbb{R}^d$ as those that are not contained in any linear subspace of $\mathbb{R}^d$ with dimension less than d. If one restricts the discussion for the case of $d = 2, 3$ (i.e. the 2-D formation control and the 3-D formation control), one can obtain the following invariance properties which are direct consequences of Theorem 1.

***Corollary** 1:* The set of collinear positions is invariant in 2-D formation systems, and the sets of collinear or coplanar positions are invariant for 3-D formation systems. That is, for 2-D formations, if all the agents start with collinear positions, they will always be in collinear positions under the gradient flow (6). Similarly, for 3-D formations, if all the agents start with coplanar (resp. collinear) positrons, then they will always be in coplanar (resp. collinear) positions under the gradient flow (6).

For the formation control system in the presence of distance mismatches (which is termed a **mismatched formation flow** [19]), the rank-preserving property still holds; see the proof in [20].

***Remark** 2:* A similar dimension-invariant result for formation control systems is shown in some recent papers [21], [22], which is termed a **path-connected property**. The authors in [21], [22] used Lie theory for proving the property. Here we are using a simpler method based on rank-preserving flows for the proof. Note that this property also holds for formation systems modelled by *directed* graph. We will provide this extension in the full version of this paper.

The question of whether rank-invariant property of the gradient flow holds in the limit of (6) for $t \to \infty$ is not immediately answerable. One can observe though that since any solution $p(t)$ of the real analytic gradient flow converges to an equilibrium point $\bar{p}$, the limit $P(\infty) = \bar{P}$ exists with $\text{rank}(P(\infty)) \leq \text{rank}(P(0))$. [1] For the following analysis, we will assume the target formation is realizable by a non-degenerate configuration in $\mathbb{R}^d$. In the case of $d = 2, 3$, the limiting rank question involves a convergence analysis of collinear/coplanar equilibria for the 2-D (and 3-D) formation control systems. These will be discussed in next sections.

V. Stability analysis for degenerate equilibria

The results in this section are extended from our previous work [4] and [7]. In [4] it shows that for a 2-D triangular formation with achievable shapes, all the agents will globally converge to the correct shape if they start with non-collinear positions. In another paper [7] we proved that for the 3-D tetrahedral formation, if all the agents start with non-coplanar positions, then they will globally converge to the correct shape. In this section we will continue the analysis to obtain very general results for formation systems with general dimensions and shapes.

In this section, for a formation control system in d dimensional spaces, we call an equilibrium point $\bar{p}$ with $\text{rank}(\bar{P}(\infty)) < d$ an *incorrect degenerate equilibrium*, according to the fact that the affine space that embeds $\bar{p}$ has dimension less than d and at such degenerate equilibrium the potential function is not zero (i.e. the target formation shape is not achieved). First we show a result concerning the existence of such incorrect degenerate equilibria.

***Lemma** 4:* For a formation system (6) with the desired target shape achievable only in d-dimensional space, there exist incorrect equilibria $\bar{p}$ for which $\text{rank}(\bar{P}) < d$. Consequently, there always exist collinear equilibria for 2-D formation systems and collinear/coplanar equilibria for 3-D formation systems.

The proof is omitted due to the space limit. Note that the above Lemma also shows that degenerate equilibria $\bar{p}$ for the d-dimensional formation system can be *attractive* when agents start at degenerate initial positions $p(0)$ with $\text{rank}(P(0)) < d$. The aim of this section is to show that such equilibria are generically unstable when agents start with generic initial positions with $\text{rank}(P(0)) = d$.

A. The equilibrium set

In this subsection we provide notations to denote different sets of equilibria of (6) (i.e. the critical points of V). [2] The set for all equilibria is described as $\mathcal{M} = \{p \in \mathbb{R}^{dN} | R(p)^\top e(p) = (E(p) \otimes I_d)p = 0\}$ while the set of correct equilibria is denoted by $\mathcal{M}_c = \{p \in \mathbb{R}^{dN} | R(p)^\top e(p) = (E(p) \otimes I_d)p = 0, e = 0\}$ and the incorrect equilibria set is denoted by $\mathcal{M}_i = \{p \in \mathbb{R}^{dN} | R(p)^\top e(p) = (E(p) \otimes I_d)p = 0, e \neq 0\}$ with $\mathcal{M} = \mathcal{M}_c \cup \mathcal{M}_i$.

