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Rigid formation control with prescribed orientation

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Abstract— **Most rigid formation controllers reported in the literature aim to only stabilize a rigid formation shape, while the formation orientation is not controlled. This paper studies the problem of controlling rigid formations with prescribed orientations in both 2-D and 3-D spaces. We propose two control approaches, one with centroid invariance and the other without preserving invariant formation centroid. The proposed controllers involve the commonly-used gradient descent control for shape stabilization, and an additional term to control the directions of certain relative position vectors associated with some chosen agents. The asymptotic convergence of the desired equilibrium is proven using the convergence property of gradient systems. Typical simulation examples are shown to support the analysis and performance of the proposed formation controllers.**

I. Introduction

Formation control for a group of autonomous mobile agents has gained much attention due to its broad applications in many areas including both civil and military fields. A key problem in this domain that receives particular interest is how to maintain the geometrical shape of formations in a distributed manner. In the recent survey paper [1], different types of formation control strategies are reviewed and compared, among which two most commonly-used approaches are

- the linear displacement-based approach: the desired formation is specified by a certain set of inter-agent *relative positions* which means that the orientation of the final formation is implicitly fixed.
- the nonlinear distance-based approach: the desired formation is specified by a certain set of inter-agent *distances*, and the orientation of the final formation is *not* implicitly or explicitly defined.

For the first approach, all the agents should be equipped with coordinate bases with the same orientation (while the origins may be different) such that the desired relative position vectors are well defined [2]. In the case that initially all the agents in the 2-D plane have different local coordinate frames, a combined control establishing coordinate frame direction alignment and linear displacement-based formation stabilization was discussed in [3]. It has also been shown in [4] that the assumption that all the agents have coordinate systems with the same orientation may not be realistic in practice as small perturbations in their local coordinate systems will cause unexpected behaviors for the displacement-based formation system. For the distance-based approach, one advantage is that any global coordinate system with a common orientation is not required, and each agent can use its local coordinate bases to achieve the formation stabilization task.

This paper focuses on the distance-based formation problem with the aim of achieving a desired rigid formation with a desired formation orientation. Rigid formation control has been discussed extensively in the literature, most of which has focused on the convergence analysis of formation shapes (see e.g. [5], [6], [7], [8], [9], [10]). Note that in many applications involving multi-agent coordination, a formation with both a desired shape and a particular orientation is required, and it may further be considered desirable to have independent control over the orientation, while retaining the shape. However, for gradient-based rigid formation control, the orientation of the final formation is not controlled and actually not well defined, [1] which may limit the practical application of shape controllers discussed in these previous papers.

The problem of stabilizing only the orientation of rigid objects subject to distance constraints were studied in [12], [13]. However, in [12], [13] the rigid shapes remain constant which are not stabilized. In our previous paper [14] we showed a feasible approach to move or reorient a rigid formation to a desired orientation by introducing distance mismatches; however, that orientation control approach, which is a by-product of the mismatched formation control problem, indicates that the final formation is slightly distorted compared to the desired formation. Furthermore, the orientation control in [14] also requires global information which is contrary to the control of using a distributed approach.

In this paper we aim to propose feasible and distributed controllers to achieve both rigid shape stabilization and formation orientation control. The basic idea underlying the controller design is to choose certain agents as *orientation agents* (definitions will become clear in the context), in

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[1]We need to distinguish different meanings of *orientation* in the context of formation control. By regarding a rigid formation as a rigid body, the formation orientation relates to the overall rigid formation. The orientation concept in e.g. [3], [11] relates to the orientation of the *local coordinate frame* for each agent. We will distinguish different meanings by referring explicitly to either formation orientation or coordinate orientation.

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which some of associated relative position vectors should achieve both desired distances and directions specified by certain vectors. Two kinds of controllers will be investigated, depending on different requirements of the underlying graph of the orientation control. These two controller types result in different formation behaviors: one controller will preserve the position of the formation centroid, and the other will not. Some preliminary results on distance-based formation control with formation orientation stabilization were discussed in [15], with a focus on proving asymptotic stability for a *2-D four-agent* formation system. In this paper we follow similar motivations as in [15], but aim to provide a general and systematic approach to *completely* solve this control problem, while the results to be discussed in this paper can be applied to general rigid formations without any restriction on agent numbers and space dimensions. In particular, we will propose two different frameworks for controlling both rigid formation shapes and orientations, depending on different requirements on the underlying graphs for orientation control.

