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Toward Robust Control of Minimally Rigid Undirected Formations

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Abstract— **Gradient control is a method based on potential functions to locally stabilize rigid undirected formations. However, when there is a mismatch in two neighboring agents' understandings of what the target distance between them is supposed to be, such a potential-based control will typically cause a rigid undirected formation to rotate in the plane. This paper investigates an appropriate modification of the gradient control, which eliminates this rotation and locally stabilizes the triangular formation with no restriction on the number of mismatches and any minimally rigid formation consisting of four or more agents with only one mismatch.**

I. Introduction

Recently multi-agent formations have found extensive applications in surveillance, search and rescue, and exploration in unknown environment, which usually can not be achieved by an individual agent [1]–[12]. By a *multi-agent formation* is meant a collection of agents in two or three dimension space whose inter-agent distances are constant over time. Based on the idea of graph rigidity, a multi-agent formation can be obtained by certain pairs of neighboring agents maintaining prescribed target distances between them. We say that the formation is *undirected* when each distance is maintained by both associated neighboring agents.

The authors of [13] have developed an elegant potential-function-based theory of formation control which provides gradient laws for asymptotically stabilizing a large class of rigid, undirected formations in two-dimensional space, assuming all agents are described by kinematic point models. This particular methodology is perhaps the most comprehensive currently in existence for maintaining undirected formations based on graph rigidity. However, the authors of [14]–[16] observed that if there is a *mismatch* between two neighboring agents' understandings of what the target distance between them is supposed to be, then the formation under consideration will almost certainly rotate at a constant angular velocity despite maintaining its approximately correct shape. This phenomena is called the *non-robustness* of undirected formations, which this paper aims to fix.

In earlier work an effort is made to fix the problem by estimating constant distance mismatches and using appropriate modifications of the gradient control [17]. The proposed method relies both on the assumption that a specific matrix is Hurwitz stable, which may not be satisfied except for special formations, and also on a certain large constant parameter appearing in the control law, whose upper bound is not easily determined. By introducing a suitably defined normalization, reference [18] resolves this issue for triangular formations. However, neither approach completely solves the non-robustness problem except for triangular formations. The main contribution of this paper is to show that the approach we are taking reduces to the classical decentralized control problem treated in [19] and [20]. It is well known that in its most general form, the classical decentralized control problem is not easily solved unless one is willing to introduce very high dimensional controllers. A second contribution of this paper is to propose a relatively simple control which cleanly resolves the non-robustness problem first for triangular formations with no restriction on the number of mismatches and second for more general formations in which only one mismatch occurs.

The rest of the paper is organized as follows: In Section II we address the non-robustness issue of the gradient control and introduce a control with the aim of estimating the constant mismatch and canceling it. In Section III, we first show that the problem of fixing the non-robustness issue can be reduced to a decentralized control problem and then propose a simple control for solving the problem. We present the main result in Section IV and conclusions in Section V.

II. Problem Formulation

We consider a formation in the plane consisting of $n \geq 3$ mobile autonomous agents labeled $1, 2, \ldots, n$. We assume the desired formation is specified by a rigid undirected graph $\mathbb{G}$ with n vertices labeled $1, 2, \ldots, n$ and m edges labeled $1, 2, \ldots, m$. We write k_{ij} for the label of that edge which connects adjacent vertices i and j. Thus $k_{ij} = k_{ji}$. We call agent j a *neighbor* of agent i if vertex j is adjacent to vertex i and we write $\mathcal{N}_i$ for the labels of agent i's neighbors. For simplicity, we assume that $\mathbb{G}$ is minimally rigid which means that $m = 2n - 3$. Let $\mathbf{n} = \{1, 2, ..., n\}$ and $\mathbf{m} = \{1, 2, ..., m\}$.

