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Rigid formation control systems modelled by double integrators: system dynamics and convergence analysis

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Abstract— **In this paper we study rigid formation control systems modelled by double integrators. Two kinds of double-integrator formation systems are considered, namely, the formation stabilization system and the flocking control system. Certain novel observations on the null space and eigenvalues of the system Jacobian matrix will be provided, which reveal important properties of system dynamics and the associated convergence results. We also establish some links between single-integrator formation systems and double-integrator formation systems via a parameterized Hamiltonian system, which further provide stability criteria for different equilibria in double-integrator formation systems by using available results in single-integrator formation systems.**

I. Introduction

Rigid formation shape control for a collection of n point agents in Euclidean space is concerned with designing distributed control laws for individual agents so that the formation can converge to a prescribed rigid shape specified by a certain set of desired inter-agent distances [1], [2]. In the literature, most results on rigid formation control are based on simple single-integrator formation models; see e.g. [1], [2], [3], [4]. In this paper, we will consider formation control systems modelled by double integrators, motivated by the fact that double-integrator system serves as a somewhat more natural model to describe many real-life control systems.

Double-integrator models have been studied extensively for flocking control of multi-agent systems, partly originated by the pioneering work [5] and [6]. Another active research topic on double-integrator models which received particular interest in recent years is the linear consensus problem; see e.g. [7], [8]. However, for rigid formation control systems modelled by double integrators, the results appear rather sparsely in the literature; see e.g. [9], [10], [11]. A recent paper [12] showed the possibility of combining a linear consensus algorithm and rigid shape control to achieve a desired rigid flocking movement. Note that all these papers only focused on some local convergence analysis. Due to the nonlinear property of the formation controller for stabilizing rigid shapes, a full characterization of the convergence analysis is quite challenging. Actually, there are many open issues for rigid formation control systems when they are modelled by double integrators, which include equilibrium properties, convergence analysis, and robustness issues, etc.

This paper aims to provide more insights on system dynamics and convergence property of double-integrator rigid formation systems. We will consider two types of formation systems: the formation stabilization system and the flocking control system. The main contributions of this paper are summarized as follows. First, we will show novel results on the null space and zero eigenvalue of the Jacobian matrix for the vector function in the double-integrator system to reveal system dynamical properties. Second, compared with the analysis and results in [9], [10], [11] and [12], we do not confine the analysis to local convergence analysis of the desired equilibrium set (i.e. those corresponding to correct formation shapes). Instead, we aim to provide more characterizations for the convergence properties of any equilibrium set, including those that do not correspond to the desired equilibrium. Third, invariant properties and links between single-integrator formation systems and double-integrator systems will be established. This will be done by employing a parameterized Hamiltonian system, an idea which was also used for power network analysis [13], [14], [15] and was briefly mentioned in a recent paper [10] on formation control. We will show how available results on characterizing equilibrium properties in single-integrator systems (e.g. [1], [4], [16], [17]) can be readily extended to the stability analysis for double-integrator formation systems.

The remaining parts of this paper are organized as follows. Section II briefly reviews some background and then introduces the relevant system equations. Section III presents results on local convergence and Jacobian matrix analysis using linearization technique. By exploring a parameterized Hamiltonian system, Section IV shows an invariance principle and certain equilibrium characterization on double-integrator formation systems by relating them to available results in single-integrator formation systems. Extensions of the results to flocking formation system are discussed in Section V. Finally, conclusions are provided in Section VI which closes this paper. Due to the space limit, we do not provide any simulation or illustration, which will be presented in the full version of this paper.

Notations. The notations used in this paper are fairly standard. The rank, determinant and null space of a matrix M are denoted by rank(M), det(M) and null(M), respectively. We use span$\{v_1, v_2, \cdots, v_k\}$ to represent the subspace

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spanned by a set of k vectors $v_1, v_2, \cdots, v_k$. We denote the $n \times n$ identity matrix and zero matrix as $I_{n\times n}$ and $\mathbf{0}_{n\times n}$ respectively. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ be the n-dimensional vectors of all ones and all zeros. When the subscripts for $I_{n\times n}$, $\mathbf{0}_{n\times n}$, $\mathbf{1}_n$ or $\mathbf{0}_n$ are omitted, their dimensions should be clear in the context. The inertia of a matrix $M \in \mathbb{R}^{n\times n}$ is given by the triple $\{\nu_+, \nu_-, \nu_0\}$, where ν_+ (respectively, ν_-) denotes the number of unstable (respectively, stable) eigenvalues of M in the open right (respectively, left) complex half plane, and ν_0 denotes the number of eigenvalues with zero real part. The symbol $\otimes$ denotes Kronecker product.

