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Stability Analysis on Four Agent Tetrahedral Formations

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***Abstract*— We consider a four agent tetrahedral formation of mobile agents in 3-dimensional Euclidean space. Each agent is required to maintain prescribed inter-agent distances from its neighbors so that they collectively form a desired formation shape, a task which is now called distance-based formation control. Under a common gradient-based control law, there exists an incorrect equilibrium set in which the agents do not achieve the desired formation shape. By investigating the linearized dynamics of the system, we prove that all incorrect equilibria are unstable, which results in that desired formation shape is almost globally asymptotically stable. Numerical simulation results are also included.**

I. Introduction

Formation control of mobile agents has attracted a lot of interest recently, [1]–[15], because of its potential applications such as satellite formation flying and mobile sensing for surveillance purposes. The mobile agents are supposed to maintain a given formation shape over the time by controlling some variables. For example, the desired formation shape could be obtained by controlling the absolute position of each agent with respect to the global reference frame, relative displacements of the agents with respect to orientation-aligned local reference frames, or inter-agent distances without orientation alignment condition for the local reference frames [16]. If we achieve the desired formation shape by controlling the inter-agent distances with relative position measurements, we call this scheme *distance-based formation control*. Note that in distance-based formation control, orientation and translational position are not controlled; thus the formation behaves like a rigid body which is free to move in space.

For formations in the plane, Olfati-Saber and Murray apply distance-based control to formations modeled by *rigid* and unfoldable graphs in which the vertex dynamics are modeled by double integrators [1] (agents correspond to graph vertices and inter-agent distances are denoted as graph edges). Motivated from the work of Olfati-Saber and Murray, Krick et al. show the asymptotic stability of *infinitesimally rigid* formations in which the agent dynamics are modeled by single integrators [2]. Oh and Ahn propose another distance-based control law in terms of inter-agent distance dynamics [3] and a new formation control law with orientation alignment algorithm [4].

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While the main results of [2] deal with formations with undirected graphs, they also provide a natural extension of their results to directed formations called *acyclic minimally persistent* formations. On the other hand, Anderson, Yu, Summers and their colleagues propose a control scheme for each agent to pursue an instantaneous target position determined by the distance constraints, and achieve local exponential convergence to the desired formation shapes with a *minimally persistent* graph [5]–[7].

However, all those results are confined to local stability analysis and global stability analysis has been rare. Using an example of the four agent complete graph (i.e., all inter-agent distances are specified), Krick et al. point out the existence of incorrect equilibria where the inter-agent distances do not accord with given desired distances. Summers, Anderson, Dasgupta and their colleagues struggled to show the instability of the incorrect equilibria, and some results are found in [8]–[10]. References [17] and [18] establish that the occurrence of incorrect equilibria is essentially unavoidable, and seek to count them in some instances.

Slightly differently from [8]–[10], we aim at analyzing global stability of four agent formations modeled by a complete graph in a 3-dimensional ambient space. Park et al. explore a particular example of an equilateral tetrahedron in [13], but their results are based on numerical calculations and require solution of complicated nonlinear algebraic equations. In this paper, we verify that any incorrect equilibria of four agent tetrahedral formations, of which the desired formation shape has nonzero volume, are unstable, which is a more general result than that of [13]. To achieve our objective, we investigate the eigenvalues of the Jacobian of the relevant nonlinear system at incorrect equilibria, and prove that there exists an unstable eigenvalue which makes any incorrect equilibrium unstable. After that, we try to characterize exact behavior of the agents near the incorrect equilibria using a geometrical interpretation relevant to the volume of the formation.

The rest of the paper is organized as follows. In Section II, we introduce some notations, and a control law to achieve the desired formation shape by maneuvering four agents. In Section III, we provide the main analysis on the instability of incorrect equilibria. Simulation results showing the repulsive behavior near the incorrect equilibria are provided in Section IV. Finally, Section V summarizes our results.