Let d_{ij} denote the prescribed target distance that agent i is to maintain from neighbor j. For two neighboring agents i and j, we do not assume that the target distances d_{ij} and d_{ji} are necessarily equal. Instead we assume that

$$|d_{ij} - d_{ji}| \leq \epsilon_{ij}$$

where ϵ_{ij} is a small nonnegative number bounding the *mismatch* between the two neighboring agents' understandings of what the target distance between them is suppose to be.

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We assume that when there is no mismatch between d_{ij} and d_{ji}, these distances are realizable by some set of points in the plane with coordinate vectors $x_1, x_2, ..., x_n \in \mathbb{R}^2$ such that the resulting framework $(x, \mathbb{G})$ is minimally infinitesimally rigid [21] at the multi-point $x = [\, x_1' \quad x_2' \quad \cdots \quad x_n' \,]'$.

Suppose each agent i's motion is described in global coordinates by a simple kinematic point model

$$\dot{x}_i = v_i, \quad i \in \mathbf{n} \tag{1}$$

We further assume each agent i is able to accurately measure the relative position $x_j - x_i$ of each of its neighbors $j \in \mathcal{N}_i$. The gradient control proposed in [13] for each agent i is

$$v_i = \sum_{j \in \mathcal{N}_i} (x_j - x_i)(||x_j - x_i||^2 - d_{ij}^2), \quad i \in \mathbf{n} \tag{2}$$

To better model the mismatch we orient each edge in $\mathbb{G}$ with a specific direction from a tail vertex to a head vertex. For the kth edge in the oriented $\mathbb{G}$ with i as the head and j as the tail, $k \in \mathbf{m}$, we let

$$z_{k_{ij}} = x_i - x_j, \quad d_k = d_{ij}, \quad e_k = ||z_k||^2 - d_k^2$$

and μ_k denote the mismatch between d_{ij} and d_{ji} defined as

$$\mu_k = d_k^2 - d_{ji}^2$$

Write $\mathcal{N}_i^+$ for the set of all $j \in \mathcal{N}_i$ for which vertex i is a head of oriented edge k_{ij}. Let $\mathcal{N}_i^-$ denote the complement of $\mathcal{N}_i^+$ in $\mathcal{N}_i$. Then the gradient control (2) becomes

$$v_i = -\sum_{j \in \mathcal{N}_i^+} z_{k_{ij}} e_{k_{ij}} + \sum_{j \in \mathcal{N}_i^-} z_{k_{ij}} (e_{k_{ij}} + \mu_{k_{ij}}), \quad i \in \mathbf{n} \tag{3}$$

It has been shown in [**?**], [14] that because of the mismatch, the trajectory of the resulting distorted but rigid formation under the above gradient control will for generic mismatch values converge exponentially fast to a closed orbit in $\mathbb{R}^2$ which is then traversed periodically at a single sinusoidal frequency. The goal of this paper is to propose a way to modify the gradient control, which eliminates the orbiting behavior while maintaining approximately correct formation shape. For this purpose we employ a similar approach to the one proposed in [17] by introducing an estimate $\widehat{\mu}_k$ for the tail agent of each edge $k \in \mathbf{m}$ whose rate of change u_k is a control signal to be determined. Thus, in place of (3) we propose using the following

$$v_i = -\sum_{j \in \mathcal{N}_i^+} z_{k_{ij}} e_{k_{ij}} + \sum_{j \in \mathcal{N}_i^-} z_{k_{ij}} (e_{k_{ij}} + \mu_{k_{ij}} - \widehat{\mu}_{k_{ij}}), \quad i \in \mathbf{n} \tag{4}$$

where

$$\dot{\widehat{\mu}}_k = u_k, \quad k \in \mathbf{m} \tag{5}$$

Recall that by the "information pattern" of a decentralized control system consisting of n agents is meant a description of the signals available to each agent i, $i \in \mathbf{n}$. For the problem at hand, what is available to agent i are the signals

$$\begin{array}{ll} z_{k_{ij}}, & j \in \mathcal{N}_i; \\ e_{k_{ij}}, & j \in \mathcal{N}_i^+; \\ e_{k_{ij}} + \mu_{k_{ij}}, \quad \widehat{\mu}_{k_{ij}} & j \in \mathcal{N}_i^-; \end{array}$$

which we call *local measurements* of agent i. For each $k \in \mathbf{m}$, we define a $\bar{k} \in \mathbf{n}$ to be the label of the tail vertex of the kth edge in the oriented $\mathbb{G}$. In the following, we will give methods for choosing each u_k as a function of the local measurements of agent $\bar{k}$.