II. Background and System Equations

A. Basic concepts on graph and rigidity theory

Consider an undirected simple graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The matrix relating the nodes to the edges is called the incidence matrix $H = \{h_{ki}\} \in \mathbb{R}^{m\times n}$, whose entries are defined as (with arbitrary edge orientations for the *undirected* formation considered here)

$$h_{ki} = \begin{cases} 1, & \text{the } k\text{-th edge sinks at node } i \\ -1, & \text{the } k\text{-th edge leaves node } i \\ 0, & \text{otherwise} \end{cases}$$

The Laplacian matrix $L(\mathcal{G})$ will often be used for matrix representation of a graph $\mathcal{G}$, which is defined as $L(\mathcal{G}) = H^T H$. For a connected undirected graph, there holds $\text{rank}(L) = n - 1$ and $\text{null}(L) = \text{null}(H) = \text{span}\{\mathbf{1}_n\}$.

Let $p_i \in \mathbb{R}^d$ where $d = \{2, 3\}$ denote a point that is assigned to $i \in \mathcal{V}$. The stacked vector $p = [p_1^T, p_2^T, \cdots, p_n^T]^T \in \mathbb{R}^{dn}$ represents the realization of $\mathcal{G}$ in $\mathbb{R}^d$. The pair $(\mathcal{G}, p)$ is said to be a framework (specifically, a *formation* in the context of our control problem) of $\mathcal{G}$ in $\mathbb{R}^d$. By introducing the matrix $\bar{H} := H \otimes I_{d\times d} \in \mathbb{R}^{dm\times dn}$, one can construct the relative position vector z as follows

$$z = \bar{H} p \tag{1}$$

where $z = [z_1^T, z_2^T, \cdots, z_m^T]^T \in \mathbb{R}^{dm}$, with $z_k \in \mathbb{R}^d$ being the relative position vector for the vertex pair defined by the k-th edge.

Define $Z(z) = \text{diag}(z_1, z_2, \cdots, z_m) \in \mathbb{R}^{dm\times m}$. With this notation at hand, we consider the smooth distance map

$$r_{\mathcal{G}} : \mathbb{R}^{dn} \longrightarrow \mathbb{R}^m, r_{\mathcal{G}}(p) = (\|p_i - p_j\|^2)_{(i,j)\in\mathcal{E}} = Z^T z. \tag{2}$$

This paper focuses on formation control of rigid shapes. The definition of graph rigidity can be found in e.g. [1]. A useful tool to study graph rigidity is the **rigidity matrix**, which is defined as the Jacobian matrix $R(p) = \frac{1}{2}\partial r_{\mathcal{G}}(p)/\partial(p)$. By inspection, $R(p)$ is an $m \times dn$ matrix given as

$$R(p) = Z(z)^T \bar{H} \tag{3}$$

Note that the entries of $R(p)$ involve only relative position vectors z, and we can rewrite it as $R(z)$.

B. Motion equations

Let $d_{k_{ij}}$ denotes the desired length of edge k which links agent i and j. We assume that the set of desired lengths is realizable, i.e., there exists a formation whose inter-agent distances correspond to the desired values. In the following, the set of all formations $(\mathcal{G}, p)$ which satisfies the distance constraints is referred to as the set of *target formations*. [1] We further define (for an arbitrary formation)

$$e_{k_{ij}} = \|p_i - p_j\|^2 - d_{k_{ij}}^2 = \|z_k\|^2 - d_{k_{ij}}^2 \tag{4}$$

to denote the squared distance error for edge k. Note we may also use e_k and d_k occasionally for notational convenience in the sequel if no confusion is expected. The squared distance error vector is denoted by $e = [e_1, e_2, \cdots, e_m]^T$.