The remaining parts of this paper are organized as follows. In Section II, we introduce some background on graph and rigidity theory as well as the problem formulation. Section III focuses on controller design for controlling both rigid shapes and desired orientations with preserving invariant formation centroid. In Section IV, a different control strategy which does not preserve centroid invariance is discussed. Some typical simulation results are shown in Section V. Finally, Section VI concludes this paper.

II. Background and problem setup

A. Graph and rigidity theory

Since formations of n mobile agents are best described in terms of graph theory, we give a brief description of some of the basic definitions and facts needed. Consider an undirected graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of node i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The matrix relating the nodes to the edges is called the incidence matrix $H = \{h_{ki}\} \in \mathbb{R}^{m \times n}$, whose entries are defined as (with arbitrary edge orientations for *undirected* formations considered here)

$$h_{ki} = \begin{cases} 1, & \text{the } k\text{-th edge sinks at node } i \\ -1, & \text{the } k\text{-th edge leaves node } i \\ 0, & \text{otherwise} \end{cases}$$

For a connected and undirected graph, one has $\text{rank}(H) = n - 1$ and $\ker(H) = \text{span}\{\mathbf{1}_n\}$.

Given a vertex element $i \in \mathcal{V}$ we associate to it a point p_i of Euclidean space $\mathbb{R}^d$. [2] The column vector $p = \text{col}(p_1, \ldots, p_n)$ thus describes a *framework* $(\mathcal{G}, p)$ of n agents, labelled by the set of vertices of $\mathcal{G}$. For any edge $k \in \mathcal{E}$ with head j and tail i which is consistent with the construction of the matrix H, consider the associated relative position vector defined as $z_k = p_j - p_i$. Let

$$z = \text{col}(z_1, z_2, \cdots, z_m) := [z_1^T, z_2^T, \cdots, z_m^T]^T \in \mathbb{R}^{dm}$$
$$D(z) = \text{diag}(z_1, z_2, \cdots, z_m) \in \mathbb{R}^{dm \times m}$$

denote the associated column vector and block diagonal matrix, respectively. Note that there holds $z = (H \otimes I_d)p$. With this notation at hand, we consider the smooth distance map

$$r_{\mathcal{G}} : \mathbb{R}^{dn} \longrightarrow \mathbb{R}^m, r_{\mathcal{G}}(p) = (\|p_i - p_j\|^2)_{(i,j) \in \mathcal{E}} = D(z)^T z. \tag{1}$$

The rigidity of frameworks is then defined as follows.

Definition 1: ([16]) A framework $(\mathcal{G}, p)$ is rigid in $\mathbb{R}^d$ if there exists a neighborhood $\mathbb{U}$ of p such that $r_{\mathcal{G}}^{-1}(r_{\mathcal{G}}(p)) \cap \mathbb{U} = r_{\mathcal{K}}^{-1}(r_{\mathcal{K}}(p)) \cap \mathbb{U}$ where $\mathcal{K}$ is the complete graph with the same vertex set as $\mathcal{G}$.

Two frameworks $(\mathcal{G}, p)$ and $(\mathcal{G}, \bar{p})$ are equivalent if $r_{\mathcal{G}}(p) = r_{\mathcal{G}}(\bar{p})$ and are congruent if $\|p_i - p_j\| = \|\bar{p}_i - \bar{p}_j\|$ for all $i, j \in \mathcal{V}$. A useful tool to study graph rigidity is **rigidity matrix**, which is defined as the Jacobian matrix $R(p) = \frac{1}{2}\partial r_{\mathcal{G}}(p)/\partial(p)$. By inspection, $R(p)$ is an $m \times dn$ matrix given as

$$R(p) = D(z)^T (H \otimes I_d) \tag{2}$$

Note that the entries of $R(p)$ only involve relative position vectors z, and we can rewrite it as $R(z)$. The rigidity matrix will be used to determine the infinitesimal rigidity of the framework, as shown in the following definition.

Definition 2: ([17]) A framework $(\mathcal{G}, p)$ is infinitesimally rigid in the d-dimensional space if

$$\text{rank}(R(p)) = dn - d(d + 1)/2 \tag{3}$$

Specifically, the framework $(\mathcal{G}, p)$ is infinitesimally rigid in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) if and only if $\text{rank}(R(p)) = 2n - 3$ (resp. $\text{rank}(R(p)) = 3n - 6$). Obviously, in order to have an infinitesimally rigid framework, the graph should have at least $2n - 3$ (resp. $3n - 6$) edges in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$). If the framework is infinitesimally rigid in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) and has exactly $2n - 3$ (resp. $3n - 6$) edges, then it is called a minimally and infinitesimally rigid framework. In this paper we will assume that the desired formation is minimally and infinitesimally rigid.