III. Methods of Choosing u_k

By applying the control (4) into (1) and letting $\bar{\mu}_k = \mu_k - \widehat{\mu}_k$, one has

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i^+} z_{k_{ij}} e_{k_{ij}} + \sum_{j \in \mathcal{N}_i^-} z_{k_{ij}} (e_{k_{ij}} + \bar{\mu}_{k_{ij}}), \quad i \in \mathbf{n} \tag{6}$$

and

$$\dot{\bar{\mu}}_k = -u_k, \quad k \in \mathbf{m} \tag{7}$$

To write the above equations into a compact form, we let $H = [h_{ij}]_{m \times n}$ be the matrix such that $h_{ij} = 1$ if j is the head of the kth edge; $h_{ij} = -1$ if j is the tail of the kth edge; $h_{ij} = 0$, otherwise. Then H is transpose of the incidence matrix of the oriented $\mathbb{G}$. By the definition of H, one has

$$z = (H \otimes I_2)x$$

where $z = [\, z_1' \quad z_2' \quad \cdots \quad z_m' \,]'$, I_2 denotes the 2×2 identity matrix and $\otimes$ denotes the kronecker product. Let R denote the rigidity matrix of the oriented $\mathbb{G}$. Then

$$R = Z'(H \otimes I_2)$$

where $Z = \text{diag}\,\{z_1, z_2, ..., z_m\}$. Let

$$S = Z'(J \otimes I_2)$$

where J is the matrix from replacing all -1s in $-H$ by 0s. Let

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}, \quad \bar{\mu} = \begin{bmatrix} \bar{\mu}_1 \\ \bar{\mu}_2 \\ \vdots \\ \bar{\mu}_m \end{bmatrix}$$

From (6) and definitions of R and S, one has

$$\dot{x} = -R'e + S'\bar{\mu} \tag{8}$$

and

$$\dot{e} = 2Z'\dot{z} = 2Z'(H \otimes I_2)\dot{x} = 2R\dot{x}$$

which imply

$$\dot{e} = -2RR'e + 2RS'\bar{\mu} \tag{9}$$

A. Formulation as a Classical Decentralized Control Problem

Let $\mathcal{E}_i^- = \{i_1, i_2, ..., i_{r_i}\}$ denote the set of labels of edges with i as the tail vertex, where $r_i = |\mathcal{E}_i^-|$. Let S_i be the $r_i \times 2n$ matrix consisting of all the kth rows of S where $k \in \mathcal{E}_i$. Let $\tilde{\mu}_i = [\, \bar{\mu}_{i_1} \quad \bar{\mu}_{i_2} \quad \cdots \quad \bar{\mu}_{i_{r_i}} \,]'$ and $\tilde{u}_i = -[\, u_{i_1} \quad u_{i_2} \quad \cdots \quad u_{i_{r_i}} \,]'$. In the special case when $\mathcal{E}_i^- = \emptyset$, S_i, $\tilde{\mu}_i$ and $\tilde{u}_i$ are empty matrices. By definitions of S, S_i, $\tilde{\mu}_i$ and $\tilde{u}_i$, one has