Most papers on rigid formation control have considered the following formation control system modelled by a single integrator

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)(p_i - p_j), \quad i = 1, \ldots, n \tag{5}$$

which defines the steepest descent gradient flow of the potential function

$$V(p) = \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)^2 \tag{6}$$

In a compact form, we can rewrite (5) as (see e.g. [17])

$$\dot{p}(t) = -\nabla_p V = -R^T(z)e(z) \tag{7}$$

Two kinds of double-integrator rigid formation systems will be considered in this paper. The first one is a model with velocity damping term, which aims to stabilize a desired rigid shape and achieve a stationary formation (the final formation should come to a rest). The formation system can be described by the following form:

$$\begin{aligned} \dot{p}_i &= v_i \\ \dot{v}_i &= -k_i v_i - \sum_{j\in\mathcal{N}_i} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)(p_i - p_j) \end{aligned} \tag{8}$$

where $v_i \in \mathbb{R}^d$ is the velocity of agent i and k_i is a positive control gain which is freely selectable by the system designer. Define

$$\psi(p, v) := \frac{1}{2} \sum_{i\in\mathcal{V}} \|v_i\|^2 + V(p) \tag{9}$$

Then in a compact form, we can rewrite the above *double-integrator formation stabilization* system as

$$\mathcal{H}_{\text{formation}} : \begin{aligned} \dot{p} &= \nabla_v \psi = v \\ \dot{v} &= -\mathcal{K}\nabla_v \psi - \nabla_p \psi \\ &= -\mathcal{K}v - R^T(z)e(z) \end{aligned} \tag{10}$$

where $\mathcal{K} = K \otimes I_{d\times d}$ and K is a diagonal gain matrix with the i-th diagonal entry being k_i.

[1] A target formation may have arbitrary centroid position, arbitrary orientation, and all agents may have the same arbitrary velocity.

The other model is for achieving a flocking behavior with both velocity consensus and shape stabilization. The overall system can be described by the following equations [2]

$$\begin{aligned}\dot{p}_i &= v_i \\ \dot{v}_i &= \sum_{j\in\mathcal{N}_i}(v_j - v_i) - \sum_{j\in\mathcal{N}_i}(\|p_i - p_j\|^2 - d_{k_{ij}}^2)(p_i - p_j)\end{aligned} \tag{11}$$

In a compact form, we can rewrite the above *double-integrator flocking control* system as

$$\mathcal{H}_{\text{flocking}}: \begin{aligned}\dot{p} &= \nabla_v \psi = v \\ \dot{v} &= -\mathcal{L}v - R^T(z)e(z)\end{aligned} \tag{12}$$

where $\mathcal{L} = L \otimes I_{d\times d}$ and L is the Laplacian matrix for the underlying undirected and connected graph.

Double-integrator formation control has been briefly mentioned in [9], [10], while the double-integrator flocking control system has been discussed in [12], [11], [18]. However, all these previous papers only discussed local convergence analysis of the desired equilibrium set. In this paper, we will explore more convergence characterizations for different equilibrium sets as well as their convergence rates.

III. System dynamics

A. General results on convergence analysis

We first describe the equilibrium set of each formation system. Suppose the formation potential (6) and the target formation are the same for the single-integrator system (7) and double-integrator systems (10) and (12). Denote the set of the equilibrium points of single-integrator systems (7) as $\mathcal{M}_S := \{p^* : \nabla_p V(p^*) = \mathbf{0}\}$. Then for the corresponding double-integrator formation system (10), the equilibrium set can be described as $\mathcal{M}_D := \{(p^*, v^*) : \nabla_p V(p^*) = \mathbf{0}, v^* = \mathbf{0} \in \mathbb{R}^{dn}\}$. Also, the equilibrium set of the corresponding double-integrator flocking system (12) can be described as $\mathcal{M}_F := \{(p^*, v^*) : \nabla_p V(p^*) = \mathbf{0}, \mathcal{L}v^* = \mathbf{0}\} = \{(p^*, v^*) : \nabla_p V(p^*) = \mathbf{0}, v^* = \mathbf{1}_n \otimes \bar{v}\}$, where $\bar{v}$ is the average velocity vector. [3]

The following convergence results for double-integrator formation systems (10) and (12) are well known, and are based on standard Lyapunov arguments and LaSalle's Invariance Principle.