II. Preliminaries

In the rest of the paper, we are going to use the following notations:

- $\mathbb{R}^n$: n-dimensional Euclidean space

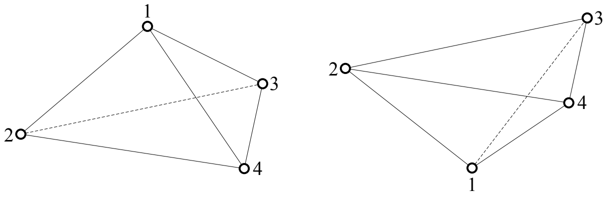


Fig. 1. Two configurations having different orientations. Note that both of them satisfy the same distance set. (Agents 2,3,4 lie in a horizontal plane and agent 1 locates above or below that plane.)

- $\mathbb{R}^{m\times n}$: the set of m by n real matrices
- $|\mathcal{S}|$: the cardinality of a set $\mathcal{S}$
- $\mathcal{A}\subsetneq\mathcal{B}$ or $\mathcal{B}\supsetneq\mathcal{A}$: $\mathcal{A}$ is a proper subset of $\mathcal{B}$
- I_n: the n by n identity matrix
- $\emptyset$: the empty set
- $A\otimes B$: the Kronecker product of matrices A and B
- $\|x\|$: the Euclidean norm of a vector x.

A. Formation representation

We use a graph, which is denoted by $\mathcal{G}$, to describe the formation of four agents. Each agent corresponds to a vertex of the graph, and the set $\mathcal{V}$ of all vertices is defined by $\mathcal{V}=\{1,2,3,4\}$. We denote the position of agent i by $p_i=[x_i\ \ y_i\ \ z_i]^T\in\mathbb{R}^3$, and then the formation shape is determined by a *realization* p defined as $p=[p_1^T\ \ \dots\ \ p_4^T]^T\in\mathbb{R}^{3|\mathcal{V}|}$. We call the pair $(\mathcal{G},p)$ a *framework*. Two different realizations p and p' are said to be *congruent* if $\|p_i-p_j\|=\|p'_i-p'_j\|$ for all $i,j\in\mathcal{V}$. The desired formation shape is given by a representative realization $\bar{p}\in\mathbb{R}^{3|\mathcal{V}|}$. We assume that the desired formation shape has nonzero volume in the space $\mathbb{R}^3$. Then, our objective is to achieve the desired formation shape determined by $\bar{p}$ up to congruence. Let $\mathcal{E}$ be the set of all edges, i.e., $\mathcal{E}=\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$. To achieve the goal under distance-based control, we are supposed to control the inter-agent distances so that for all $(i,j)\in\mathcal{E}$, $\|p_i-p_j\|\to\|\bar{p}_i-\bar{p}_j\|$ as time goes on.[1] Note that there could be different configurations realized from the same distance set (see Fig. 1) which are not obtainable by translation or rotation. A (discontinuous) reflection is required.

The relative displacements corresponding to the edges are defined as $r_{ij}=p_i-p_j$ for all $(i,j)\in\mathcal{E}$. Let $e(p)$ be an error function defined by $e(p)=[e_{12}\ \ e_{13}\ \ e_{14}\ \ e_{23}\ \ e_{24}\ \ e_{34}]^T$, where $e_{ij}=d_{ij}^2-\bar{d}_{ij}^2$, $d_{ij}=\|p_i-p_j\|$, and $\bar{d}_{ij}=\|\bar{p}_i-\bar{p}_j\|$. Then the desired formation set $\mathcal{P}_d$ is defined by

$$\mathcal{P}_d=\left\{p\in\mathbb{R}^{3|\mathcal{V}|}\colon e(p)=0\right\}.$$

B. Equations of motion

We assume that the motion of each agent is modeled by a single integrator, i.e.,

$$\dot{p}=u,$$

[1]Note that the desired distance set is *realizable* in $\mathbb{R}^3$ because the distances are determined from the given realization.

where $u=[u_1^T\ \ \dots\ \ u_4^T]^T$. To design a control law which makes $\mathcal{P}_d$ stable, we consider a potential function defined by

$$V(p)=\frac{1}{4}\sum_{(i,j)\in\mathcal{E}}e_{ij}^2.$$

From the fact that V is positive definite with respect to $e(p)$, we use the gradient-descent law which is also proposed in several papers [2], [9]–[12]. Thus we have

$$\dot{p}=-\left[\frac{\partial V}{\partial p}\right]^T \tag{1a}$$
$$=-[R(p)]^T e(p) \tag{1b}$$
$$=-(E(p)\otimes I_3)p. \tag{1c}$$