$$S'\bar{\mu} = \sum_{i=1}^{n} S_i'\tilde{\mu}_i \tag{10}$$

and

$$\dot{\tilde{\mu}}_i = \tilde{u}_i$$

Substituting (10) into (9) implies

$$\dot{e} = -2RR'e + 2R\sum_{i=1}^{n} S_i'\tilde{\mu}_i \tag{11}$$

Let $A = -2RR'$ and $B_i = RS_i'$, $i \in \mathbf{n}$. Since each entry of RR' and RS_i' proves to be a continuously differentiable function of e around $e = 0$ [**?**], then A and B_i are continuously differentiable function of e. In the following we can write $A(e)$ and $B_i(e)$, $i \in \mathbf{n}$, instead. By linearization of (11) around $e = 0$ and $\tilde{\mu}_i = 0$, $i \in n$, one has

$$\dot{e} = A(0)e + \sum_{i=1}^{n} B_i(0)\tilde{\mu}_i \tag{12}$$

where

$$\dot{\tilde{\mu}}_i = \tilde{u}_i \tag{13}$$

Next we review the local measurements available at each agent i in terms of e and $\tilde{\mu}_i$. Let $\mathcal{E}_i$ denote the set of labels of all edges in $\mathbb{G}$ adjacent to vertex i and $q_i = |\mathcal{E}_i|$. Then $q_i = |\mathcal{N}_i|$ and $q_i \geq r_i$. Let $\bar{C}_i$ be the $q_i \times m$ matrix consisting of all the kth rows of the $m \times m$ identity matrix, where $k \in \mathcal{E}_i$. Let $\widehat{C}_i$ denote the $q_i \times r_i$ matrix consisting of all the kth columns of $\bar{C}_i$ where $k \in \mathcal{E}_i^-$. Then agent i is able to measure

$$[\,\bar{C}_i \quad \widehat{C}_i\,] \begin{bmatrix} e \\ \mu_{i_1} \\ \mu_{i_2} \\ \vdots \\ \mu_{i_{r_i}} \end{bmatrix} \tag{14}$$

Since $\widehat{\mu}_{i_1}, \widehat{\mu}_{i_2}, \cdots, \widehat{\mu}_{i_{r_i}}$ belong to agent i's local measurements, then agent i knows

$$\widehat{C}_i \begin{bmatrix} \widehat{\mu}_{i_1} \\ \widehat{\mu}_{i_2} \\ \vdots \\ \widehat{\mu}_{i_{r_i}} \end{bmatrix}$$

which is equal to

$$[\,\bar{C}_i \quad \widehat{C}_i\,] \begin{bmatrix} 0 \\ \widehat{\mu}_{i_1} \\ \widehat{\mu}_{i_2} \\ \vdots \\ \widehat{\mu}_{i_{r_i}} \end{bmatrix}. \tag{15}$$

By subtracting (15) from (14), one has each agent i is able to know

$$y_i = C_i \begin{bmatrix} e \\ \tilde{\mu}_i \end{bmatrix}, \quad i \in \mathbf{n} \tag{16}$$

where $C_i = [\,\bar{C}_i \quad \widehat{C}_i\,]$. The problem we are interested in becomes the following decentralized control problem.

Decentralized Control Problem: For each $i \in n$, devise a control $\tilde{u}_i$, depending only on y_i, so that the overall linearized system (12)-(13) is exponentially stable.

We refer to [20], [22], [23] for some general methods for solving this problem. In the following, we will give one choice of u_k, $k \in \mathbf{m}$ depending only on local measurements of agent $\bar{k}$.

B. One Choice for u_k

Recall that

$$\dot{e} = -2RR'e + 2RS'\bar{\mu} \tag{17}$$

where

$$\dot{\bar{\mu}}_k = -u_k, \quad k \in \mathbf{m}$$

The key idea employed in the following is choosing u_k to ensure that

$$\dot{\bar{\mu}} = 2SR'e - 2SS'\bar{\mu} \tag{18}$$

For this purpose we let

$$u_k = 2z_k' v_{\bar{k}}, \quad k \in \mathbf{m} \tag{19}$$

where $\bar{k} \in \mathbf{n}$ is the label of the tail vertex of the kth edge. Note immediately by (4) that each $v_{\bar{k}}$ depends only on local measurements of agent $\bar{k}$ and so does each u_k, $k \in \mathbf{m}$.

