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Undirected Rigid Formations are Problematic

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***Abstract*— By an *undirected rigid formation* of mobile autonomous agents is meant a formation based on graph rigidity in which each pair of "neighboring" agents is responsible for maintaining a prescribed target distance between them. In a recent paper, a systematic method was proposed for devising gradient control laws for asymptotically stabilizing a large class of rigid, undirected formations in two-dimensional space assuming all agents are described by kinematic point models. The aim of this paper is to explain what happens to such formations if neighboring agents have slightly different understandings of what the desired distance between them is supposed to be, or equivalently, if neighboring agents have differing estimates of what the actual distance between them is. In either case, what one would expect would be a gradual distortion of the formation from its target shape as discrepancies in desired or sensed distances increase. While this is observed for the gradient laws in question, something else quite unexpected happens at the same time. It is shown that for any rigidity-based, undirected formation which is comprised of three or more agents, that if some neighboring agents have slightly different understandings of what the desired distances between them are suppose to be, then almost for certain, the trajectory of the resulting distorted but rigid formation will converge exponentially fast to a closed circular orbit in two-dimensional space which is traversed periodically at a constant angular speed.**

I. INTRODUCTION

The problem of coordinating a large network of mobile autonomous agents by means of distributed control has raised a number of issues concerned with the forming, maintenance and real-time modification of multi-agent networks of all types. One of the most natural and useful tasks along these lines is to organize a network of agents into an application specific "formation" which might be used for such tasks as environmental monitoring, search, or simply moving the agents efficiently from one location to another. By a multi-agent formation is usually meant a collection of agents in real two or three dimensional space whose inter-agent distances are all essentially constant over time, at least under ideal conditions. One approach to maintaining such formations is based on the idea of "graph rigidity" [1], [2]. Rigid formations can be "directed" [3], [4] "undirected" [5], [6], or some combination of the two. The appeal of the rigidity based approach is that it has the potential for providing control laws which are totally distributed in that the only information which each agent needs to sense is the relative positions of its nearby neighbors.

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By an *undirected rigid formation* of mobile autonomous agents is meant a formation based on graph rigidity in which each pair of "neighboring" agents i and j is responsible for maintaining the prescribed target distance d_{ij} between them. In [5] a systematic method was proposed for devising gradient control laws for asymptotically stabilizing a large class of rigid, undirected formations in two-dimensional space assuming all agents are described by kinematic point models. This particular methodology is perhaps the most comprehensive currently in existence for maintaining undirected formations based on graph rigidity. In [7] an effort was made to understand what happens to such formations if neighboring agents i and j have slightly different understandings of what the desired distance d_{ij} between them is supposed to be. The question is relevant because no two positioning controls can be expected to move agents to precisely specified positions because of inevitable imprecision in the physical comparators used to compute the positioning errors. The question is also relevant because it is mathematically equivalent to determining what happens if neighboring agents i and j have differing estimates of what the actual distance between them is. In either case, what one might expect would be a gradual distortion of the formation from its target shape as discrepancies in desired or sensed distances increase. While this is observed for the gradient laws in question, something else quite unexpected happens at the same time. In particular, it turns out for any rigidity-based, undirected formation of the type considered in [5] which is comprised of three or more agents, that if some neighboring agents have slightly different understandings of what the desired distances between them are suppose to be, then almost for certain, the trajectory of the resulting distorted but rigid formation will converge exponentially fast to a closed circular orbit in $\mathbb{R}^2$ which is traversed periodically at a constant angular speed. In [7] this was shown to be so for the special case of a three agent triangular formation. The aim of this paper is to explain why this same phenomenon also occurs with any undirected rigid formation in the plane consisting of three or more agents.

II. UNDIRECTED FORMATIONS

We consider a formation in the plane consisting of $n \geq 3$ mobile autonomous agents $\{$eg, robots$\}$ labeled $1, 2, \ldots, n$.

We assume the desired formation is specified in part, by a rigid graph $\mathbb{G}$ with n vertices labeled $1, 2, \ldots, n$ and m edges labeled $1, 2, \ldots, m$. For simplicity, we assume that $\mathbb{G}$ is minimally rigid which means that $m = 2n - 3$ [8]. We write k_{ij} for the label of that edge which connects adjacent vertices i and j. Thus $k_{ij} = k_{ji}$. We call agent j a *neighbor* of agent i if vertex j is adjacent to vertex i and we write $\mathcal{N}_i$ for the labels of agent i's neighbors.

We assume that the desired *target distance* between agent i and neighbor j is d_{ij} where d_{ij} is a positive number. We assume that agent i is tasked with the job of maintaining the specified target distances to each of its neighbors. However, unlike [5] we do not assume that the target distances d_{ij} and d_{ji} are necessarily equal. Instead we assume that $|d_{ji} - d_{ij}| \leq \beta_{k_{ij}}$ were $\beta_{k_{ij}}$ is a small nonnegative number bounding the discrepancy in the two agents understanding of what the desired distance between them is suppose to be. We assume that in the unperturbed case when there is no discrepancy between d_{ij} and d_{ji}, these distances are realizable by some set of points in the plane with coordinate vectors $x_1, x_2, \ldots, x_n$ such that at the *multi-point* $x = [\, x_1' \quad x_2' \quad \ldots \quad x_n' \,]'$, the resulting formation or "framework" $\{\mathbb{G}, x\}$ is minimally infinitesimally rigid [1].