The matrix $R(p)$ is the Jacobian of $e(p)/2$ which is known as the *rigidity matrix* and defined by

$$R(p)=\begin{bmatrix} p_1^T-p_2^T & p_2^T-p_1^T & 0 & 0\\ p_1^T-p_3^T & 0 & p_3^T-p_1^T & 0\\ p_1^T-p_4^T & 0 & 0 & p_4^T-p_1^T\\ 0 & p_2^T-p_3^T & p_3^T-p_2^T & 0\\ 0 & p_2^T-p_4^T & 0 & p_4^T-p_2^T\\ 0 & 0 & p_3^T-p_4^T & p_4^T-p_3^T \end{bmatrix},$$

and $E(p)$ is given by

$$E(p)=\begin{bmatrix} e_{12}+e_{13}+e_{14} & -e_{12} & -e_{13} & -e_{14}\\ -e_{12} & e_{12}+e_{23}+e_{24} & -e_{23} & -e_{24}\\ -e_{13} & -e_{23} & e_{13}+e_{23}+e_{34} & -e_{34}\\ -e_{14} & -e_{24} & -e_{34} & e_{14}+e_{24}+e_{34} \end{bmatrix}.$$

Depending on the rank of the rigidity matrix $R(p)$, there could be nonzero $e(p)$ which makes the right side of (1b) become zero. Accordingly, we define the incorrect equilibrium set $\mathcal{P}_i$ by

$$\mathcal{P}_i=\left\{p\in\mathbb{R}^{3|\mathcal{V}|}\colon \frac{\partial V}{\partial p}=0,\ e\neq 0\right\}.$$

Note that $e(p)=0$ implies that the agents are at equilibrium under (1b) so the desired formation set $\mathcal{P}_d$ is an equilibrium set. Further, $\mathcal{P}_d$ and the incorrect equilibrium set $\mathcal{P}_i$ partition the set of all equilibria.

C. Local exponential stability of error system

Suppose that a realization p^* is in the desired formation set $\mathcal{P}_d$. Then any translation of $(\mathcal{G},p^*)$ in $\mathbb{R}^3$ produces another realization of which the inter-agent distances are the same as those of p^* so the new realization is also in $\mathcal{P}_d$. Thus, $\mathcal{P}_d$ is unbounded and non-compact, which makes the stability analysis somewhat complicated. If we transform the overall system into the error system, we obtain

$$\dot{e}=-2R(p)[R(p)]^T e.$$

Let $A=-2R(p)[R(p)]^T$. Then the equilibrium set of e corresponding to $\mathcal{P}_d$ is $\{e\colon e=0\}$ which is closed and bounded. Further, Sun and his colleagues show that each element of A can be expressed in terms of e, and prove that the origin of the error system

$$\dot{e} = A(e)e \tag{2}$$

is exponentially stable [12, Lemma 1]. Moreover, the exponential stability guarantees that p converges to $\mathcal{P}_d$.

In the proof of the exponential stability, an important assumption is that the desired formation shape has nonzero volume, which is equivalent in the case of a four agent formation to the assumption that the desired formation is infinitesimally rigid. The infinitesimal rigidity of the desired formation guarantees that $R(p)$ has full row rank near the desired formation from the following proposition.

Theorem 1 (Hendrickson (1992), [19]): A framework $(\mathcal{G}, p)$ with N vertices is infinitesimally rigid in $\mathbb{R}^n$ with $N \geq n$ if and only if $\text{rank} R(p) = Nn - n(n+1)/2$.

In our case, we have $N = 4$ and $n = 3$ so the rigidity matrix $R(p)$ has rank 6 with an infinitesimally rigid framework $(\mathcal{G}, p)$.

III. Analysis of the incorrect equilibrium points

From (1b), we know that there may be p^* such that $[R(p^*)]^T e(p^*) = 0$ but $e(p^*) \neq 0$ if $R(p^*)$ does not have full rank, i.e., $(\mathcal{G}, p^*)$ is not infinitesimally rigid. In our problem with four agent tetrahedral formation, those incorrect equilibria take place only when the agents are coplanar [13, Lemma 2]. Note that the stability of equilibria is not dependent on rotation and translation of the formation because only the relative displacements matter. Thus, without loss of generality, we analyze the stability of the incorrect equilibrium set under the assumption that the formations corresponding to the incorrect equilibria are in the x–y plane.