Now we explain how we reach (18) from (19). By (19) one has

$$\dot{\bar{\mu}} = -2Z' \begin{bmatrix} v_{\bar{1}} \\ v_{\bar{2}} \\ \vdots \\ v_{\bar{m}} \end{bmatrix} \tag{20}$$

By definitions of J and $\bar{k}$, one has

$$\begin{bmatrix} v_{\bar{1}} \\ v_{\bar{2}} \\ \vdots \\ v_{\bar{m}} \end{bmatrix} = (J \otimes I_2) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (J \otimes I_2)\dot{x} \tag{21}$$

From (20), (21) and $S = Z'(J \otimes I_2)$, one has

$$\dot{\bar{\mu}} = -2S\dot{x} \tag{22}$$

which and $\dot{x} = -R'e + S'\bar{\mu}$ lead to (18).

IV. Main Result

In this section, we will present our main result under the control (4)- (5) with u_k given in (19), which leads to the system consisting of (17) and (18). For convenience of analysis, we let

$$\phi = \begin{bmatrix} e \\ \bar{\mu} \end{bmatrix}, \quad Q = 2\begin{bmatrix} RR' & -RS' \\ -SR' & SS' \end{bmatrix}$$

Then

$$\dot{\phi} = -Q\phi \tag{23}$$

Note that

$$Q = 2\,[\,R' \quad -S'\,]'\,[\,R' \quad -S'\,],$$

which is clearly positive semi-definite. To prove that $\phi = 0$ is an exponentially stable equilibrium of (23), it is sufficient to show the matrix $\begin{bmatrix} R' & -S' \end{bmatrix}$ has full column rank. In the next two subsections, we distinguish three-agent formations with no restriction on the number of mismatches, and formations of four or more agents with just one mismatch.

A. The Triangular Formation

In the case that $\mathbb{G}$ is a triangular formation, we have the following lemma

Lemma 1: If $\mathbb{G}$ is a triangular formation and $(x, \mathbb{G})$ is infinitesimally rigid, $[\,R' \quad -S'\,]$ has full column rank.

Proof of Lemma 1: Note that the oriented $\mathbb{G}$ is either a cyclic triangle or an acyclic triangle. We will prove Lemma 1 is true by showing

$$\ker [\,R' \quad -S'\,] = 0$$

in both cases.

First we consider the case that the oriented $\mathbb{G}$ is a cyclic triangle. Without loss of generality we assume the edges are oriented such that the edge vectors are $z_1 = x_2 - x_1, z_2 = x_3 - x_2, z_3 = x_1 - x_3$. Then one has

$$R = \begin{bmatrix} -z_1' & z_1' & 0 \\ 0 & -z_2' & z_2' \\ z_3' & 0 & -z_3' \end{bmatrix}, \quad -S = \begin{bmatrix} -z_1' & 0 & 0 \\ 0 & -z_2' & 0 \\ 0 & 0 & -z_3' \end{bmatrix}$$

Suppose $p = [\,p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6\,]' \in \ker [\,R' \quad -S'\,]$. Then one has

$$-z_1 p_1 + z_3 p_3 - z_1 p_4 = 0$$
$$z_1 p_1 - z_2 p_2 - z_2 p_5 = 0$$
$$z_2 p_2 - z_3 p_3 - z_3 p_6 = 0$$

Since $(x, \mathbb{G})$ is infinitesimally rigid, then any two of z_1, z_2, z_3 are linearly independent. Thus $p = 0$ and $\ker [\,R' \quad -S'\,] = 0$.