***Lemma** 1:* Each trajectory of the double-integrator formation system (10) converges to an invariant set in the equilibrium set $\mathcal{M}_D$. Also, each trajectory of the double-integrator flocking system (12) converges to an invariant set in the equilibrium set $\mathcal{M}_F$.

[2] Note that there are two types of velocity consensus algorithms depending on different underlying graphs: one is based on undirected underlying graph and the other is based on directed graph (for achieving a leader-following control). In this paper we focus on the first one (with undirected underlying graph for the velocity consensus) for convenience.

[3] Due to the fact that the shape potential V is invariant under arbitrary Euclidean group transformations, any rigid formation control system possesses *continuum equilibria*. For example, for 2-D single-integrator rigid formation systems, the equilibrium set $\mathcal{M}_S$ in which the correct shape is attained is a 3-D manifold [1]. The same property also applies to the equilibrium sets $\mathcal{M}_D$ or $\mathcal{M}_F$ of double-integrator systems.

The proof can be shown by following similar steps from [11], [18] and is omitted here. Furthermore, the following result characterizes the convergence of the desired equilibrium.

***Corollary** 1:* The desired equilibrium set $\bar{\mathcal{M}}_D = \{(p^*, \mathbf{0}) : e(p^*) = \mathbf{0}\}$ of the double-integrator formation system (10) is locally asymptotically stable and the trajectory of (10) converges locally to an equilibrium point in $\bar{\mathcal{M}}_D$. Also, the desired equilibrium set $\bar{\mathcal{M}}_F = \{(p^*, v^*) : e(p^*) = \mathbf{0}, v^* = \mathbf{1}_n \otimes \bar{v}\}$ (which corresponds to multiple orbits of equilibrium points) of the double-integrator flocking system (12) is locally asymptotically stable.

Note that the above result only concerns the local convergence property of the desired equilibrium set. In general, multiple equilibrium sets (which include those corresponding to incorrect shapes) exist for both the double-integrator formation stabilization system (10) and flocking control system (12) due to the nonlinearity of the controller. In the following, we will show some more results on general equilibria by analyzing the Jacobian matrix and by relating the double-integrator formation system to the single-integrator system.

B. Jacobian matrix analysis

From this subsection we will focus on the analysis of the double-integrator formation control system (7), but note that most results can be extended to the corresponding flocking model (12) with slight modifications. Indeed, in Section VI, we will show how to extend the results from formation control system (10) to the flocking control system (12).

Denote the Jacobian matrix of the right-hand vector function of (7) at an equilibrium point p^* as

$$\begin{aligned}J_{p^*} &= \frac{\partial(-R^T(p)e(p))}{\partial p}|_{p=p^*} \\ &= \left(-\frac{\partial R^T(p)}{\partial p}e(p) - R^T(p)\frac{\partial e(p)}{\partial p}\right)|_{p=p^*}\end{aligned} \tag{13}$$

Note that because (7) describes a gradient descent flow of V, J_{p^*} is actually the Hessian matrix of $-V$ and thus is symmetric. Define a diagonal matrix $E = \text{diag}(e_1, e_2, \cdots, e_m) \in \mathbb{R}^{m\times m}$. Then an explicit expression of J_{p^*} can be obtained as (see e.g. [17])

$$\begin{aligned}J_{p^*} &= -2R(p^*)^T R(p^*) - (H^T E(p^*) H) \otimes I_d \\ &= \bar{H}^T\left(-2Z(p^*)Z(p^*)^T - E(p^*) \otimes I_d\right)\bar{H}\end{aligned} \tag{14}$$

Note that, as can be seen from the above formula (14), the null space of J_{p^*} is always non-empty. For example, in the 2-D case, the dimension of null(J_{p^*}) is at least three, with two dimensions reflecting the translation invariance and one dimension reflecting the rotation invariance of a rigid shape.

The Jacobian matrix of the double-integrator formation system (10) at an equilibrium point $(p^*, \mathbf{0})$ can be calculated as

$$J(p^*, \mathbf{0}) = \begin{bmatrix}\mathbf{0}_{dn\times dn} & I_{dn\times dn} \\ J_{p^*} & -\mathcal{K}\end{bmatrix} \tag{15}$$

where J_{p^*} is the Jacobian matrix of (7) defined in (13), and $\mathcal{K}$ is the diagonal gain matrix defined in (10).

The following lemma characterizes the null space of $J(p^*, \mathbf{0})$.