B. Gradient-based formation controller and problem formulation

Let $d_{k_{ij}}$ denote the desired distance of edge k which links agent i and j. We further define

$$e_{k_{ij}} = \|p_i - p_j\|^2 - (d_{k_{ij}})^2 \tag{4}$$

to denote the squared distance error for edge k. Note we may also use e_k and d_k occasionally for notational convenience in the sequel if no confusion is expected. The squared distance error vector is denoted by $e = [e_1, e_2, \cdots, e_m]^T$. In this paper, we suppose that each agent is modeled by a single

[2] In this paper we will focus on 2-D and 3-D formations, i.e. $d = 2, 3$.

integrator $\dot{p}_i = u_i$ where u_i is the controller to be designed for achieving the formation control objective.

In [5], the following formation control system was proposed:

$$\dot{p}_i = -\sum_{j \in \mathcal{N}_i} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)(p_i - p_j), \ i = 1, \ldots, n \quad (5)$$

The above control describes a steepest descent gradient flow of the following potential function

$$V_1(p) = \frac{1}{4} \sum_{(i,j) \in \mathcal{E}} (\|p_i - p_j\|^2 - d_{k_{ij}}^2)^2 \quad (6)$$

This potential function (6) for rigid shape stabilization and the associated gradient flow (5) have been extensively studied in the literature (see e.g. [5], [6], [7], [8], [18], [19]). More general forms of formation controllers and the associated convergence results which extend the control form of (5) have been discussed in [20]. However, the above control and its extensions studied in these previous papers only stabilize the rigid formation shapes, while the orientation of the formation is not specified. In this paper we will consider the problem on how to simultaneously stabilize a rigid shape and achieve a desired orientation for a target formation. It is expected that the overall controller takes the following form

$$u_i = u_i^{\text{shape}} + u_i^{\text{orientation}} \quad (7)$$

where u_i^{shape} is to *stabilize rigid shapes* and $u_i^{\text{orientation}}$ is to *control the orientation* of a rigid formation to a desired one. In Section III and Section IV, certain different controllers to achieve this formation control task will be proposed and discussed in detail.

III. Formation and Orientation Control with Invariant Centroid

In this section we will consider the controller design problem to stabilize a rigid formation shape and control its orientation with the condition that the position of the formation centroid should be preserved. As mentioned in the above section, the commonly-used gradient-based controller does not control the orientation and there are certain degrees of freedom relating to rotations for a converged formation. Intuitively, by regarding the rigid formation as a rigid body and specifying certain directions of some edges, the orientation of the overall rigid formation can be fixed. This will be the basic idea in the controller design discussed in the sequel.

For simplifying the controller design and implementation, we choose one agent and one (or two, depending on different dimensions) of its neighboring agent(s) in the group as the specified agents to implement the additional orientation control task, and the associated edges between them will be assigned with both distance constraints and orientation constraints. We term these agents with additional orientation control as *orientation agents*, and other agents as *non-orientation agents*. Note that there exists reflection

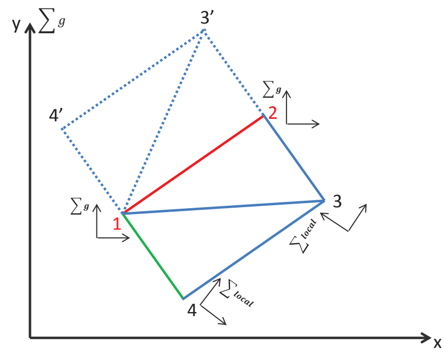

Fig. 1. An example of controlling a 2-D rigid formation with prescribed orientation. Agent 1 and one of its neighbors, agent 2, are chosen as orientation agents. The relative position vector $p_2 - p_1$ associated with edge (1,2) is used to describe the desired orientation, which is denoted by red color (in this example $(1, 2) \in \mathcal{E}_O$). Note that specifying the direction of one edge is not enough to uniquely describe the orientation of a 2-D formation as a reflected formation exists. In this example the reflected formation is obtained by reflection via the mirror edge (1,2), which is denoted by the dotted blue line. Such reflection ambiguity can be avoided by specifying the direction of an additional relative position vector such as the one associated with edge (1,4); by doing so a unique formation with the desired orientation will be obtained. However, if the initial formation shape starts close to the desired one, the shape will converge to the desired one instead of converging to the reflected one (by the property of a formation gradient system). Thus, in the following formation controller design, we can still use one edge in a 2-D formation to specify the desired orientation.