We assume that agent i's motion is described in global coordinates by the simple kinematic point model

$$\dot{x}_i = u_i, \quad i \in \mathbf{n} \tag{1}$$

where $\mathbf{n} = \{1, 2, \ldots, n\}$. We further assume that for $i \in \mathbf{n}$, agent i can measure the relative position $x_j - x_i$ of each of its neighbors $j \in \mathcal{N}_i$. The control law proposed in [5] for agent i is then simply

$$u_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)(||x_j - x_i||^2 - d_{ij}^2)$$

Application of such controls to the agent models (1) yields the equations

$$\dot{x}_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)(||x_j - x_i||^2 - d_{ij}^2), \quad i \in \mathbf{n} \tag{2}$$

Our aim is to express these equations in state space form. To do this, it is convenient to assume that each edge in $\mathbb{G}$ is "oriented" with a specific direction: one end of the edge being its 'head' and the other being its 'tail.' To proceed, let us write $H_{m\times n}$ for that matrix whose kith entry is $h_{ki} = 1$ if vertex i is the head of oriented edge k, $h_{ki} = -1$ if vertex i is the tail of oriented edge k and $h_{ki} = 0$ otherwise. Thus H is a matrix of 1s, -1s and 0s with exactly one 1 and one -1 in each row. Note that H is the transpose of the incidence matrix of the oriented graph $\mathbb{G}$; because $\mathbb{G}$ is connected, the rank of H is $n-1$ [9]. Next define for each edge k_{ij},

$$z_{k_{ij}} = \psi_{ij}(x_i - x_j) \tag{3}$$

where $\psi_{ij} = 1$ if i is the head of edge k_{ij} or $\psi_{ij} = -1$ if i is the tail of edge k_{ij}. The definition of H implies that

$$z = \bar{H}x \tag{4}$$

where $z = [\, z_1' \quad z_2' \quad \cdots z_m' \,]'$, $\bar{H}_{2m\times 2n} = H \otimes I_{2\times 2}$, $I_{2\times 2}$ is the 2×2 identity and $\otimes$ is the Kronecker product.

Next define $d_{k_{ij}} = d_{ij}$ and $\mu_{k_{ij}} = d_{ij}^2 - d_{ji}^2$ for all adjacent vertex pairs (i, j) for which i is the head of edge k_{ij}; clearly

$$d_{ij}^2 = d_{k_{ij}}^2 \qquad \text{and} \qquad d_{ji}^2 = d_{k_{ij}}^2 - \mu_{k_{ij}}$$

for all such pairs. Let $e_k : \mathbb{R}^m \to \mathbb{R}$ denote the *kth error function*

$$e_k(z) = ||z_k||^2 - d_k^2, \quad k \in \mathbf{m} \tag{5}$$

where $\mathbf{m} = \{1, 2, \ldots, m\}$. Write $\mathcal{N}_i^+$ for the set of all $j \in \mathcal{N}_i$ for which vertex i is a head of oriented edge k_{ij}. Let $\mathcal{N}_i^-$ denote the complement of $\mathcal{N}_i^+$ in $\mathcal{N}_i$. With the $z_{k_{ij}}$ and z as defined in (3) and (4) respectively, the system of equations given in (2) can be written as

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i^+} z_{k_{ij}} e_{k_{ij}}(z) + \sum_{j \in \mathcal{N}_i^-} z_{k_{ij}}(e_{k_{ij}}(z) + \mu_{k_{ij}}), \quad i \in \mathbf{n} \tag{6}$$

These equations in turn can be written compactly in the form

$$\dot{x} = -R'(z)e(z) + S'(z)\mu \tag{7}$$

where μ is the *mismatch error* $\mu = [\, \mu_1 \quad \mu_2 \quad \cdots \mu_m \,]'$, $e(z) = [\, e_1(z) \quad e_2(z) \quad \cdots \quad e_m(z) \,]'$, $R_{m\times 2n}(z) = D'(z)\bar{H}$, $S_{m\times 2n}(z) = D'(z)\bar{J}$, $D_{2m\times m}(z) = \text{diagonal}\{z_1, z_2, \ldots, z_m\}$, and $\bar{J}_{2m\times 2n}$ is what results when the positive elements in $\bar{H}$ are replaced by zeros. It is easy to verify that $R(z)|_{z=Hx}$ is the rigidity matrix for the formation $\{\mathbb{G}, x\}$ [1]. Note that because of (4), (7) is a smooth self-contained dynamical system of the form $\dot{x} = f(x, \mu)$. We shall refer to (7) $\{$with $z = \bar{H}x\}$ as the *overall system.*

