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Controlling Formations with Double Integrator and Passive Actuation

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Abstract

This paper considers the problem where a group of agents must achieve a rigid formation specified by a subset of interagent distances. They must do so by being only able to sense the positions of their neighbors, agents with whom they share a specified desired distance, and knowing only their own velocities. The paper builds upon the work of [12] and [16]. The former assumes that the agents are modeled as single integrators. The latter assumes double integrator dynamics and assumes that agents can also sense the velocities of their neighbors, increasing the communication/sensing burden. In contrast this paper assumes that the agent velocities are generated by the actuation signals through a Positive Real dynamics. Double integrator dynamics happen to be a special case of this. Further, unlike [16] no agent needs its neighbor's velocities. We enunciate a control law consistent with our specification, argue that no law for this problem can be globally stable, and as is done in [12] and [16] for the laws therein, prove its local stability.

1. Introduction

This paper concerns the cooperative attainment and maintenance of rigid formations, [1] specified by a subset of interagent distances. The goal is to induce the agents to achieve and preserve the specified interagent distances in a decentralized manner through limited information exchange between them. Such an objective is driven by important applications. For example, in high resolution earth and deep-space imaging, optimal sensing requires sensors must organize into a formation that has a fixed shape.

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As is now standard we regard each formation as a graph. Each agent is treated as a node. If the distance between two agents is specified then the graph has an edge between the two corresponding nodes. An agent communicates only with those agents with whom it shares an edge.

Past work includes [2], [3] and [4] all of which use graph rigidity theory to study the information exchange architecture underlying such control. Related papers are [5]-[9], that adopt a directed information architecture. Specifically, only one agent of a pair with an edge between them is responsible for maintaining the interagent distance, and has access to the other's position.

The paper directly motivating this work is [12] which formulates a control law and shows that it locally achieves the desired formation shape. The information architecture is rigid and undirected. Specifically, agents at the end points of each edge sense each other's position and use a gradient descent algorithm to move to set the edge to the desired distance. The special case of a four agent architecture represented by a complete graph has been considered by [13], [14] and [15].

As with a vast majority of the references above, [12] assumes a single integrator agent dynamics. In many cases agents do not have a single integrator dynamics and the actuation process maybe much more complicated. This fact motivates the present paper. In fact we go beyond [16], which extends [12] to the case where the agents have a double integrator dynamics.

In its approach [16] recognizes that a prerequisite to achieving the desired formation is that all agents acquire a common velocity. Thus, it layers a velocity consensus law over the distance control objective. Consequently, in addition to exchanging position information, agents additionally exchange velocity information, complicating the information exchange architecture. The result is local attainment of the desired formation and a consensus velocity that is the average of the initial velocities of the agents.

In contrast we assume an actuation dynamics that is passive, [17]- [19]. More specifically, we assume that the agent velocities are the output of a Positive Real

(PR), [20], system whose inputs are the actuating signals. The double integrator dynamics are in fact a special case of our actauation model. Unlike [16], in our scheme no agent needs to know the velocities of the other agents, though we assume that it does know its own velocity. The initial control law is shown to result in locally achieving the desired formation while the velocities converge to zero. We later comment on how the control law can be trivially modified to be able to achieve a prespecified common nonzero velocity.

The rest of this paper is organized as follows. Section 2 reviews pertinent facts from the theory of rigid graphs. Section 3 specifies the problem. Section 4 formulates the control law that achieves a zero limiting velocity. Section 5 analyzes stability and describes a modified law that induces the agents to achieve a common but arbitrary specified velocity. Section 6 concludes.

2. Rigid Formations

We consider n-agents, the i-th having position $p_i = [x_i, y_i]^\top \in \mathbb{R}^2$. We assume that the agents must collectively attain and maintain a rigid formation specified by a subset of interagent distances. In the sequel define $p \in \mathbb{R}^{2n}$ as

$$p = [p_1^\top, \cdots, p_n^\top]. \tag{1}$$

We will associate an *undirected* graph $G = (V, E)$ with the formation $F = (G, p)$ if an edge exists in G between vertices $i, j \in V$ whenever the desired distance $d_{ij} = \|p_i - p_j\|$ is provided in the formation specification. In the tradition of [12], the edge set E also represents the information architecture of the control law we present here, in the sense that in its decentralized execution an agent i can access the position of agent j iff j is its neighbor, i.e. there is an edge in G between the vertices i and j. Unlike [16] agents have access only to their own velocities, as opposed to that of their neighbors. We also say that $p : V \to \mathbb{R}^{2n}$ is a *representation* of the graph G, in that the vertex $i \in V$ has the position p_i in the formation $F = (G, p)$.