According to Lemma 4, a subset of $\mathcal{M}_i$ is the set of degenerate equilibria denoted by

$$\mathcal{M}_{\text{d}} = \{p \in \mathbb{R}^{dN} | R(p)^\top e(p) = (E(p) \otimes I_d)p = 0, \text{rank}(P) < d\} \tag{14}$$

Our particular interest will be on 2-D and 3-D formations, with the collinear equilibria (for both 2-D and 3-D formations) denoted by $\mathcal{M}_{\text{collinear}} = \{p \in \mathbb{R}^{dN} | R(p)^\top e(p) = (E(p) \otimes I_d)p = 0, \text{rank}(P) = 1\}$ and the set of incorrect coplanar equilibria (for 3-D formations) denoted by $\mathcal{M}_{\text{coplanar}} = \{p \in \mathbb{R}^{dN} | R(p)^\top e(p) = (E(p) \otimes I_d)p = 0, \text{rank}(P) = 2\}$ Note that for a general 2-D formation with $N \geq 4$ agents, one has $\mathcal{M}_{\text{collinear}} \subset \mathcal{M}_i$, and for a general 3-D formation with $N \geq 5$ agents, there holds $(\mathcal{M}_{\text{collinear}} \cup \mathcal{M}_{\text{coplanar}}) \subset \mathcal{M}_i \subset \mathcal{M}$. This means that for a 2-D/3-D general formation there exists additional incorrect equilibrium set other than the collinear/coplanar equilibria (see [10], [2]). It is an open problem to determine the existence and properties of *stable incorrect* equilibria in rigid formation control. The following analysis can be seen as a further step towards tackling this problem, and we will show that all the degenerate equilibria are unstable.

B. Eigenvalue property of E and the Hessian

We first show a result on the eigenvalue property of E which holds for formation systems in arbitrary dimensions.

***Lemma** 5:* The matrix $E(\bar{p})$ has at least one negative eigenvalue at an incorrect equilibrium $\bar{p} \in \mathcal{M}_i$.

The proof is a generalization of the 2-D three-agent case in [23] and the 3-D four-agent case in [7] and is omitted here. In the sequel, we will then use Lemma 5 to study the stability of degenerate equilibria for a general formation system, which include collinear/coplanar equilibria for 2-D/3-D formations as special cases. Note that the closed-loop formation control system (6) describes a descent gradient flow of the potential function V defined in (5), and the equilibrium points of (6) are the same as the critical points of V. The Jacobian of the right hand side of the formation system (6) is the same as the negative of the Hessian of V, which is denoted as $H_V(p)$. We show there is a nice and very useful expression for the Hessian $H_V(p)$ (see e.g. [2], [23]):

$$H_V(p) = 2R^\top(p)R(p) + (E(p) \otimes I_d) \tag{15}$$

The core idea is the eigenstructure analysis of the Hessian $H_V(p)$ of the potential function V, which will be detailed in next subsections.

C. Degenerate equilibria are unstable

The nature of an equilibrium (of being a local minimum, a saddle point or a local maximum) can be determined by the signs of the eigenvalues of the Hessian at that equilibrium. Firstly, we introduce a transformation matrix $T \in \mathbb{R}^{dN \times dN}$,

[1]One typical example of $\text{rank}(P(\infty)) < \text{rank}(P(0))$ comes from the formation control problem with unrealizable shapes [23]: If the triangle inequality does not hold for the desired distances in a triangular shape control problem, then all the agents will converge to a *stable collinear equilibrium* for which $\text{rank}(P(\infty)) = 1$, even if they start with noncollinear positions with $\text{rank}(P(0)) = 2$. Note that for such flows the rank-preserving property still holds for any finite time but at the limit $t = \infty$ the rank reduces.