***Lemma** 2:* Suppose the null space of J_{p^*} is spanned by a set of l linearly independent vectors ν_j and denote the null space by $\text{null}(J_{p^*}) = \text{span}\{\nu_1, \cdots, \nu_j, \cdots, \nu_l\}$. Define the corresponding vector $\upsilon_j = [\nu_j^T, \mathbf{0}^T]^T$. Then there holds $\text{null}(J(p^*, \mathbf{0})) = \text{span}\{\upsilon_1, \cdots, \upsilon_j, \cdots, \upsilon_l\}$.

The proof is omitted due to the space limit. The above analysis further leads to the following lemma, which can be easily proved by observing the size of the matrices and the dimensions of the null spaces.

***Lemma** 3:* Suppose at an equilibrium p^*, the rank of the Jacobian matrix J_{p^*} for the single-integrator system (7) is k. Then at the corresponding equilibrium $(p^*, \mathbf{0})$ the rank of the Jacobian matrix $J(p^*, \mathbf{0})$ for the double-integrator system (10) is $dn + k$.

We next show a result on an eigenvalue property of $J(p^*, \mathbf{0})$.

***Lemma** 4:* With the positive definite gain matrix $\mathcal{K}$ as defined previously, the eigenvalues of $J(p^*, \mathbf{0})$ cannot be purely imaginary.

The proof is omitted due to the space limit.

Remark 1: Note that when $\mathcal{K} = 0$ (the damping term is zero) then the above double-integrator system (10) describes a Hamiltonian system. We note one consequence of the above Lemma 4: The non-existence of purely imaginary eigenvalues of the Jacobian $J(p^*, \mathbf{0})$ implies that Hopf bifurcation in the double-integrator formation system (10) cannot occur.

One of the main aims of this paper is to characterize the convergence properties of the double-integrator system (10) (and (12)) in terms of those of the reduced-order single-integrator formation system (7). This aspect will be further explored in next section by studying a parameterized Hamiltonian system.

IV. Perspectives from a parameterized double-integrator formation system

In this section we will consider a family of parameterized formation systems which provides a bridge between the single-integrator gradient formation system (7) and the double-integrator formation system (10) (and further the flocking control system (12)). This parameterization-based idea is inspired by [13], [14] and [15], in which similar approaches were employed for the stability and convergence analysis of Hamiltonian-like power systems ([13], [14]) and oscillator networks ([15]). Consider the double-integrator formation system with a parameter $\lambda \in [0, 1]$ in the following form

$$\mathcal{H}_\lambda : \begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} -\lambda I & (1-\lambda)I \\ -(1-\lambda)I & -\mathcal{K} \end{bmatrix}}_{=:S_\lambda} \begin{bmatrix} \nabla_p \psi \\ \nabla_v \psi \end{bmatrix} \quad (16)$$

In the case that $\lambda = 0$, the above system (16) reduces to the double-integrator formation stabilization system shown in (10). In the case that $\lambda = 1$, the above system (16) then reduces to the following uncoupled gradient system:

$$\begin{aligned} \dot{p} &= -\nabla_p \psi \\ \dot{v} &= -\mathcal{K} \nabla_v \psi \end{aligned} \quad (17)$$

A. Invariance principles

In the following, we will prove an invariance result which relates a gradient system (17) and a double-integrator formation stabilization system (10) via the parameterized Hamiltonian system (16).

***Lemma** 5:* For the one-parameter family of dynamical systems $\mathcal{H}_\lambda$ in (16), the following statements hold:

- **Invariance of equilibria set**: For all $\lambda \in [0, 1]$, the equilibrium set of $\mathcal{H}_\lambda$ is given by the set of critical points of the potential function ψ (i.e. the critical points of (17)) and is independent of λ;
- **Invariance of local stability**: For any equilibrium of $\mathcal{H}_\lambda$ for all $\lambda \in [0, 1]$, the inertia of the Jacobian of the vector function of $\mathcal{H}_\lambda$ is equal to that of the Hessian of ψ and is independent of λ.