A graph is called *rigid* if for almost all representations, all continuous motions preserving the distances between vertices connected by an edge, preserve the distance between every vertex pair. It is globally rigid if even discontinuous motions preserving the distances between vertices connected by an edge, preserve the distance between every vertex pair. A formation $F = (G, p)$ is rigid if G is a rigid graph. By specifying the edge lengths in a rigid formation one specifies the formation to within a translation, rotation and certain ambiguities like a flip ambiguity. Thus Figure 1 depicts two rigid formations with identical edge lengths. However, one cannot move to the other through con-

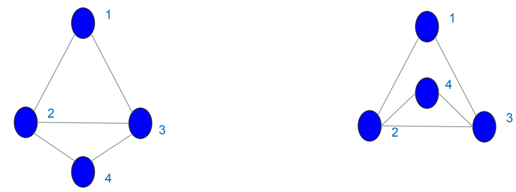

Figure 1. Two rigid formations that correspond to identical edge lengths. One cannot move from one to the other through continuous movements that preserve the edge lengths.

tinuous movements that preserve the edge lengths. The movement of vertex 4, constitutes a flip ambiguity. By contrast, a globally rigid formation is specified by the edge lengths to within a rotation and a translation only, and other amibiguities are precluded.

While several characterizations of rigidity are available in the literature, the one of primary interest here is the *Rigidity Matrix*. For the formation $F = (G, p)$, the rigidity matrix $R(p) : p \to \mathbb{R}^{|E| \times 2n}$ has one row for every edge in G. If the edge is between vertices i and j, then the corresponding row has $(p_i - p_j)^T$ and $(p_j - p_i)^T$ in the $2i-1, 2i$ and $2j-1, 2j$ locations, respectively. The remaining elements of the row are zero. Thus for either graph in Figure 1 $R(p)$ is given by:

$$\begin{bmatrix} (p_1 - p_2)^\top & (p_2 - p_1)^\top & 0_2^\top & 0_2^\top \\ (p_1 - p_3)^\top & 0_2^\top & (p_3 - p_1)^\top & 0_2^\top \\ 0_2^\top & (p_2 - p_3)^\top & (p_3 - p_2)^\top & 0_2^\top \\ 0_2^\top & (p_2 - p_4)^\top & 0_2^\top & (p_4 - p_2)^\top \\ 0_2^\top & 0_2^\top & (p_3 - p_4)^\top & (p_4 - p_3)^\top \end{bmatrix}$$

Then a graph G is rigid iff for almost all representations it has rank $2n-3$. Finally a formation is *minimally rigid* if it is rigid and has precisely $2n-3$ edges.

3. Problem Specification and Assumptions

As in [12] we consider a rigid undirected graph $G = (V, E)$. We will say that $F = (G, \bar{p})$ is a desired formation if it satisfies a given set of desired distance relations $\bar{d}_{ij}^2 = \|\bar{p}_i - \bar{p}_j\|^2$ for all $\{i, j\} \in E$. The vector $D(\bar{p}) \in \mathbb{R}_+^{|E|}$, will comprise the elements $\bar{d}_{ij}^2$. The set of neighbors of i is defined as

$$\mathcal{N}(i) = \{j | \{i, j\} \in E\} \tag{2}$$

Unlike, [12] that assumes that each agent has single integratot dynamics, and [16] that assumes a double integrator dynamics, this paper assumes that these dynamics are more complicated. Thus, with $v_i = \dot{p}_i$ and $v = [v_1^\top, \cdots, v_n^\top]^\top$, the dynamics are represented by:

$$\dot{p} = v, \tag{3}$$

and $v(\cdot)$ is the output of a system with transfer function $g(s)$ whose input is the vector of actuation signals $u = [u_1, \cdots, u_n]^\top : \mathbb{R} \to \mathbb{R}^{2n}$, respectively. Further the transfer function $g(s)$ satisfies the following assumption.