III. Error System

Our aim is to study the geometry of the overall system. Towards this end, first note that

$$\dot{z} = -\bar{H}R'(z)e(z) + \bar{H}S'(z)\mu \tag{8}$$

because of (4) and (7). This equation and the definitions of the e_k in (5) enable one to write

$$\dot{e} = -2R(z)R'(z)e + 2R(z)S'(z)\mu \tag{9}$$

An important property of a rigid formation is that its shape does not change under "translations" and "rotations." To make precise what is meant by this let us agree to say that a *translation* of a multi-point $x = [\, x_1' \quad x_2' \quad \cdots \quad x_n' \,]'$ is a function of the form $[\, x_1' \quad x_2' \quad \cdots \quad x_n' \,]' \longmapsto [\, x_1' + y' \quad x_2' + y' \quad \cdots \quad x_n' + y' \,]'$ where y is a vector in $\mathbb{R}^2$. Similarly, a *rotation* of a multi-point x is a function of the form $[\, x_1' \quad x_2' \quad \cdots \quad x_n' \,]' \longmapsto [\, (Tx_1)' \quad (Tx_2)' \quad \cdots \quad (Tx_n)' \,]'$ where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation matrix. The set of all such translations and rotations together with composition forms a transformation group which we denote by $\mathfrak{G}$; this group is isomorphic to the special Euclidean group $SE(2)$. To say that a rigid formation $\{\mathbb{G}, x\}$ retains its "shape" under the action of the group means that for any $\gamma \in \mathfrak{G}$, the formations $\{\mathbb{G}, x\}$

and $\{\mathbb{G}, \gamma \circ x\}$ are congruent where $\gamma \circ x$ denotes the multi-point which results after γ is applied to x. More is true. Examination of (9) reveals that it is translation and rotation invariant in that for any given $x \in \mathbb{R}^{2n}$, the vector $(2R(z)R'(z)e(z) - 2R(z)S'(z)\mu)|_{z=\bar{H}(\gamma \circ x)}$ is the same for all $\gamma \in \mathfrak{G}$.

A key step in the analysis of the gradient law proposed in [5] is to show that along trajectories of the overall system (7), the error vector $e(z)|_{z=\bar{H}x}$ also satisfies a differential equation of the form $\dot{\epsilon} = g(\epsilon, \mu)$ where g is a continuously differentiable function of just ϵ and μ and not z. As we will see, this can be shown to be true locally on the orbit under $\mathfrak{G}$ of an open set $\mathcal{A}$ containing any value of x for which $e(z)|_{z=\bar{H}x} = 0$. The precise technical result is as follows.

Theorem 1: Let y be any point in $\mathbb{R}^{2n}$ at which $e(z)|_{z=\bar{H}y} = 0$. There exists an open set $\mathcal{A} \subset \mathbb{R}^{2n}$ containing y and a continuously differentiable function $g : e(\bar{H}\mathcal{A}) \times \mathbb{R}^m \to \mathbb{R}^m$ for which the following is true. If $x(t)$ is a solution to the overall system (7) for which $x(t) \in \mathcal{A}$ on some time interval $[t_0, t_1)$, then on the same time interval, the error vector $e(\bar{H}x(t))$ satisfies the self-contained differential equation

$$\dot{\epsilon} = g(\epsilon, \mu) \tag{10}$$

We call (10) the *error system* at y and we say that $\mathcal{A}$ is an *ambient space* on which it is valid. Suppose $\mathcal{C}$ is the set of all multi-points $y \in \mathbb{R}^{2n}$ for which $\{\mathbb{G}, y\}$ has edge lengths $d_1, d_2, \ldots, d_m$. In the light of the fact that for any given $x \in \mathbb{R}^{2n}$, the vector $(2R(z)R'(z)e(z) - 2R(z)S'(z)\mu)|_{z=\bar{H}(\gamma \circ x)}$ is the same for all $\gamma \in \mathfrak{G}$, it is easy to see that the error system at y must be same as the error system at any multi-point in $\mathcal{C}$ which is translation/rotation equivalent to y, so the error system is really determined by the translation/rotation equivalence class of y rather than just y. The number of such equivalence classes in $\mathcal{C}$ is know to be finite [10] so there are a finite number of error systems for any given minimally rigid graph $\mathbb{G}$ and corresponding list of desired edge distances $\{d_1, d_2, \ldots, d_m\}$.

In the full length version of this paper it is shown that for each multi-point $x \in \mathbb{R}^{2n}$, each term in the matrices $R(z)R'(z)|_{z=\bar{H}x}$ and $R(z)S'(z)|_{z=\bar{H}x}$ which appear in (9), can be written as a linear combination of norm-squared terms of the form $||x_i - x_j||^2$ for $i, j \in \{1, 2, \ldots, n\}$. Note that if i and j are the labels of adjacent vertices in $\mathbb{G}$, then

$$||x_i - x_j||^2 = ||z_{k_{ij}}||^2$$

because of (3). From this and the definition of $e_{k_{ij}}$ in (5), it follows that $||x_i - x_j||^2 = e_{k_{ij}} + d_{k_{ij}}^2$. In other words, for each pair of adjacent vertices $i, j \in \mathbb{G}$, the norm square $||x_i - x_j||^2$ can be expressed as a smooth function of e. Therefore if all pairs of vertices in $\mathbb{G}$ are adjacent, then both $R(z)R'(z)$ and $R(z)S'(z)$ can be expressed as smooth functions of e. Therefore if $\mathbb{G}$ is a complete graph, then there exists a smooth function g for which Theorem 1 holds globally with $\mathcal{A} = \mathbb{R}^{2n}$