ambiguity by specifying only one (resp. two) relative position vector(s) with desired direction(s) in a 2-D (resp. 3-D) formation, where a reflected formation is regarded as a different formation compared to the desired one with the same set of inter-agent distances. However, this can be avoided by carefully choosing initial positions of all the agents. Otherwise, one can choose two (resp. three) linearly independent relative position vectors with desired directions associated with a common agent in 2-D (resp. 3-D) formations to specify a *unique* orientation of the overall formation. For the convenience of later analysis, we denote $\mathcal{G}_O$ as the underlying graph of the orientation control to distinguish with the underlying graph $\mathcal{G}$ of the formation shape control. If the edge (i, j) associated with agent i and j is chosen in the orientation control in $\mathcal{G}_O$, we denote it as $(i, j) \in \mathcal{E}_O$. The set of neighboring agents of orientation agent i chosen in the orientation control is defined as $\mathcal{N}_i^O := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}_O\}$. The desired orientation for the relative position vector $p_j - p_i$ for edge $(i, j) \in \mathcal{E}_O$ is denoted as $\hat{p}_{ji} := \hat{p}_j - \hat{p}_i$. Thus, the orientation control is to additionally stabilize the relative position $p_j - p_i$ to the desired one $\hat{p}_j - \hat{p}_i$ with $(i, j) \in \mathcal{E}_O$. Note that any two agents associated with one edge can be chosen as orientation agents. The election of an orientation control edge in a graph is little different from the standard computer science notion of leader election, which can occur in a decentralized way. We show two examples, a 2-D four-agent rectangular formation and a 3-D tetrahedral formation depicted in Fig. 1 and Fig. 2, respectively, to illustrate the formation control framework.

To continue the analysis, we need the following assumption.

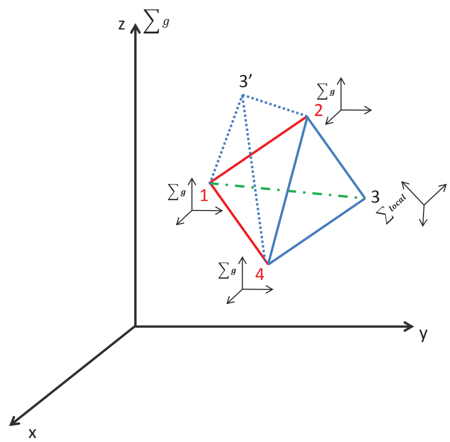

Fig. 2. An example of controlling a 3-D rigid formation with prescribed orientation. Agent 1 and two of its neighbors, agents 2 and 4, are chosen as orientation agents. The relative position vectors $p_2 - p_1$ and $p_4 - p_1$ associated with edges (1,2) and (1,4) are used to describe the desired orientation, which are denoted by red color (in this example $(1,2),(1,4) \in \mathcal{E}_O$). Specifying directions of two edges in a 3-D formation is not enough to fix a unique formation with the prescribed orientation as an additional formation exists by reflection via the mirror plane spanned by that two relative vectors. In this example the mirror plane is spanned by that two relative vectors in edges (1,2), (1,4) and the reflected formation is denoted by the dotted blue line. Such reflection ambiguity can be avoided by specifying the direction of an additional relative position vector such as the one associated with edge (1,3), and a unique 3-D formation with the desired orientation will be obtained. However, if the initial formation shape starts close to the desired one, the shape will converge to the desired one instead of converging to the reflected one. Thus, we can still use two edges in a 3-D formation to specify the desired orientation.

***Assumption** 1:* All orientation agents should be equipped with coordinate systems aligned with the global coordinate system.

Taking the formation control formulation in Fig. 1 as an example, since agents 1 and 2 are chosen as orientation agents, their coordinate systems should be the same to the global coordinate system denoted by $\sum_g$. Such a global coordinate system is required to define the desired vector $(\hat{p}_j - \hat{p}_i)$ for $(i,j) \in \mathcal{E}_O$. Thus Assumption 1 provides a necessary condition for the controller design and implementation.

We propose the following combined controller:

$$\dot{p}_i(t) = \underbrace{\sum_{j \in \mathcal{N}_i} (p_j(t) - p_i(t))(\|p_j(t) - p_i(t)\|^2 - d_{k_{ij}}^2)}_{\text{shape control term, if } (i,j) \in \mathcal{E}} + \underbrace{\sum_{j \in \mathcal{N}_i^O} ((p_j(t) - p_i(t)) - (\hat{p}_j - \hat{p}_i))}_{\text{orientation control term, if } (i,j) \in \mathcal{E}_O} \quad (8)$$

with the condition that $\|(\hat{p}_j - \hat{p}_i)\| = d_{k_{ij}}$. It is obvious from Eq. (8) that the control is distributed since only local information from neighboring agents is needed.