A. Linearization and the Hessian of the potential function

We are aiming at showing the instability of the incorrect equilibrium set. Accordingly, we linearize the system in (1a) with respect to p to investigate the behavior of the agents near the incorrect equilibrium set. Since the right side in (1a) is the negative gradient of the potential function V, the Jacobian $J_f(p)$ of the right side in (1a) is the same as the negative Hessian of V. Let $H_V(p)$ denote the Hessian of V. Then we have

$$H_V(p) = 2[R(p)]^T R(p) + E(p) \otimes I_3 = -J_f(p),$$

which is a trivial extension of the 2-dimensional case in [8]–[10]. Thus, if J_f has a positive eigenvalue at the incorrect equilibrium set, i.e., if H_V has a negative eigenvalue at an incorrect equilibrium, then we could conclude that the incorrect equilibrium is unstable.

To investigate the eigenvalues of H_V, we introduce a column-reordering transformation T such that

$$RT = [R_x \quad R_y \quad R_z] = \bar{R},$$

where $R_i \in \mathbb{R}^{6\times 4}$ is the matrix whose columns consist of the columns of R corresponding to coordinate i (see (3) at the top of the next page). The transformed Hessian matrix $\bar{H}$ is given by

$$\begin{aligned}\bar{H}_V(p) &= T^T H_V T \\ &= 2\bar{R}^T \bar{R} + I_3 \otimes E \\ &= 2\begin{bmatrix} R_x^T R_x + \frac{1}{2}E & R_x^T R_y & R_x^T R_z \\ R_y^T R_x & R_y^T R_y + \frac{1}{2}E & R_y^T R_z \\ R_z^T R_x & R_z^T R_y & R_z^T R_z + \frac{1}{2}E \end{bmatrix}.\end{aligned}$$

Since T is a permutation matrix which is orthogonal, the eigenvalues of H_V and $\bar{H}_V$ are same; thus we shall consider the eigenvalues of $\bar{H}_V$.

Before we go further, we verify that E is not positive semidefinite at an incorrect equilibrium.

Lemma 1: Let p^* be an incorrect equilibrium point of (1). Then $E(p^*)$ has at least one negative eigenvalue.

Proof: Suppose that all eigenvalues of $E(p^*)$ are nonnegative. Then for any vector $w \in \mathbb{R}^{3|\mathcal{V}|}$, it must be true that $w^T[E(p^*) \otimes I_3]w \geq 0$. Consider $w = \bar{p}$. Then we have

$$\bar{p}^T[E(p^*) \otimes I_3]\bar{p} = \sum_{(i,j)\in\mathcal{E}} e_{ij}(p^*)\|\bar{p}_i - \bar{p}_j\|^2.$$

Note that $[p^*]^T[E(p^*) \otimes I_3]p^* = 0$ because p^* is an equilibrium point. As a result, we have

$$\begin{aligned}\bar{p}^T[E(p^*) \otimes I_3]\bar{p} &= \bar{p}^T[E(p^*) \otimes I_3]\bar{p} - [p^*]^T[E(p^*) \otimes I_3]p^* \\ &= \sum_{(i,j)\in\mathcal{E}} e_{ij}(p^*)\left[\|\bar{p}_i - \bar{p}_j\|^2 - \|p_i^* - p_j^*\|^2\right] \\ &= -\sum_{(i,j)\in\mathcal{E}} [e_{ij}(p^*)]^2 < 0, \end{aligned} \tag{4}$$

which contradicts to the assumption that $E(p^*)$ is positive semidefinite. Note that the strict inequality in (4) holds because p^* is an incorrect equilibrium point resulting in nonzero distance errors. ■

B. Instability of incorrect equilibrium set

Now, using the transformed Hessian $\bar{H}_V$ and Lemma 1, we can formulate the following proposition.

Theorem 2: Any incorrect equilibrium point p^* of (1) is unstable.