To deal with the more general case when vertices such as i and j are not necessarily adjacent, it is enough to show that squared distances such as $||x_i - x_j||^2$ can be expressed as continuously differentiable functions of the squared edge distances within a formation $\{\mathbb{G}, x\}$. Equivalently, it is enough to show that such $||x_i - x_j||^2$ can be expressed as continuously differentiable functions of $e(\bar{H}x)$. For this to be possible for all possible vertex pairs, $\{\mathbb{G}, x\}$ clearly must be at least rigid. The hypothesis $e(z)|_{z=\bar{H}y} = 0$ in Theorem 1 ensures that the distances between the y_i must be the prescribed target distances so the formation $\{\mathbb{G}, y\}$ is infinitesimally rigid. In the full length version of this paper it is shown that for any $x = [\, x_1' \quad x_2' \quad \cdots \quad x_n' \,]'$ in a suitably defined open set of multi-points x for which $\{\mathbb{G}, x\}$ is infinitesimally rigid, it is possible to express the squared distances between each pair of points x_i, x_j in terms of $e(\bar{H}x)$. Since infinitesimal rigidity demands among other things that for at least one pair of points p and q, $x_p \neq x_q$, nothing is lost by excluding from consideration at the outset, values of x for which $x_p = x_q$. For simplicity we will assume the vertices are labeled so that $x_1 \neq x_2$. Accordingly, let $\mathcal{X}$ denote the set of all $x \in \mathbb{R}^{2n}$ for which $x_1 \neq x_2$.

To proceed, write $\delta : \mathcal{X} \to \mathbb{R}^{\frac{n(n-1)}{2}}$ for the *squared distance function*

$$x \longmapsto [\, ||x_1 - x_2||^2 \quad ||x_1 - x_3||^2 \quad \cdots \quad ||x_{n-1} - x_n||^2 \,]'$$

In view of the preceding discussion, to establish the existence of $\mathcal{A}$ and g for which Theorem 1 holds, it is enough to prove the following lemma.

Lemma 1: Let y be any vector in $\mathcal{X}$ for which $\{\mathbb{G}, y\}$ is an infinitesimally rigid formation. There exists an open set $\mathcal{A} \subset \mathcal{X}$ containing y and a continuously differentiable function
$f : e(\bar{H}\mathcal{A}) \to \mathbb{R}^{\frac{n(n-1)}{2}}$ for which

$$\delta(x) = f(e(\bar{H}x)), \ \ x \in \mathcal{A} \tag{11}$$

In proving the lemma as well as in subsequent sections of the paper, use is made of the function $\rho : \mathcal{X} \to \mathbb{R}^m$ defined by $x \longmapsto e(\bar{H}x)$. Note that $\rho(x)$ and $e(\bar{H}x)$ have the same value at every point $x \in \mathcal{X}$, although they are clearly different functions. To further distinguish ρ from e, we will always refer to ρ as the *induced error map* whereas we have already defined e to be the error function of the system.

A. Exponential Stability of the Unperturbed Error System

In this section we shall study the stability of the error system at y for the special case when $\mu = 0$. It is clear from (9) that in this case, the zero state $\epsilon = 0$ is an equilibrium state of the unperturbed error system. The follow theorem states that this is in fact an *exponentially stable* equilibrium.

Theorem 2: Let y be any value of $x \in \mathcal{X}$ for which $e(z)|_{z=\bar{H}y} = 0$. The equilibrium state $\epsilon = 0$ of the unperturbed error system $\dot{\epsilon} = g(\epsilon, 0)$ at y is exponentially stable.

As is well known, a critically important property of exponential stability is its *robust property*. We now explain exactly what this means for the error system under consideration:

Corollary 1: On any sufficiently small open neighborhood $\mathcal{M} \subset \mathbb{R}^m$ about $\mu = 0$, there is a continuously differentiable function $\mu \longmapsto \epsilon_\mu$ such that $\epsilon_0 = 0$ and for each $\mu \in \mathcal{M}$, ϵ_μ is an exponentially stable equilibrium state of the error system $\dot{\epsilon} = g(\epsilon, \mu)$.

At this point we have established for any given $y \in \mathcal{X}$ such that $e(\bar{H}y) = 0$, the existence of an open subset $\mathcal{A} \subset \mathcal{X}$ and a self-contained error system $\dot{\epsilon} = g(\epsilon, \mu)$ and shown that for any solution $x(t)$ of the overall system, the signal $e(\bar{H}x(t))$ coincides with a solution $\epsilon(t)$ of the error system for values of t such that $x(t) \in \mathcal{A}$. We have also shown that the error system has an exponentially stable equilibrium ϵ_μ provided μ is sufficiently small. The following corollary asserts that if $x(t)$ starts at a state $x(0)$ for which $e(\bar{H}x(0))$ is sufficiently close to ϵ_μ, then $x(t)$ will in fact exist and remain within $\mathcal{A}$ for all time.