[2]The equilibria form sets, with each set obtainable by translation and rotation of any point in the set (see discussions in e.g. [8], [10]). For example, for all equilibria for the 2-D formations, there will always be three eigenvalues of $H_V(p)$ which are zero, corresponding to the rotational and translational invariance.

which is used to transform the rigidity matrix into another form:

$$RT = [R_1\, R_2\, \cdots R_m \cdots R_d] = \bar{R}$$

where $R_m \in \mathbb{R}^{M \times N}$ is the sub-matrix of R whose columns consist of the columns of R corresponding to the m-th coordinate. Taking 2-D formation case as an example, T is constructed to transform the rigidity matrix: $RT = [R_x,\, R_y] = \bar{R}$, where R_x and R_y are the sub-matrices whose columns consist of the columns of R corresponding to the coordinates x and y, respectively. By doing this, one can obtain a transformed Hessian matrix $\bar{H}_V(p)$ as

$$\begin{aligned}\bar{H}_V(p) &= T^\top H_V(p) T \\ &= 2 \begin{bmatrix} R_1^\top \\ R_2^\top \\ \vdots \\ R_d^\top \end{bmatrix} [R_1\ R_2 \cdots\ R_d] + I_d \otimes E(p) \end{aligned} \qquad (16)$$

Note that T is a permutation matrix and thus is orthogonal (i.e. $TT^\top = I$); its expression is omitted here due to the space limit. Hence, the eigenvalues of $H_V(p)$ and $\bar{H}_V(p)$ are the same, and in the following we shall consider the eigenvalues of $\bar{H}_V(p)$.

In the following analysis, without loss of generality, we will study the stability of the degenerate equilibria $\bar{p}$ with $\text{rank}(\bar{P}) = d-1$; the case for $\text{rank}(\bar{P}) < d-1$ can be proceeded similarly. Further note that the stability of an equilibrium point is independent of the action of the group $SE(d)$. In particular, only relative positions matter. We further suppose, again without loss of generality, that the degenerate equilibrium formation lives in the $d-1$ dimensional space spanned by the basis of the first $d-1$ coordinates. That is, $R_d = 0$ and one obtains the special expression of the Hessian in (17) (shown in next page), which will be the key to identify the property of degenerate equilibria.

The following result is a consequence of Lemma 5.

***Lemma** 6:* The matrix $\bar{H}_V(\bar{p})$ and consequently the Hessian $H_V(\bar{p})$ have at least one negative eigenvalue at an incorrect degenerate equilibrium $\bar{p} \in \mathcal{M}_d$. Thus the degenerate equilibria are unstable saddle points.

The proof is omitted here due to the space limit and will be provided in the full version of this paper. All the above lemmas then lead to the following main result.

***Theorem** 2:* Suppose the set of desired distances $d_{k_{ij}}$ is realizable by a non-degenerate formation in $\mathbb{R}^d$. Starting with generic initial positions in the d-dimensional space, agents' trajectories will not converge to any degenerate equilibrium and their positions will be bounded away from any degenerate positions in a lower dimensional space.

Proof: The linearized version of the formation system (10) at an equilibrium point $\bar{p}$ can be described as

$$\dot{p} = -H_V(\bar{p})p = -T\bar{H}_V(\bar{p})T^\top p \qquad (18)$$

In Lemma 6 it proves that there exists at least one negative eigenvalue for the Hessian $H_V(\bar{p})$ at a degenerate equilibrium $\bar{p}$. We denote the corresponding eigenvector as $v^-(H_V)$, where the unstable subspace of the linearized system (18) at $\bar{p}$ associated with the eigenvector $v^-(H_V)$ can be described as $p = \bar{p} + \text{span}(v^-(H_V))$. This indicates that the stable subspace, with initial positions chosen in this subspace converging to $\bar{p}$, will have dimension at most $d-1$, let alone the fact that there could be more negative eigenvalues in $H_V(\bar{p})$. It follows that such set of initial positions is a thin set and actually has measure zero in the d dimensional space. In fact, the Center Manifold Theorem [24] implies for the nonlinear system (10) that in a small neighborhood of the degenerate equilibrium $\bar{p}$ there exists an invariant local stable (resp. unstable) manifold tangent to the stable (resp. unstable) subspace of (18). The set of initial points which lie in the invariant stable manifold and converge to $\bar{p}$ does not contain an interior point of $\mathbb{R}^d$ and is a set of first category. By the Baire category theorem [25, Page 34], the union of the stable manifolds of the finitely many incorrect degenerate equilibria is nowhere dense, i.e. it is contained in the complement of a residual set. This shows that the gradient flow (10) for generic initial conditions does not converge to an incorrect degenerate equilibrium point.