Proof: Note that $\mathcal{K}$ is positive definite and is invertible. By the Schur Determinant Formula [19, Theorem 1.1] it follows that $\det(S_\lambda) = \det(-\mathcal{K})\det(-\lambda I + (1-\lambda)^2(-\mathcal{K})^{-1})$, which is non-zero for all $\lambda \in [0, 1]$. Hence, the matrix S_λ is nonsingular for all $\lambda \in [0, 1]$, which implies that the equilibrium set of $\mathcal{H}_\lambda$ for which the right side of (16) is zero is identical with the equilibrium set for which the right-hand side of (17) is zero, irrespective of the value of $\lambda \in [0, 1]$. That is, the equilibrium set of $\mathcal{H}_\lambda$ is given by the critical points of ψ: $\mathcal{M}_{\mathcal{H}_\lambda} = \{(p^*, v^*) : \nabla_p \psi(p^*) = \mathbf{0}, \nabla_v \psi(v^*) = \mathbf{0}\}$ which is independent of the value of λ. Thus the first statement is proved.

The proof of the second statement is inspired by [15]. Denote the Hessian of ψ as $\nabla^2_{p,v}\psi$. The Jacobian of the vector function on the right-hand side of (16) can be written as $J_{\mathcal{H}_\lambda} = S_\lambda \nabla^2_{p,v}\psi$. Denote $A := -J^T_{\mathcal{H}_\lambda}$, $P := \nabla^2_{p,v}\psi$. By noting that P is symmetric, one has

$$\begin{aligned} Q : &= (AP + PA^T) = P\left(-S_\lambda - S_\lambda^T\right)P \\ &= P \begin{bmatrix} 2\lambda I_{dn \times dn} & \mathbf{0}_{dn \times dn} \\ \mathbf{0}_{dn \times dn} & 2\mathcal{K} \end{bmatrix} P \end{aligned} \quad (18)$$

It is obvious that $Q \succeq 0$ for $\lambda \geq 0$, and for $\lambda \neq 0$ we have $\text{null}(Q) = \text{null}(P)$. Hence, by [20, Theorem 5] one can conclude that the non-zero inertia (i.e. the number of $\{\nu_+, \nu_-\}$) of A is the same as the non-zero inertia of P. Also note that $-J_{\mathcal{H}_\lambda}$ and $-J^T_{\mathcal{H}_\lambda}$ have the same set of eigenvalues. These facts further imply that the non-zero inertia of $-J_{\mathcal{H}_\lambda}$ is determined by $\nabla^2_{p,v}\psi$ and is independent of $\lambda \in (0, 1]$. We then consider the case of $\lambda = 0$. Note that the eigenspace corresponding to the eigenvalues with zero real parts of $-J_{\mathcal{H}_\lambda}$ equals the eigenspace of zero eigenvalues of $\nabla^2_{p,v}\psi$ for all $\lambda \in [0, 1]$, which implies that the non-zero inertia of $J_{\mathcal{H}_\lambda}$ cannot change when $\lambda = 0$. In summary, the inertia of $-J_{\mathcal{H}_\lambda}$ equals the inertia of $\nabla^2_{p,v}\psi$ for all $\lambda \in [0, 1]$, and thus the stability property of an equilibrium for (16) is determined by the inertia of the Hessian $\nabla^2_{p,v}\psi$ at that equilibrium and is independent of λ. ■

Remark 2: The above Lemma 5 is not new, but is a reinterpretation of [10, Theorem 4.1] (which again is an extension of [15, Theorem 5.1]). This lemma was presented and used in [10] to show the local asymptotic convergence of a formation stabilization system with double integrators. However, no proof was shown in [10]. We note that a proof for the above result is non-trivial. Also, a by-product of this Lemma 5 shows the equivalence between the eigenspace of the eigenvalues with zero real parts of $J(p^*, \mathbf{0})$ and the null space of $S_\lambda \nabla^2_{p,v}\psi$, which generalizes the result in Lemma 2.

B. Relating double-integrator formation systems to single-integrator formation systems

We now show more results to characterize different equilibrium set of the double-integrator formation systems, by relating them to the available results in single-integrator formation systems (e.g. [1], [4], [17]). The following results are direct consequences of Lemma 5.

***Corollary** 2:* Suppose at an equilibrium p^* in the equilibrium set $\mathcal{M}_S$, the Jacobian matrix J_{p^*} for the single-integrator system (7) has k (resp. j) eigenvalues with positive (resp. negative) real parts, then at the corresponding equilibrium $(p^*, \mathbf{0})$ in the equilibrium set $\mathcal{M}_D$ for the double-integrator system (10), the Jacobian matrix $J(p^*, \mathbf{0})$ has k (resp. $dn + j$) eigenvalues with positive (resp. negative) real parts.