Second, we consider the acyclic case. Without loss of generality we assume the edges in the oriented $\mathbb{G}$ are such that $z_1 = x_1 - x_2, z_2 = x_3 - x_2, z_3 = x_1 - x_3$. Then

$$R = \begin{bmatrix} z_1' & -z_1' & 0 \\ 0 & -z_2' & z_2' \\ z_3' & 0 & -z_3' \end{bmatrix}, \quad -S = \begin{bmatrix} 0 & -z_1' & 0 \\ 0 & -z_2' & 0 \\ 0 & 0 & -z_3' \end{bmatrix}$$

Note that any two of z_1, z_2, z_3 are linearly independent. Using a similar argument as in the cyclic case, one still has $\ker [\,R' \quad -S'\,] = 0$. We complete the proof. ■

B. Formations of Four or More Agents

If $\mathbb{G}$ is an n-agent minimally rigid formation with $n \geq 4$, one has $m > n$. Then $[\,R' \quad -S'\,]_{2n \times 2m}$ can not have full column rank. In order for the proposed control to work, we limit ourselves to the case that there is only one mismatch for $n \geq 4$. Without loss of generality, we assume that the mismatch takes place on the 1st edge $(1, 2)$, that is, $\mu_1 \neq 0$ and $\mu_k = 0$ for $k = 2, 3, ..., m$. We orient this edge with a direction from 1 to 2. Then one only needs to introduce $\widehat{\mu}_1$ in the control of agent 1 and all other agents still use the standard gradient-based control. That is,

$$v_1 = -\sum_{j \in \mathcal{N}_1^+} z_{k_{ij}} e_{k_{ij}} + \sum_{j \in \mathcal{N}_1^-, k \neq 1} z_{k_{ij}} e_{k_{ij}} + z_1(e_1 + \mu_1 - \widehat{\mu}_1) \tag{24}$$

where

$$\dot{\widehat{\mu}}_1 = 2 z_1' v_1$$

and

$$v_i = -\sum_{j \in \mathcal{N}_i^+} z_{k_{ij}} e_{k_{ij}} + \sum_{j \in \mathcal{N}_i^-} z_{k_{ij}} e_{k_{ij}}, \quad i = 2, 3, ..., n$$

The system in the form of (23) will still be obtained with R as the rigidity matrix and S being given as $S = [\,z_1' \quad 0\,]_{1 \times 2n}$.

Lemma 2: If $\mathbb{G}$ is a formation of four or more agents with only one mismatch, and $(x, \mathbb{G})$ is infinitesimally rigid, then $[\,R' \quad -S'\,]$ has full column rank.

Proof of Lemma 2: Let σ_i be the ith column of the $2n \times 2n$ identity matrix, $i = 1, 2, ..., 2n$. Let $\mathcal{U}$ be the subspace spanned by $\{\sigma_3, \sigma_4, ..., \sigma_{2n}\}$. Let $\mathcal{W}$ be the subspace spanned by $w_1 = \sum_{i=1,3,..,2n-1} \sigma_i$ and $w_2 = \sum_{i=2,4,..,2n} \sigma_i$. Then

$$\mathcal{W} + \mathcal{U} = \mathbb{R}^{2n} \tag{25}$$

Since $S = \left[\ z_1' \quad 0\ \right]_{1 \times 2n}$ and $z_1 \in \mathbb{R}^2$, then

$$\mathcal{U} \subset \ker S \tag{26}$$

From $R w_1 = 0$ and $R w_2 = 0$, one has

$$\mathcal{W} \subset \ker R \tag{27}$$

From (25), (26) and (27) one has

$$\ker R + \ker S = \mathbb{R}^{2n}$$

which implies that

$$\text{Im } R' \cap \text{Im } S' = 0 \tag{28}$$

Suppose

$$\begin{bmatrix} p \\ q \end{bmatrix} \in \ker [\,R' \quad -S'\,]$$

where $p \in \mathbb{R}^m, q \in \mathbb{R}$. Then

$$R' p = S' q$$

which together with (28) implies

$$R' p = 0, \quad S' q = 0$$

Since the formation is infinitesimally rigid, one has $\ker R' = 0$ and $z_1 \neq 0$, the latter of which implies $\ker S' = 0$. Then $p = 0, q = 0$. Thus $\ker [\,R' \quad -S'\,] = 0$ by which one concludes $[\,R' \quad -S'\,]_{2n \times (m+1)}$ has full column rank. ■

C. Main Result

In this subsection, we will present our main result by applying Lemma 1 and Lemma 2 just derived above.