Corollary 2: Let y be any state in $\mathcal{X}$ at which $e(\bar{H}y) = 0$. Let $\dot{\epsilon} = g(\epsilon, \mu)$ be the error system at y and let $\mathcal{A} \subset \mathcal{X}$ be an ambient space on which it is valid. Let $\mathcal{B}$ be any ball in $\mathbb{R}^m$, centered at $\rho(y) = 0$, which is small enough so that $\rho^{-1}(\mathcal{B}) \subset \mathcal{A}$ and also so that $\{\mathbb{G}, x\}$ is infinitesimally rigid for all $x \in \rho^{-1}(\mathcal{B})$. For each μ in any sufficiently small open neighborhood $\mathcal{M}$ in $\mathbb{R}^m$ about $\mu = 0$, and each initial state $x(0) \in \mathcal{X}$ for which $||e(\bar{H}x(0)) - \epsilon_\mu||$ is sufficiently small, the trajectory of the overall system starting at $x(0)$ exists for all time and lies within $\rho^{-1}(\mathcal{B})$. Moreover the error signal $e(\bar{H}x(t))$ is a solution to the error system equation $\dot{\epsilon} = g(\epsilon, \mu)$ and converges exponentially fast to the error system's equilibrium state ϵ_μ.

IV. Square Subsystem

In this section we derive a special 2×2 "square" subsystem whose behavior along trajectories of the overall system will enable us to easily predict the behavior of $x(t)$. Suppose that during some time period $[t_0, t_1)$, the state $x(t)$ of the overall system is "close" to a state y for which $e(\bar{H}y) = 0$. Since sufficient closeness of $x(t)$ and y would mean that the formation in $\{\mathbb{G}, x(t)\}$ is infinitesimally rigid, it is natural to expect that if $x(t)$ and y are close enough during the period $[t_0, t_1)$, then over this period the behavior of all of the z_i will depend on only a few of the z_i. As we will soon see, this is indeed the case. To explain why this is so, we will make use of the fact that the z system in (8) can also be written as

$$[\,\dot{z}_1 \quad \dot{z}_2 \quad \cdots \quad \dot{z}_m\,] = [\,z_1 \quad z_2 \quad \cdots \quad z_m\,]\, M(e(z), \mu) \tag{12}$$

where $M(e, \mu)$ is a $m \times m$ matrix depending linearly on the pair (e, μ). This is a direct consequence of the definition of the z_i in (3) and the fact that the x_i satisfy (6). We can now state the following proposition.

Proposition 1: Let y be any point in $\mathbb{R}^{2n}$ at which $e(\bar{H}y) = 0$. Let $\mathcal{B}$ be an open ball in $\mathbb{R}^m$ which is centered at 0. There are integers $p, q \in \mathbf{m}$ depending on y for which $z_p(y)$ and $z_q(y)$ are linearly independent. Moreover, if $\mathcal{B}$ is sufficiently small, then the matrix $Z(z) = [\,z_p \quad z_q\,]$ is nonsingular on $\bar{H}\rho^{-1}(\mathcal{B})$ and there is a continuously differentiable matrix - valued function $Q : e(\bar{H}\rho^{-1}(\mathcal{B})) \to \mathbb{R}^{2\times m}$ for which

$$[\,z_1 \quad z_2 \quad \ldots \quad z_m\,] = Z(z)Q(e(z)), \;\; z \in \bar{H}\rho^{-1}(\mathcal{B}) \tag{13}$$

If $x(t)$ is a solution to the overall system (7) for which $x(t) \in \rho^{-1}(\mathcal{B})$ on some time interval $[t_0, t_1)$, then on the same time interval, $Z(\bar{H}x(t))$ is nonsingular and satisfies

$$\dot{Z} = ZA(e(z), \mu) \tag{14}$$

where $A(e, \mu) = Q(e)M(e, \mu)L$ and L is the $m \times 2$ matrix whose columns are the pth and qth unit vectors in $\mathbb{R}^m$. Moreover

$$Q(\epsilon_\mu)M(\epsilon_\mu, \mu) = A(\epsilon_\mu, \mu)Q(\epsilon_\mu) \tag{15}$$

where ϵ_μ is the equilibrium state of the error system at y.

The proposition clearly implies that on the time interval $[t_0, t_1)$, the behavior of the entire vector z is determined by the behavior of the *square subsystem* defined by (13) and (14).

V. Analysis of the Overall System

In view of Corollary 2, we now know that for any mismatch error μ with small norm and any initial state $x(0)$ for which $e(\bar{H}x(0))$ is close to the error system equilibrium state ϵ_μ, the error signal $e(\bar{H}x(t))$ must converge exponentially fast to ϵ_μ and $\dot{x}(t)$ must be bounded on $[0, \infty)$. But what about $x(t)$ itself? The aim of the remainder of this paper is to answer this question. We will address the question in two steps. First in Section V-A we will consider the situation when $e(\bar{H}x(t))$ has already converged ϵ_μ. Then in Section V-B we will elaborate on the case when $e(\bar{H}x(t))$ starts out close to ϵ_μ.