Assumption 3.1 *The scalar transfer function $g(s)$ is minimal strictly proper, minimum phase, and postive real with degree m, and all poles in the open left half plane except possibly one at zero. Further it has minimal realization $\{A, b, c\}$.*

We observe that the double integrator dynamics of [16] is a special case of (3, 6) under Assumption 3.1 as $g(s) = 1/s$ satisfies Assumption 3.1. Define

$$\mathscr{A} = A \oplus A = \text{ diag } \{A, A\}, \mathscr{B} = b \oplus b, \mathscr{C} = c^\top \oplus c^\top. \tag{4}$$

Then in addition to (3) with state vectors $z_i : \mathbb{R} \to \mathbb{R}^{2m}$ the actruation dynamics are:

$$\begin{aligned} \dot{z}_i &= \mathscr{A} z_i + \mathscr{B} u_i & (5) \\ v_i &= \mathscr{C} z_i. & (6) \end{aligned}$$

Define P_i as the position vectors of the neighbors of i. The objective is to obtain a control law that locally achieves:

$$\lim_{t \to \infty} D(p(t)) = D(\bar{p}). \tag{7}$$

We impose the requirement of decentralization in that information exchange architecture must mirror the archtecture of G. Thus, we require that the control input to the i-th agent be restricted to be of the form:

$$u_i = f(P_i, v_i). \tag{8}$$

Thus each agent has access only to the positions of its neighbors and its own velocity. The information architecture is undirected as it mirrors the undirected graph G. There is clearly a noncompact manifold

$$S(\bar{p}) = \{p | D(p) = D(\bar{p})\}. \tag{9}$$

Thus, in general one cannot expect global stability. Even the control law postulated in [12] under single integrator dynamics is only locally stable. We make the following assumption.

Assumption 3.2 *Suppose $F = (G, \bar{p})$ is a desired formation. Then the rigidity matrix $R(\bar{p})$ has rank $2n-3$.*

4. The control law

A natural cost function to address this problem is

$$J(p) = \frac{\|D(p) - D(\bar{p})\|^2}{4}. \tag{10}$$

Clearly it is zero iff $p \in S(\bar{p})$. In the sequel, the gradient of $J(p)$ plays a crucial role. It is readily seen, see also, [15] that

$$\nabla J(p) = R^\top(p)(D(p) - D(\bar{p})). \tag{11}$$

The actuation model used in [12] assumes the single integrator model $\dot{p}_i = u_i(t)$. The control law it proposes then simply selects $u = -R^\top(p)(D(p) - D(\bar{p}))$. As the model here is given by (3-6), we use instead:

$$u = -v - R^\top(p)(D(p) - D(\bar{p})). \tag{12}$$

Evidently we thus have

$$u_i = -v_i - \sum_{j \in \mathscr{N}(i)} \left(d_{ij}^2 - \bar{d}_{ij}^2\right)(p_i - p_j) \tag{13}$$

Thus to execute its control law each agent requires the kowledge of its velocity, its relative position with neighbors and the distance specifications it is involved with.

While the law of [12] is pure gradient descent, being

$$\dot{p} = -\nabla J(p), \tag{14}$$

(see (11)) the law enunciated here is more complicated. Even in the special case of $g(s) = 1/s$, the closed loop becomes:

$$\begin{aligned} \dot{p} &= v \\ \dot{v} &= -\nabla J(p) - v. & (15) \end{aligned}$$

In contrast, to additionally ensure velocity consensus, a prerequisite to achieving a formation through (7), [16] uses a control law that requires a more extensive exchange of velocity information among neighbors.

A key property exploited by [12] is that under (14) the centroid of the agents does not move, as

$$R(p)\mathbf{1} = 0, \tag{16}$$

where $\mathbf{1}$ is a vector of all 1s. With velocity updates the centroid can move though its motion can be studied using (16).

5. Stability

Krick et al prove local stability of (14) by invoking center manifold theory. Instead we use here Lasalle's

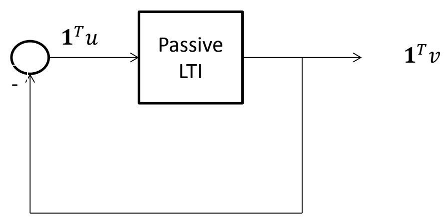

Figure 2. Block diagram useful in proving Lemma 5.1. Forward path is passive, the feedback path is strictly passive.

invariance principle. We begin with some well known facts about positive real (PR) systems. The first is the Kalman-Yacubovich-Popov (KYP) lemma, [20], because of which, under Assumption 3.1 there exist a positive definite symmetric $Q \in \mathbb{R}^{m\times m}$ and $l \in \mathbb{R}^m$ such that:

$$A^\top Q + QA = -ll^\top \quad (17)$$

$$Qb - c = 0. \quad (18)$$

In the context of (4-6) we thus have that with $\mathscr{Q} = Q \oplus Q$ and $\mathscr{L} = l \oplus l$, (17) and (18) lead to:

$$\mathscr{A}^\top \mathscr{Q} + \mathscr{Q}\mathscr{A} = -\mathscr{L}\mathscr{L}^\top \quad (19)$$