***Corollary** 3:* **(Property of equilibria in single- and double-integrator formation systems)** Suppose for the single-integrator formation system (7), a particular equilibrium manifold $\mathcal{M}_S(p^*)$ is stable (resp. unstable); then for the double-integrator formation system (10), the corresponding equilibrium manifold $\mathcal{M}_D(p^*, \mathbf{0})$ is stable (resp. unstable).

As shown in Lemma 2, the Jacobian matrix $J(p^*, \mathbf{0})$ for (10) is always singular at every equilibrium point. Also, the Jacobian matrix J_{p^*} is always singular, with parts of the null space induced by the rotation and translation invariance of rigid formation shapes. Actually, each equilibrium set is a manifold and the above Corollary 3 concerns the stability of such equilibrium manifold. To this end available results on stability analysis in single-integrator systems (e.g. [1], [4], [16], [17]) can be applied here to determine whether an equilibrium set $\mathcal{M}_D(p^*, \mathbf{0})$ is stable. The following result in particular shows the convergence to a point in an equilibrium manifold and its convergence rate in our rigid formation control problem. With a slight abuse of terminology, we will call the matrix J_{p^*} intrinsically non-singular, if its null space only consists of the subspace induced by shape invariance.

***Lemma** 6:* **(Local exponential convergence)** The trajectory of (10) converges locally *exponentially fast* to an equilibrium point $(p^*, \mathbf{0})$ in an equilibrium manifold $\bar{\mathcal{M}}_D(p^*, \mathbf{0})$, if and only if J_{p^*} is intrinsically positive definite at p^* in the vector space orthogonal to the null space induced by shape invariance.

The proof for Lemma 6 is omitted but can be inferred from Lemma 5 as well as Corollary 2 and Corollary 3. The following lemma further clarifies the topological conjugacy [21, Chapter 4.7] of the two trajectories between (10) and (17), which indicates that they can be continuously deformed to match each other while preserving the parameterization of time. A detailed proof is omitted here due to the space limit.

***Lemma** 7:* **(Topological conjugacy)** Consider the double-integrator formation system (10) and the *uncoupled* single-integrator formation gradient system (17) modelled by the same undirected and rigid underlying graph. Then locally near the desired equilibrium set, their trajectories are topologically conjugate.

Remark 3: Let us take 2-D formation systems as an example to interpret the above Lemma 6 and 7. From the characterization of the null space of the Jacobian matrix, the inertia of the Jacobian matrix at a point in a particular equilibrium set is independent of the choice of that point. If at an equilibrium point p^* in a particular equilibrium set, the Jacobian J_{p^*} of a 2-D single-integrator system (7) has inertia as $\{\nu_+, \nu_-, \nu_0\} = \{0, 2n-3, 3\}$, then such equilibrium set is locally exponentially stable for (7). Actually, by invoking the Center Manifold Theorem [22] and by assuming that the target formation is *infinitesimally rigid* (for definitions see e.g. [1]), it has been proven in [1] that the position of each agent in the gradient system $\dot{p} = -\nabla_p\psi$ in (17) converges *locally exponentially fast* to an equilibrium point in the desired equilibrium set, with the equilibrium point dependent on the initial condition. [4] By combining this result with the result in Lemma 6, one can show that if at an equilibrium point $(p^*, \mathbf{0})$ in a particular equilibrium set $\mathcal{M}_D(p^*, \mathbf{0})$, the Jacobian $J(p^*, \mathbf{0})$ of a 2-D double-integrator system (10) has inertia as $\{\nu_+, \nu_-, \nu_0\} = \{0, 4n-3, 3\}$, then such equilibrium set is locally exponentially stable and the trajectory of (10) converges locally exponentially fast to an equilibrium point in this equilibrium set. In particular, if the target formation is *infinitesimally rigid*, then the double-integrator formation system (10) converges *locally exponentially fast* to an equilibrium point in the desired equilibrium set with desired formation shape. Furthermore, according to the definition of the distance error vector e in (4), we can conclude from the above discussion that inter-agent distances also converge locally exponentially fast to the desired values. It is worth mentioning that there exist other ways for proving local convergence for double-integrator formation systems (see e.g. [10], [11] by invoking either Lojasiewicz's inequality for gradient flow or Barbalat's Lemma), without however showing how fast the convergence is. Note that the *exponential convergence* is a crucial property for studying robustness issues in rigid formation control systems [24], [23]. The robustness analysis will be considered in the future work.