Toward this end, we need to make use of infinitesimal rigidity. Note that $(x, \mathbb{G})$ is infinitesimally rigid at $e = 0$. By the definition of infinitesimal rigidity, one has that there exists a constant $\rho > 0$ such that $(x, \mathbb{G})$ is infinitesimally rigid for $||e||^2 \leq \rho$.

Let $\phi(t)$ be the solution to the system (23) starting at any given state $||\phi(0)||^2 \leq \rho$. Let $[0, T)$ denote this solution's maximal interval of existence. It is easy to verify that the rate of change of the function $V = ||\phi||^2$ along this solution must satisfy

$$\dot{V} = -2\phi' Q \phi$$

Since Q is at least positive semi-definite no matter what the value of ϕ, V must be non-increasing. Thus $V(t) \leq V(0)$. In view of the definition of V and the assumption that $||\phi(0)||^2 \leq \rho$ it must be true that $V(t) \leq \rho$. This implies that

$$||e(t)||^2 \leq \rho$$

and thus $(x(t), \mathbb{G})$ is infinitesimally rigid for $t \in [0, T)$. Since $V(t)$ is bounded, one concludes that the solution in question is bounded on $[0, T)$. From this it follows by a standard argument that $T = \infty$. Then for $t \in [0, \infty)$, $(x(t), \mathbb{G})$ is infinitesimally rigid. Using Lemma 1 and Lemma 2, one has the matrix $[\, R' \quad -S' \,]$ has full column rank in the case that $\mathbb{G}$ is a triangular formation with multiple mismatches or $\mathbb{G}$ is a formation of four or more agents with only one mismatch. Thus $Q(t)$ is positive definite for $t \in [0, \infty)$. Let λ denote the minimum of the smallest eigenvalue of $Q(t)$. Then one has

$$\dot{V} = -2\phi' Q\phi \leq -2\lambda||\phi||^2 = -2\lambda V$$

This clearly implies for the system (23) that any trajectory starting such that $||e(0)||^2 \leq \rho$ must approach the equilibrium $\phi = 0$ as fast as $e^{-2\lambda t}$ approaches 0.

Since e is bounded, R, S are bounded, which together with $\dot{x} = -R'e + S'\bar{\mu}$ and the fact that e and $\bar{\mu}$ converge to 0 exponentially fast implies that $\dot{x}$ converges to 0 and x converges to a constant vector exponentially fast. We summarize.

Theorem 1: Given a triangular formation with any number of mismatches, or a minimally rigid formation of four or more agents with only one mismatch. Suppose there exists a positive constant ρ such that the formation is infinitesimally rigid for $||e||^2 \leq \rho$. Suppose $x(0)$ and $\widehat{\mu}(0)$ are such that $||e(0)||^2 + ||\bar{\mu}(0)||^2 \leq \rho$. Then e and $\dot{x}$ converge to 0 exponentially fast under the control (4)- (5) with u_k given in (19).

V. Conclusion

By using the idea of estimating the constant mismatch and canceling it, we have fixed the non-robustness issue for triangular formations and minimally rigid undirected formations of four or more agents with only one mismatch. It is worth mentioning that the control proposed in this paper does not involve any undetermined parameter as in [17] or requires the formation to be generated by only vertex addition as in [18]. Future work will focus on developing methods for all minimally rigid formations with no restriction on the number of mismatches.

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