A. Equilibrium Analysis

Let y be any state in $\mathbb{R}^{2n}$ at which $e(\bar{H}y) = 0$. The aim of this section is to determine the behavior of the formation $\{\mathbb{G}, x(t)\}$ over time for mismatch errors from a suitably defined "generic" set, assuming that for each such value of μ, the error $e(\bar{H}x(t))$ is constant and equal to the equilibrium state ϵ_μ of the error system at y . We will do this by first determining in Section V-A.1, a set of values of μ for which z and x are nonconstant for all $t \geq 0$. Then in Section V-A.2 we will show that for such values of μ, the distorted but infinitesimally rigid formation $\{\mathbb{G}, x(t)\}$ moves in a circular orbits about the origin in $\mathbb{R}^2$ at a fixed angular speed ω_μ.

1) Mismatch Errors for which z is Nonconstant: The aim of this sub-section is to show that once the error $e(\bar{H}x(t))$ has converged to a constant value, neither z nor x will be constant for small $||\mu||$ other than possibly for certain exceptional values. We shall do this assuming that the unperturbed formation $\{\mathbb{G}, y\}$ is "unaligned" where by an *unaligned* formation is meant formation $\{\mathbb{G}, x\}$ with multipoint $x = [\,x_1' \quad x_2' \quad \cdots x_n'\,]' \in \mathbb{R}^{2n}$ which does not contain a set of points x_i, x_j, x_k, x_l in $\mathbb{R}^2$ for which the line between x_i and x_j is parallel to the line between x_k and x_l. This is equivalent to saying that $\{\mathbb{G}, x\}$ is unaligned if, for every set of four points x_i, x_j, x_k, x_l within x, $(x_i - x_j) \wedge (x_k - x_l) \neq 0$. It is clear that the set of multi-points x for which $\{\mathbb{G}, x\}$ is unaligned is open and dense in $\mathbb{R}^{2n}$.

Throughout this sub-section we assume that $\mathcal{A}$, $\mathcal{B}$, ρ, and $\mathcal{M}$ are as in Corollary 2 and that for each $\mu \in \mathcal{M}$, $x(t, \mu)$ is a solution to the overall system for which $e(\bar{H}x(t, \mu)) = \epsilon_\mu$ where ϵ_μ is the exponentially stable equilibrium state of the error system $\dot{\epsilon} = g(\epsilon, \mu)$ at y. Our ultimate goal is to show that z is nonconstant for small normed but otherwise "generic" values of μ. The following Proposition enables us to make precise what is meant by a generic value.

Proposition 2: If $\{\mathbb{G}, y\}$ is an unaligned formation, there is an open set $\mathcal{M}_0 \subset \mathcal{M}$ about $\mu = 0$ within which the set of values of μ for which $z(x(t, \mu))$ is nonconstant on $[0, \infty)$, is open and dense in $\mathcal{M}_0$.

What the proposition is saying is that for almost any value of μ within any sufficiently small open subset of $\mathbb{R}^m$ which contains the origin, $z(x(t, \mu))$ will be nonconstant along any trajectory of the overall system for which e is fixed at the equilibrium state ϵ_μ of the error system at y. Thus if μ has a sufficiently small norm and otherwise chosen at random, it is almost for certain that $z(x(t, \mu))$ will be nonconstant. It is natural to say that μ is *generic*, if it is a value in $\mathcal{M}_0$ for which $z(x(t, \mu))$ is nonconstant.

2) Equilibrium Solutions: The aim of this section is to discuss the evolution of the formation $\{\mathbb{G}, x(t, \mu)\}$ along an "equilibrium solution" to the overall system assuming that μ is fixed at any value in $\mathcal{M}$. By an *equilibrium solution*, written $\bar{x}(t)$, is meant any solution to (7) for which $e(\bar{H}\bar{x}(t)) = \epsilon_\mu,\ t \geq 0$, where ϵ_μ is the equilibrium state of the error system at y. For simplicity we write $\bar{z}(t)$ for = $z(\bar{x}(t))$ and let $\bar{z}_i(t),\ i \in \mathbf{m}$, be the sub-vectors in $\mathbb{R}^2$ comprising $\bar{z}(t)$; i.e., $\bar{z}(t) = [\, \bar{z}_1'(t) \quad \bar{z}_2'(t) \quad \cdots \quad \bar{z}_n'(t) \,]'$.