V. Extensions to double-integrator flocking systems

In this section we provide brief discussions to show that the obtained results in previous sections on the double-

[4]Another way for proving the exponential convergence is to focus on the relative position dynamics or distance error dynamics; see e.g. [4], [23].

integrator formation stabilization system (10) can be extended to the formation flocking system (12). Bearing in mind that $\dot{e} = 2R\dot{p}$, we note that the system (10) can be equivalently stated in the following equations which describe the evolution of the distance error e and the velocity term v:

$$\begin{aligned} \dot{e} &= 2R(z)v \\ \dot{v} &= -\mathcal{K}v - R^T(z)e(z) \end{aligned} \tag{19}$$

We then analyze the formation flocking system (12). Let $\bar{v} \in \mathbb{R}^d$ denote the average velocity of all the agents, i.e., $\bar{v}(t) = \frac{1}{n}\sum_{i=1}^n v_i(t)$. Observe that $\bar{v}(t)$ is time-invariant, since a simple calculation using (12) shows that $\dot{\bar{v}} = 0$. We introduce the *velocity disagreement vector* $\delta = [\delta_1^T, \delta_2^T, \cdots, \delta_n^T]^T$, where δ_i is defined as $\delta_i = v_i - \bar{v}$. Then one has $\dot{\delta}_i = \dot{v}_i$ since $\bar{v}$ is constant. Also note that $v_i - v_j = \delta_i - \delta_j$. Therefore, there holds $Rv = Z^T\bar{H}v = Z^T\bar{H}\delta = R\delta$. Hence, one can transform (12) into the following equation:

$$\begin{aligned} \dot{e} &= 2R\delta \\ \dot{\delta}_i &= \sum_{j\in\mathcal{N}_i}(\delta_j - \delta_i) + \sum_{j\in\mathcal{N}_i} e_{k_{ij}}(p_j - p_i) \end{aligned} \tag{20}$$

In a compact form, it can be written as

$$\begin{aligned} \dot{e} &= 2R\delta \\ \dot{\delta} &= -\mathcal{L}\delta - R^T(z)e(z) \end{aligned} \tag{21}$$

Since $\text{null}(L) = \text{span}\{\mathbf{1}_n\}$, there holds $\text{null}(\mathcal{L}) = \text{span}\{\mathbf{1}_n \otimes I_{d\times d}\}$. Thus $\mathcal{L}$ is *positive definite* by restricting it to the vector space of $\text{span}(\mathbf{1}_n \otimes I_{d\times d})^\perp$ in which the velocity disagreement vector δ lives. From this viewpoint, $\mathcal{L}$ serves the same role as the positive definite gain matrix $\mathcal{K}$ in (19). Note that the equilibrium set $\mathcal{M}_F$ can be restated as $\mathcal{M}_F = \{(p^*, \delta^*) : \nabla_p V(p^*) = \mathbf{0}, \delta^* = \mathbf{0}\}$. Thus, the transformed system (21) has *the same structure* as the double-integrator formation system (19), and all previous results on the equilibrium properties and parameterized system analysis for the double-integrator formation system (19) can be readily extended to (21).

VI. Conclusions

In this paper we have considered formation control systems modelled by double integrators, which include the formation stabilization model and flocking control model. Due to the multi-equilibrium property caused by the nonlinear rigid formation controller, a complete analysis of the convergence is quite challenging. This paper serves as a further step to understand the dynamical behavior of such formation control systems. Some properties of the system dynamics and convergence analysis for different equilibria are discussed by analyzing certain properties of the linear system obtained by linearizing around the equilibrium and by relating it to the corresponding Jacobian matrix in the single-integrator counterpart. We then further explore several relationships between the extensively-studied single-integrator formation system and the double-integrator system via a parameterized system; this reveals several novel results on characterizing stability property and convergence rate of different equilibria in double-integrator formation systems.

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