$$\mathscr{Q}\mathscr{B} - \mathscr{C} = 0. \quad (20)$$

A direct consequence of this, [20], is the following. Define the storage function for the i-th agent

$$W(z_i) = z_i^\top \mathscr{Q} z_i. \quad (21)$$

Then

$$\dot{W}(z_i(t)) \leq v_i^\top(t) u_i(t),\ \forall t \geq 0. \quad (22)$$

Recall that a consequence of (16) is that under (14) the centroid of the agent positions does not move but that in our setting this may not hold. The first Lemma asserts that under (12) the centroid is still bounded. Its proof uses the fact that the relation depicted in Figure 2 obtains, and thus, [20], $\mathbf{1}^\top v$. converges to zero

Lemma 5.1 *Consider (3-6) under (12) and Assumption 3.1. Then the centroid of the agents*

$$c(p) = \frac{p^\top \mathbf{1}}{n}$$

converges exponentially to a point.

Call

$$z = [z_1^\top, \cdots, z_n^\top]^\top. \quad (23)$$

We proceed using the following Lyapunov like function:

$$L(p,z) = J(p) + \int_0^t v^\top(\tau) u(\tau) d\tau. \quad (24)$$

Then Lasalle's Theorem, [20], can be used to prove the uniform convergence of $\nabla J(p)$ to zero, in fact the convergence to a *point* on the critical surface of $J(p)$.

Theorem 5.1 *Under the conditions of Lemma 5.1 and Assumption 3.2, p and v converge uniformly to the trajectory*

$$v \equiv 0,\ z_i \equiv 0 \text{ and } \nabla J(p) \equiv 0. \quad (25)$$

Further $\dot{L}(p,z) \leq 0$ and for all t

$$\int_0^t v^\top(\tau) u(\tau) d\tau \geq -\sum_{i=1}^n W(z_i(0)), \quad (26)$$

where $W(\cdot)$ is as in (21).

Thus convergence to a critical manifold of $J(p)$ is assured. As also $v \equiv 0$, the convergence is to a point. Of course (25) does not guarantee convergence to the desired manifold $S(\bar{p})$. As noted earlier even (25) is only locally stable. Indeed the theorem below asserts that (12) also leads to local stability.

Theorem 5.2 *Under the conditions of Theorem 5.1 there is a neighborhood $U(\bar{p})$ of $S(\bar{p})$ and a ball $B(\varepsilon) = \{z_i \| \|z_i\| \leq \varepsilon\}$ such that (7) holds for all $p(0) \in U(\bar{p})$ and $z_i(0) \in B(\varepsilon)$.*

As presented, (12) assures that a formation is locally attained with zero limiting velocity. The work of [16] permits attainment of formations with nonzero constant velocities. It does so at the expense of further information exchange. Further, the velocity attained is the average of the initial velocities of the agents. We now show how (12) can be modified to achieve a prescribed common desired velocity, known to each agent.

Suppose this desired velocity vector is $v^* \in \mathbb{R}^2$. Define $\mathscr{V} \in \mathbb{R}^{2n}$ as

$$\mathscr{V} = [v^{*\top}, v^{*\top}, \cdots, v^{*\top}]^\top. \quad (27)$$

Then all it takes to meet the objective is to change (12) to:

$$u = -v - R^\top(p)(D(p) - D(\bar{p})) + \mathscr{V}. \quad (28)$$

That this achieves the desired objective is readily seen from the following facts.

(A) If one replaces p_i by $p_i - v^* t$ and v by $v - \mathscr{V}$ then the input to state model is identical to that produced by (12).

(B)

$$p \in S(\bar{p}) \Leftrightarrow p - \mathscr{V}t \in S(\bar{p}).$$

Thus under (28), (7) and (29) occur locally uniformly.

$$\lim_{t\to\infty} v(t) = \mathscr{V} \tag{29}$$

6. Conclusion

We have considered the problem where a group of agents cooperatively achieve a rigid formation, an objective considered in both [12] and [16]. While [12] assumes a single integrator actuation dynamics, [16] assumes a double integrator dynamics. Our actuation model is more general, being LTI but passive, and captures the dynamics of [16] as a special case. Unlike [16] that assumes that each agent senses its neighbor's position and velocity, the law enunciated here assumes that each agent, while sensing its neighbor's position, only needs its own velocity. As is done in [12] and [16] for the laws therein, we prove the local stability of our control law. We believe that the linearity assumption on the actuation dynamics can be readily relaxed, as long as it is passive. Finding a control law for a more general actuation dynamics is an open issue.

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