Note that since $e(\bar{H}\bar{x}(t)) \in \mathcal{B}$ and $\rho(\bar{x}(t)) = e(\bar{H}\bar{x}(t))$, $\bar{x}(t) \in \rho^{-1}(\mathcal{B})$ and therefore $\bar{z}(t) \in \bar{H}\rho^{-1}(\mathcal{B}),\ t \geq 0$. Thus, in view of Proposition 1, there are integers $p, q \in \mathbf{m}$ for which the matrix $\bar{Z}(t) = [\, \bar{z}_p(t) \quad \bar{z}_q(t) \,]$ is nonsingular for $t \geq 0$. Moreover

$$[\, \bar{z}_1 \quad \bar{z}_2 \quad \ldots \quad \bar{z}_m \,] = \bar{Z}(t)\bar{Q},\ \ t \geq 0 \tag{16}$$

and

$$\dot{\bar{Z}} = \bar{Z}\bar{A},\ \ t \geq 0 \tag{17}$$

where $\bar{Q}$ and $\bar{A}$ are the *constant* matrices $\bar{Q} = Q(\epsilon_\mu)$ and $\bar{A} = A(\epsilon_\mu, \mu)$. It follows that the Gramian $\bar{Z}'\bar{Z}$ must satisfy

$$\dot{\overline{\bar{Z}'\bar{Z}}} = \bar{A}'\bar{Z}'\bar{Z} + \bar{Z}'\bar{Z}\bar{A} \tag{18}$$

In view of the definition of the z_i in (3), we see that the four entries in $\bar{Z}'\bar{Z}$ are of the form $(\bar{x}_i(t) - \bar{x}_j(t))'(\bar{x}_k(t) - \bar{x}_l(t))$ for various values of i, j, k and l. It can be shown that each such term is equal to a term is a continuously differentiable function of of the form ϵ_μ which is constant. Therefore $\bar{Z}'\bar{Z}$ is constant on $[0, \infty)$. Hence

$$\bar{A}'\bar{Z}'\bar{Z} + \bar{Z}'\bar{Z}\bar{A} = 0 \tag{19}$$

because of (18). Clearly

$$(\bar{Z}\bar{A}\bar{Z}^{-1})' + \bar{Z}\bar{A}\bar{Z}^{-1} = 0$$

Evidently the 2×2 matrix $\bar{Z}\bar{A}\bar{Z}^{-1}$ is skew symmetric so its spectrum must be $\{j\omega, -j\omega\}$ for some real number $\omega \geq 0$. But $\bar{A}$ is similar to $\bar{Z}\bar{A}\bar{Z}^{-1}$ so $\bar{A}$ must have the same spectrum.

We claim that $\bar{A} = 0$ and consequently that $\omega = 0$ if and only if $\bar{z}$ is constant. To understand why this is so, note first that if $\bar{z}$ is constant, then $\dot{\bar{Z}} = 0$. On the other hand, if $\dot{\bar{Z}} = 0$ then $\bar{z}$ must be constant because of (16). Meanwhile $\dot{\bar{Z}} = 0$ if and only if $\bar{A} = 0$ because of (17) and the fact that $\bar{Z}$ is nonsingular. Thus the claim is true.

Suppose $\omega > 0$ in which case in which case $\bar{z}_p$ and $\bar{z}_q(t)$ must be sinusoidal vectors varying at a single frequency ω. Moreover the same must also be true of the remaining $\bar{z}_i$ because of (16). Additionally, each $\bar{z}_i$ must have a constant norm because for all $t \geq 0$, $||\bar{z}_i(t)||^2 = e_i(\bar{H}\bar{x}(t)) + d_i^2,\ i \in \mathbf{m}$, and $e(\bar{H}\bar{x}(t)) = \epsilon_\mu$. These properties imply that z_i must be of the form

$$\bar{z}_k(t) = (\bar{\epsilon}_k + d_k^2)^{\frac{1}{2}} \begin{bmatrix} \cos(\omega t + \phi_k) \\ \sigma_k \sin(\omega t + \phi_k) \end{bmatrix}$$

where σ_k equals either 1 or -1. We claim that all of the σ_k be equal. To understand why this is so, observe that for all $i, j \in \mathbf{m}$,

$$\frac{d(\bar{z}_i'\bar{z}_j)}{dt} = \omega(\bar{\epsilon}_i + d_i^2)^{\frac{1}{2}}(\bar{\epsilon}_j + d_i^2)^{\frac{1}{2}}(\sigma_i\sigma_j - 1)\sin(2\omega t + \phi_i + \phi_i)$$

Since each $\bar{z}_i'\bar{z}_j$ is constant and $\omega(\bar{\epsilon}_i + d_i^2)^{\frac{1}{2}}(\bar{\epsilon}_j + d_i^2)^{\frac{1}{2}} \neq 0$, it must be true that $\sigma_i\sigma_j - 1 = 0,\ \ i, j \in \mathbf{m}$. Therefore $\sigma_i = \sigma_j,\ \ i, j \in \mathbf{m}$ so all of the σ_k have the same value.

The following theorem characterizes equilibrium solutions $\bar{x}$.

Theorem 3: Let $\mu \in \mathcal{M}$ be fixed and suppose that y is a vector in $\mathbb{R}^{2n}$ for which $e(\bar{H}y) = 0$. Let $\bar{x}$ be a solution to the overall system along which $e(\bar{H}\bar{x}(t)) = \epsilon_\mu$.