The above formation control system can be written in a compact form

$$\dot{p} = -R^T e - \bar{L}\bar{p} \quad (9)$$

where $\bar{L} = L \otimes I_d \in \mathbb{R}^{dn \times dn}$ and L is the Laplacian matrix of the underlying undirected graph $\mathcal{G}_O$ for the orientation control, and the vector $\bar{p} = [\bar{p}_1^T, \bar{p}_2^T, \cdots, \bar{p}_n^T]^T$ is defined as $\bar{p}_i = \mathbf{0}$, if $(i,j) \notin \mathcal{E}_O$ and $\bar{p}_i = p_i - \hat{p}_i$, if $(i,j) \in \mathcal{E}_O$. [3]

We will show that the proposed control will asymptotically locally stabilize the rigid formation to the desired shape with the desired orientation.

***Theorem** 1:* Suppose the desired formation is minimally and infinitesimally rigid and initial positions of all the agents are chosen such that the initial formation is close to the desired formation. Then by the above control (8), the formation system will locally asymptotically converge to the desired rigid formation with the prescribed orientation.

Proof: We choose the same potential function in (6) as the potential for the shape control, and the following potential function

$$V_2 = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}_O} ((p_j(t) - p_i(t)) - (\hat{p}_j - \hat{p}_i))^2 = \frac{1}{2} \bar{p}^T \bar{L} \bar{p} \quad (10)$$

for the orientation control. Define a compact level set $\mathcal{B}(\rho) = \{e : V_1(e) \leq \rho\}$ for some small ρ ($\rho > 0$), such that for all the points in the set $\mathcal{B}(\rho)$ the formation is infinitesimally and minimally rigid, while a sufficiently small ρ also indicates the initial formation shape accordingly must be sufficiently close to the target one. Note that the set does not need to be arbitrarily small since the infinitesimal rigidity of the formation is a generic property.

The composite potential function is then defined as

$$V = V_1 + V_2$$

The dynamical system for the relative position is

$$\dot{z} = H\dot{p} = -HR^T e - H\bar{L}\bar{p} \quad (11)$$

and the distance error system

$$\dot{e} = 2R\dot{p} = -2RR^T e - 2R\bar{L}\bar{p} \quad (12)$$

Note that the potential functions V_1 and V_2 are functions involving only relative position vectors in terms of z and e rather than the absolute position vector p. [4] Thus, we can write the potential as $V(z)$. Furthermore, the sub-level set of the potential V is compact with respect to the self-contained z system and e system. We then calculate the derivative of the potential V_1 and V_2 along the trajectories of system (11) and (12):

$$\begin{aligned} \dot{V}_1 &= \frac{1}{2} e^T \dot{e} = e^T(-RR^T e - R\bar{L}\bar{p}) \\ &= -e^T RR^T e - e^T R\bar{L}\bar{p} \end{aligned} \quad (13)$$

and

$$\begin{aligned} \dot{V}_2 &= \bar{p}^T \bar{L}\dot{\bar{p}} = \bar{p}^T \bar{L}(-R^T e - \bar{L}\bar{p}) \\ &= -\bar{p}^T \bar{L} R^T e - \bar{p}^T \bar{L}\bar{L}\bar{p} \end{aligned} \quad (14)$$

[3] Note that the vector $\bar{p}_i$ may not be a control input as $\hat{p}_i$ may not be available for agent i (the real control term is $\hat{p}_{ji} := \hat{p}_j - \hat{p}_i$). The introduction of $\bar{p}_i$ is for the convenience of analysis and for writing a compact form of the formation system as in (9).

[4] Also note that the distance error vector e can be written in terms of z according to the definition of e in (4).

The derivative of V can be computed as

$$\begin{aligned}\dot{V} &= \dot{V}_1 + \dot{V}_2 \\ &= -e^T R R^T e - e^T R \bar{L}\bar{p} - \bar{p}^T \bar{L} R^T e - \bar{p}^T \bar{L}\bar{L}\bar{p} \\ &= -e^T R R^T e - 2e^T R \bar{L}\bar{p} - \bar{p}^T \bar{L}\bar{L}\bar{p} \\ &= -(R^T e + \bar{L}\bar{p})^T (R^T e + \bar{L}\bar{p}) \leq 0 \end{aligned} \tag{15}$$