Proof: We know that any incorrect equilibrium formation is coplanar, and the stability is not dependent on rotation and translation of the formation. Suppose that the formation defined by p^* is in the x–y plane, i.e., $z_i^* = 0$ for all $i \in \mathcal{V}$. Then we have

$$\bar{H}_V(p^*) = 2\begin{bmatrix} R_x^T R_x + \frac{1}{2}E(p^*) & R_x^T R_y & 0 \\ R_y^T R_x & R_y^T R_y + \frac{1}{2}E(p^*) & 0 \\ 0 & 0 & \frac{1}{2}E(p^*) \end{bmatrix}.$$

Since $E(p^*)$ has a negative eigenvalue from Lemma 1, $\bar{H}_V(p^*)$ also has a negative eigenvalue, and so $H_V(p^*)$ does. Therefore, the incorrect equilibrium point is unstable. ■

Note that the above argument, postulating a coplanar incorrect equilibirium, actually subsumes collinear and collocated equilibria (all agents at the same point) as special cases. The Hessian corresponding to a collocated equilibrium has no positive eigenvalues, so the equilibrium is not a saddle.

$$\bar{R} = \left[\begin{array}{cccc|cccc|cccc} x_1-x_2 & x_2-x_1 & 0 & 0 & y_1-y_2 & y_2-y_1 & 0 & 0 & z_1-z_2 & z_2-z_1 & 0 & 0 \\ x_1-x_3 & 0 & x_3-x_1 & 0 & y_1-y_3 & 0 & y_3-y_1 & 0 & z_1-z_3 & 0 & z_3-z_1 & 0 \\ x_1-x_4 & 0 & 0 & x_4-x_1 & y_1-y_4 & 0 & 0 & y_4-y_1 & z_1-z_4 & 0 & 0 & z_4-z_1 \\ 0 & x_2-x_3 & x_3-x_2 & 0 & 0 & y_2-y_3 & y_3-y_2 & 0 & 0 & z_2-z_3 & z_3-z_2 & 0 \\ 0 & x_2-x_4 & 0 & x_4-x_2 & 0 & y_2-y_4 & 0 & y_4-y_2 & 0 & z_2-z_4 & 0 & z_4-z_2 \\ 0 & 0 & x_3-x_4 & x_4-x_3 & 0 & 0 & y_3-y_4 & y_4-y_3 & 0 & 0 & z_3-z_4 & z_4-z_3 \end{array}\right]. \quad (3)$$

Theorem 3: For almost every initial condition in $\mathbb{R}^{3|\mathcal{V}|}$, the trajectory $p(t)$ of (1) converges to the desired equilibrium set $\mathcal{P}_d$.

Proof: By taking the derivative of V, we have

$$\dot{V} = \frac{\partial V}{\partial p}\dot{p} = -\left\|\frac{\partial V}{\partial p}\right\|^2 \leq 0,$$

which results in that r_{ij} and e_{ij} are bounded for all $i, j \in \mathcal{V}$. From the boundedness of r_{ij} and e_{ij}, we can also show that $\ddot{V}$ is bounded so $\dot{V}(p(t))$ is uniformly continuous in t on $[t_0,\infty)$ with an initial time t_0. Since $V(p(t))$ is a non-increasing lower bounded function, the limit of $V(p(t))$ exists. Therefore, $\dot{V}(p(t))$ converges to 0 as $t \to \infty$ from Barbalat's lemma [20, Lemma 8.2], which means that $p(t)$ approaches either $\mathcal{P}_d$ or $\mathcal{P}_i$. The local asymptotic stability of $\mathcal{P}_d$ is guaranteed from the local exponential stability of the origin of (2) [12], or direct results of Oh and Ahn [11]. Besides, the instability of the incorrect equilibrium set $\mathcal{P}_i$ has been proved in Theorem 2. Therefore, for almost every initial condition in $\mathbb{R}^{3|\mathcal{V}|}$, $p(t)$ approaches $\mathcal{P}_d$. Moreover, from the exponential stability of $\{e : e = 0\}$ we can conclude the right side of (1b) converges exponentially fast to 0 because $R(p)$ consists of r_{ij} which is bounded. As a consequence, the trajectory $p(t)$ converges to $\mathcal{P}_d$. ■

C. Repulsiveness of incorrect equilibrium formations

Consider a formation at an initial time, prior to application of the control law, such that the four agents are all coplanar. Thus, the set of all affine combinations of $p_1(t_0)$, $p_2(t_0)$, $p_3(t_0)$, and $p_4(t_0)$ define a two-dimensional affine subspace of $\mathbb{R}^3$. Since, for each $i \in \mathcal{V}$, $\dot{p}_i$ is a linear combination of r_{ij} (refer to (1b)), every agent moves on the plane defined by the positions at t_0 of the agents. Thus, the set of all realizations of the coplanar agents is positively invariant. We are going to investigate the property of the positively invariant set.