1) If μ is a mismatch error for which $\bar{z}$ is constant, then depending on the value of μ, all points within the time-varying, infinitesimally rigid formation $\{\mathbb{G}, \bar{x}(t)\}$ with distorted edge distances $(\bar{e}_i + d_i^2)^{\frac{1}{2}},\ i \in \mathbf{m}$, are either fixed in position or move off to infinity at the same constant velocity.
2) If μ is a mismatch error for which $\bar{z}$ is nonconstant, then all points within $\{\mathbb{G}, \bar{x}(t)\}$ rotate in either a clockwise or counterclockwise direction with the same constant angular speed $\omega > 0$ along circles centered at some point q in the plane, as does the distorted formation itself. Moreover if the undistorted formation $\{\mathbb{G}, y\}$ is unaligned, almost any mismatch error μ will cause this behavior to occur provided the norm of μ is sufficiently small.

B. Non-Equilibrium Analysis

Fix $\mu \in \mathcal{M}$ and let y be any state in $\mathbb{R}^{2n}$ for which the error $e(\bar{H}y) = 0$. In this section we will consider the situation when a solution $x(t)$ of the overall system starts out with an error signal $e(\bar{H}x(t))$ which is initially close to the exponentially stable equilibrium state ϵ_μ of the error system at y. As in section V-A.2, we let $\bar{x}(t)$ denote an equilibrium solution of the over all system and we write $\bar{z}(t) = z(\bar{H}\bar{x}(t))$. There may of course be many equilibrium solutions $\bar{x}(t)$ to the overall system along which $e = \epsilon_\mu$. Our aim is to show that any solution $x(t)$ to the overall system starting with $e(\bar{H}x(0))$ sufficiently close to ϵ_μ converges exponentially fast to such an equilibrium solution.

The detailed analysis of the non-equilibrium case, which relies heavily on the findings of section V-A.2, is carried out in the full length version of this paper. The main result is as follows.

Theorem 4: Let $\mu \in \mathcal{M}$ be fixed and let y be any state in $\mathbb{R}^{2n}$ for which $e(\bar{H}y) = 0$. Let $x(t)$ be any solution of the overall system starting in a state for which the error $e(\bar{H}x(0))$ is in the domain of attraction of the exponentially stable equilibrium state ϵ_μ of the error system $\dot{\epsilon} = g(\epsilon, \mu)$ at y. There exists a solution $\bar{x}$ to the overall system along which $e(\bar{H}\bar{x}(t) = \epsilon_\mu$, to which $x(t)$ converges exponentially fast.

1) If μ is a mismatch error for which $\bar{z}$ is constant, then depending on the value of μ, all points within the time-varying, infinitesimally rigid formation $\{\mathbb{G}, x(t)\}$ either converge exponentially fast to constant values or drift off to infinity.
2) If μ is a mismatch error for which $\bar{z}$ is nonconstant, then all points within $\{\mathbb{G}, x(t)\}$ converge exponentially fast to the points in a formation which rotates in either a clockwise or counterclockwise direction at a constant angular speed $\omega > 0$ along a circle centered at some fixed point in the plane. Moreover if the undistorted formation $\{\mathbb{G}, y\}$ is unaligned, almost any mismatch error μ will cause this behavior to occur provided the norm of μ is sufficiently small.

VI. Concluding Remarks

In this paper we have identified a basic robustness problem with the type of formation control proposed in [5]. A natural question to ask is if the problematic behavior can be eliminated by modifying the control laws? Simulations suggest that introducing delays or dead zones will not help. On the other hand, progress has been made to achieve robustness by introducing controls which estimate the mismatch error and take appropriate corrective action similar in spirit to what is typically done in adaptive control [11]. While results exploiting this idea are limited in scope [12], [13], they do nonetheless suggest that the approach may indeed resolve the problem.

We see no roadblocks to extending the findings of this paper to three dimensional formations. All of the material in Sections II through IV is readily generalizable without any surprising changes, although the square subsystem in Section IV will of course have to be 3×3 rather than 2×2. This change in size has an important consequence. This implication is that the skew symmetric matrix $\bar{Z}\bar{A}\bar{Z}^{-1}$ used in Section V-A.2 to characterize the spectrum of $\bar{A}$, will be 3×3 rather than 2×2. Thus in the three dimensional case, if $\bar{Z}\bar{A}\bar{Z}^{-1}$ is nonzero, its spectrum and consequently $\bar{A}$'s, must contain an a eigenvalue at 0 in addition to a pair of imaginary numbers $j\omega$ and $-j\omega$. Thus the corresponding formation will not only rotate at an angular speed ω, but it will also drift linearly with time. More precisely, in the three dimensional case, a mismatch errors can cause formation to move off to infinity along a helical trajectory.

Other questions remain. For example, it is natural to wonder how these findings might change for formations with more realistic agent models which are dynamic and nonholonomic. We conjecture that more elaborate agent models will not significantly alter the findings of this paper, although actually proving this will likely be challenging, especially in the realistic case when the parameters in the models of different agents are not identical.

Finally we point out that robustness issues raised here have broader implications extending well beyond formation maintenance to the entire field of distributed optimization and control. In particular, this research illustrates that when assessing the efficacy of a particular distributed algorithm, one must consider the consequences of distinct agents having slightly different understandings of what the values of shared data between them is suppose to be. For without the protection of exponential stability/convergence, it is likely that such discrepancies will cause significant misbehavior to occur.

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