The above derivative calculation also implies that the formation system (8) describes a gradient descent flow for the composite potential V. The solution of the formation system converges to the largest invariant set in the set

$$\begin{aligned}\mathcal{O}(z) &= \{(e, z) | \dot{V} = 0\} \\ &= \{(e, z) | R^T e + \bar{L}\bar{p} = 0\}\end{aligned} \tag{16}$$

Note that in the set $\mathcal{O}$, the desired equilibrium point at which $R^T e = 0$ (or equivalently $e = 0$ due to the minimal and infinitesimal rigidity of the target formation) and $\bar{L}\bar{p} = 0$ (or equivalently $p_i - p_i = \hat{p}_i - \hat{p}_j$ for $(i, j) \in \mathcal{E}_O$) is the *global minimum* of the potential V. It is obvious that the potential V is real analytic; thus any local/global minimum of V is an asymptotically stable equilibrium of the gradient system [21, Theorem 3]. Also note that the desired equilibrium at which $e = 0$, $p_i - p_j = \hat{p}_i - \hat{p}_j$ for $(i, j) \in \mathcal{E}_O$ is isolated from all other equilibria of $V(z)$ because the sublevel set of $V(z)$ is compact with respect to the relative position vector z. Since the desired equilibrium is the global minimum of the real analytic potential V, it follows that there exists a neighbourhood around the minimum (which can be chosen small enough to be inside the set $\mathcal{B}(\rho)$) in which the desired minimum equilibrium point is asymptotically stable. According to the definition of the rigid framework, this implies that at the minimum equilibrium point the whole formation is at the desired orientation. ■

***Remark** 1:* Note that this is a local convergence result, which requires that initial positions of all the agents should be close to the target formation and the reflection ambiguity is avoided. Some relevant discussions on *local* convergence in rigid formation control can be found in [9], [22]. The reflection avoidance is due to the property of the descent gradient system, as the convergence to a reflected formation would require at some point on the trajectory an increase of the potential function, which is impossible if the initial formation is close to the target one.

In the following, we show several properties of the proposed control (8).

***Lemma** 1:* The position of the formation centroid is preserved by the above control law.

***Lemma** 2:* For all non-orientation agents, their local coordinate systems will be sufficient to implement the control law.

The proofs for the above two lemmas are omitted due to space limit and will be provided in the full version of this paper.

This controller property has been illustrated in Fig.1 and Fig. 2. In the example shown in Fig. 1, agents 3 and 4 are non-orientation agents and the directions of their coordinate systems do not need to be aligned with the global coordinate system.

***Remark** 2:* One can straightforwardly modify the designed controller (8) into the following form

$$\dot{p}_i(t) = \sum_{j \in \mathcal{N}_i} (p_j(t) - p_i(t))(\|p_j(t) - p_i(t)\|^2 - d_{k_{ij}}^2) + \underbrace{\sum_{j \in \mathcal{N}_i^O} \left(\frac{(p_j(t) - p_i(t))}{\|(p_j(t) - p_i(t))\|} - (\hat{p}_j - \hat{p}_i) \right)}_{\text{orientation control term, if } (i,j) \in \mathcal{E}_O} \tag{17}$$

where $(\hat{p}_j - \hat{p}_i)$ is a *unit* vector indicating the desired direction for the relative position vector $p_j - p_i$. Note that in this case the additive orientation control term is to control only the direction of the specified relative position vectors, and the distance information is not involved.

***Remark** 3:* One may ask what happens if the formation is initially with a correct shape but needs to adjust the orientation with the designed controller (8). As can be seen from (12), $e(0) = 0$ does not imply $\dot{e}(t) = 0$ when the combined controller (8) is applied. Thus, during the orientation adjustment the formation shape will be temporarily lost until the formation converges to the desired shape and orientation. If the formation shape should remain unchanged during the orientation adjustment, a sufficient condition for the controller design is to ensure that the z system takes the form as $\dot{z}_i = \omega \times z_i$ (where ω indicates the angular velocity and $\times$ denotes the cross product) which guarantees a constant norm of z and thus a constant formation shape (see [12]).