By combining the results from Theorem 1 and the above argument, one can conclude that generic trajectory that starts with generic initial position $p(0)$ in $\mathbb{R}^d$ will always be bounded away from any degenerate positions for which $\text{rank}(P(t)) < d$. ■

The following result is a direct consequence of Theorem 2, which greatly simplifies some previous results reported in e.g. [4], [5], [7].

***Corollary** 2:* For 2-D formations, agents' trajectories with generic initial positions in the 2-D space will not approach an incorrect collinear equilibrium.

***Corollary** 3:* For 3-D formations, agents' trajectories with generic initial positions in the 3-D space will not approach an incorrect collinear/coplanar equilibrium.

***Remark** 3:* One may be concerned with the physical interpretations and applications of the above result for the case of $d > 3$, as real-life implementable formation systems are always in the physical space that has $d = 1, 2, 3$ dimension(s). Here we mention that the results may have implications in other fields. One closely related area is the multidimensional scaling (MDS) research using an s-stress function [13] (where d can take values greater than 3). Note that the stress function usually takes the same form of the potential function described in (5) as what people in control community often use for formation control (see an application in [15]). The MDS problem aims to find a desired configuration with a specific embedding dimension to minimize a stress function. Suppose there exists a desired configuration in $\mathbb{R}^d$ which minimizes a given stress function to zero. The results in this paper indicate that any degenerate critical point of such stress function in an embedding subspace of dimension less than d is unstable.

VI. Conclusions and some open problems

In this paper we have shown two main results for rigid formation control systems in arbitrary dimensions. The first one is a rank-preserving property, which indicates that if

$$\bar{H}_V = \begin{bmatrix} 2R_1^\top R_1 + E & 2R_1^\top R_2 & \cdots & 2R_1^\top R_{d-1} & \mathbf{0} \\ 2R_2^\top R_1 & 2R_2^\top R_2 + E & \cdots & 2R_2^\top R_{d-1} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2R_{d-1}^\top R_1 & 2R_{d-1}^\top R_2 & \cdots & 2R_{d-1}^\top R_{d-1} + E & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & E \end{bmatrix} \tag{17}$$

all the agents' positions start in the Euclidian space with a specific dimension, then their trajectories will live in that space with the same dimension. The second main result is on the stability of the degenerate equilibria, which are proven to be unstable. The consequence is that if all the agents start with generic positions in the full dimensional space, then their trajectories will be strictly bounded away from the set of degenerate formations at any finite or infinite time.

This paper shows a further step to understand the formation gradient system and the property of the critical points of the potential function. We also list some open problems in the future research on rigid formation control:

- It has not been determined whether there exist local stable minima for a formation system with a general target shape. Some recent efforts have been devoted to understanding formation systems with a simple shape like triangular shape [4], tetrahedral shape [7] and rectangular shape [6]. One may start with the special case of a 2-D K4 formation, and determine the uniqueness of local/global minima.
- It is desirable to study other types of invariants apart from the rank and centroid such that more properties of the formation system can be explored.
- An outstanding issue is to establish the Morse function property when the desired formation is in a $d \geq 2$ dimensional space. It has been established for one-dimensional formations [2] and for *complete* formations in higher dimensions [26]. The analysis in [10] provided some bounds on the number critical points of two gradient-type formation systems based on the assumption that the potential is an equivariant Morse function. However, it is still unknown whether the potential is generically an equivariant Morse function.
- The general issue is to find algebraic invariants of the gradient flow in terms of computer algebra methods, which may facilitate the analysis with the aim of available computation algorithms.

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