IV. Formation and orientation control without preserving centroid invariance

In this section we aim to propose alternative controllers where the position of the formation centroid is not necessarily preserved. Differently to the problem setting in Section III, we suppose in this section that one agent in the group is chosen as the orientation agent with the additional orientation task, while the relative positions relating to one (or two) incident edges should converge to not only the desired distances but also the desired directions. The main difference is that the underlying graph for the orientation control in this case is a *directed* graph, i.e. $(i, j) \in \mathcal{E}_O$ does not mean $(j, i) \in \mathcal{E}_O$. The formation controller takes a similar form as (8) as follows

$$\dot{p}_i(t) = \underbrace{\sum_{j \in \mathcal{N}_i} (p_j(t) - p_i(t))(\|p_j(t) - p_i(t)\|^2 - d_{k_{ij}}^2)}_{\text{shape control term, if} (i,j) \in \mathcal{E}} + \underbrace{\sum_{j \in \mathcal{N}_i^O} (p_j(t) - p_i(t)) - (\hat{p}_j - \hat{p}_i)}_{\text{orientation control term, if} (i,j) \in \mathcal{E}_O} \tag{18}$$

with the condition that $\|(\hat{p}_j - \hat{p}_i)\| = d_{k_{ij}}$ and the difference that $(i, j) \in \mathcal{E}_O$ is an edge in the directed graph $\mathcal{G}_O$.

***Remark** 4:* Compared to the controllers designed in Section III, an advantage of the orientation control strategy in this section is that only one agent is involved in the task

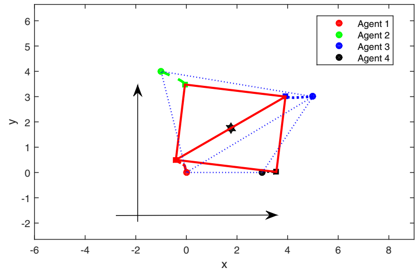

Fig. 3. Stabilization of a rigid rectangular formation without orientation control. The initial and final positions are denoted by circles and squares, respectively. The initial formation is denoted by dotted blue lines, and the final formation is denoted by red solid lines.

of specifying and controlling formation orientations. Also, only one agent (the chosen orientation agent) is required to have its coordinate orientation to be aligned with the global coordinate, which further simplifies the controller design and implementation.

The above formation control system can be written in a compact form

$$\dot{p} = -R^T e - \bar{L}\bar{p} \tag{19}$$

where $\bar{L} \in \mathbb{R}^{dn \times dn}$ is the Laplacian matrix for the underlying *directed* graph $\mathcal{G}_O$ for the orientation control.

***Theorem** 2:* Suppose the desired formation is minimally and infinitesimally rigid and initial positions of all the agents are chosen such that the initial formation shape is close to the desired formation shape. Then by the above control (18), the formation system will locally asymptotically converge to the desired rigid formation with the prescribed orientation.

The proof is similar to the one in Section III by modifying the potential V_2 and is omitted here. In the following, we show several properties of the proposed control (18).

***Lemma** 3:* The formation centroid moves at the velocity given by $-\frac{1}{n}(\mathbf{1}_n \otimes I_d)^T \bar{L}\bar{p}$ and will be stationary in the limit.

Similar to Lemma 2, the following lemma shows the requirement of coordinate systems for implementing the control (18).

***Lemma** 4:* For all non-orientation agents, their local coordinate systems will be sufficient to implement the control law designed in (18).

The proofs for the above two lemmas will be provided in the full version of this paper. Furthermore, similarly to (17), one can also modify the controller (18) into an alternative form which involves a unit vector indicating the desired direction of the desired relative position vector.

V. Illustrative Examples

In this section we provide several simulations to show formation behaviors and controller performance of the proposed control. Consider a 4-agent formation system, with the desired distances given as $d_{12}^* = d_{34}^* = 3$, $d_{23}^* = d_{14}^* = 4$, $d_{13}^* = 5$ corresponding to a rectangular shape. The initial positions for each agent are chosen as $p_1(0) = [0,\ 0]^T$, $p_2(0) = [-1,\ 4]^T$, $p_3(0) = [5,\ 3]^T$ and $p_4(0) = [3,\ 0]^T$, so

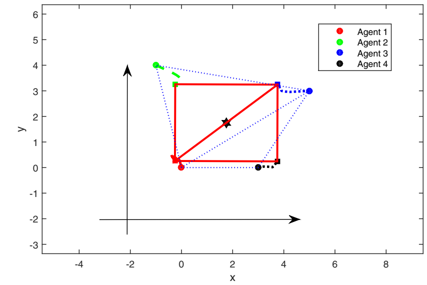

Fig. 4. Stabilization of a rigid rectangular formation with centroid-preserving orientation control. The initial and final positions are denoted by circles and squares, respectively. The initial formation is denoted by dotted blue lines, and the final formation is denoted by red solid lines. The black star denotes the formation centroid.

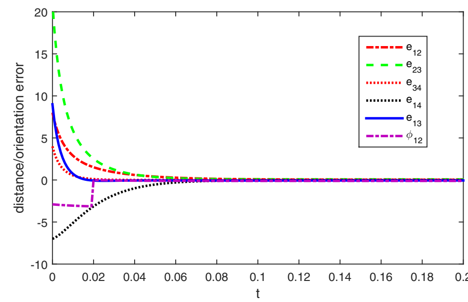

Fig. 5. Convergence of the distance/orentation errors for the combined controller (8) with invariant formation centroid. The orientation error ϕ_{12} is defined as $\phi_{12} = \arctan((p_{2,y} - p_{1,y})/(p_{2,x} - p_{1,x})) - \pi/2$.

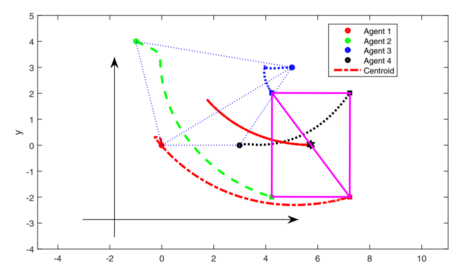

Fig. 6. Stabilization of a rigid rectangular formation with orientation control without centroid invariance. The initial and final positions are denoted by circles and squares, respectively. The initial formation is denoted by dotted blue lines, and the final formation is denoted by pink solid lines. The red line denotes the formation centroid.

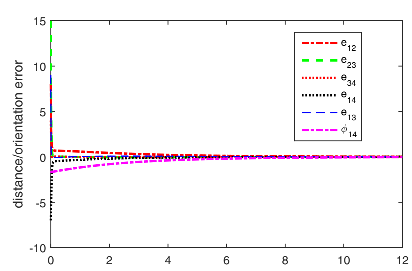

Fig. 7. Convergence of the distance/orentation errors for the combined controller (18) without centroid invariance. The orientation error ϕ_{14} is defined as $\phi_{14} = \arctan((p_{4,y} - p_{1,y})/(p_{4,x} - p_{1,x})) - \pi/2$.

that the initial formation shape is close to the target shape. When the original controller (5) is used, the trajectories of each agent and the final shape are depicted in Fig. 3, from which it can be seen that although the desired shape is achieved, the formation orientation is undefined.

We then consider the simulation using the proposed controller (8). We suppose the target formation should be the one with the rigid rectangular shape in addition that the relative position vector $p_2 - p_1$ associated to edge $(1, 2)$ should be aligned with the direction of the y-axis and the relative position vector $p_4 - p_1$ associated to the edge $(1, 4)$ should be aligned with the direction of the x-axis in the global coordinate. The desired relative vector for edge $(1, 2)$ is set as $\hat{p}_2 - \hat{p}_1 = (0, 3)^T$ and the initial positions are chosen as the same as in Fig. 3, which can avoid the reflected formation. The trajectories of each agent and the final shape are depicted in Fig. 4, which clearly show that the desired formation shape with the correct orientation is achieved and the formation centroid is preserved. The trajectories of each distance error and the orientation error for the edge $(1, 2)$ are depicted in Fig. 5.

We then perform simulations using the combined controller (18) without invariant centroid. We suppose that the target formation should be the one where the edge $(1, 4)$ is aligned with the direction of the y-axis and the edge $(1, 2)$ is aligned with the direction of the $-x$-axis in the global coordinate. The additional orientation control term is then assigned to agent 1, while two relative positions associated with these two edges $(1, 4)$ and $(1, 2)$ are both controlled as it is not obvious to avoid the reflection ambiguity by controlling one edge with the chosen initial position. The trajectories of each agent and the final shape are depicted in Fig. 6, which shows the achieved formation with the correct orientation, while the position of formation centroid moves in the stabilizing process. The trajectories of each distance error and the orientation error for the edge $(1, 4)$ are depicted in Fig. 7, all of which converge to the origin as desired.

VI. CONCLUSION

In this paper we have discussed the formation control problem to achieve both the desired rigid shape and formation orientation. The designed controller combines the advantages of displacement-based approach and distance-based approach, by specifying a small number of agents as orientation agents which are tasked to control relative position vectors associated with them to desired directions. For all non-orientation agents, the global coordinate system is not required for them to implement the control. Two different controllers are proposed, depending on different requirements on the underlying orientation graph and centroid preservation. Both controllers are distributed in that only relative measurements from neighboring agents are required. Certain simulation examples are provided to demonstrate the effectiveness of the proposed formation controllers. Future topics include further analysis on the convergence rate of the formation system with combined controllers, and extensions of current results on minimally rigid formations to more general formations.

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