Let $Z(p) = [r_{12} \;\; r_{13} \;\; r_{14}] \in \mathbb{R}^{3\times 3}$ and $\Delta(p) = \det Z$. Then the set of all realizations of coplanar agents is defined by

$$\mathcal{C} = \left\{p \in \mathbb{R}^{3|\mathcal{V}|} :\ \operatorname{rank} Z(p) < 3\right\}.$$

Note that the absolute value of Δ is twice the volume of the formation in $\mathbb{R}^3$, and $\Delta(p) = 0$ for any p in $\mathcal{C}$. In addition, the incorrect equilibrium set $\mathcal{P}_i$ is a subset of $\mathcal{C}$. By taking the time derivative of Δ, we have

$$\begin{aligned}\dot{\Delta} &= \det[\dot{r}_{12} \;\; r_{13} \;\; r_{14}] + \det[r_{12} \;\; \dot{r}_{13} \;\; r_{14}] + \det[r_{12} \;\; r_{13} \;\; \dot{r}_{14}] \\ &= -\operatorname{trace}(E)\Delta. \end{aligned} \quad (5)$$

From (5), Δ is expressed by

$$\Delta(p(t)) = \exp\left[-\int_{t_0}^{t} \operatorname{trace}[E(p(s))]\mathrm{d}s\right]\Delta(p(t_0)). \quad (6)$$

Suppose that $p(t_0) \in \mathcal{C}$ at initial time t_0. Then we know that $p(t) \in \mathcal{C}$ for each $t \in [t_0,\infty)$. Now, to explore what happens if $p(t_0) \notin \mathcal{C}$, we first partition $\mathcal{C}$ into three subsets as follows;

$$\mathcal{C}_i = \{p \in \mathcal{C} :\ \operatorname{rank} Z(p) = i\}, \quad i \in \{0,1,2\}.$$

Lemma 2: Let p^* be an incorrect equilibrium point, and assume that the corresponding agents are coplanar but not collinear, i.e., $p^* \in \mathcal{P}_i \cap \mathcal{C}_2$. Then $E(p^*)$ has rank 1 and is negative semidefinite.

Proof: Since p^* is an equilibrium point, $(E(p^*) \otimes I_3)p^* = 0$. By rebuilding $(E(p^*) \otimes I_3)p^* = 0$, we have $E(p^*)M = 0$, where

$$M = \begin{bmatrix} 1 & x_1^* & y_1^* & z_1^* \\ 1 & x_2^* & y_2^* & z_2^* \\ 1 & x_3^* & y_3^* & z_3^* \\ 1 & x_4^* & y_4^* & z_4^* \end{bmatrix}. \quad (7)$$

From the fact that $\operatorname{rank} Z(p^*) = 2$, M has rank 3 so $E(p^*)$ has rank less than or equal to 1. Since $E(p^*)$ is not a zero matrix, the rank cannot be zero, which means that the rank of $E(p^*)$ is 1. As a consequence, $E(p^*)$ is negative semidefinite by a negative eigenvalue (see Lemma 1). ■

From Lemma 2, it is obvious that $\operatorname{trace} E(p^*) < 0$ for each $p^* \in \mathcal{P}_i \cap \mathcal{C}_2$. In addition to the case where $\operatorname{rank} Z = 2$, if $\operatorname{rank} Z = 0$, i.e., if all agents are collocated, then we have $\operatorname{trace} E(p^*) = -2\sum_{(i,j)\in\mathcal{E}} d_{ij}^2 < 0$ for any $p^* \in \mathcal{P}_i \cap \mathcal{C}_0$.

Lemma 3: Suppose that an initial point $p(t_0)$ is not in $\mathcal{C}$. Then any solution trajectories of (1) do not approach $\mathcal{P}_i \cap (\mathcal{C}_0 \cup \mathcal{C}_2)$.

Proof: Suppose that $p(t)$ approaches $\mathcal{P}_i \cap (\mathcal{C}_0 \cup \mathcal{C}_2)$ as $t \to \infty$ even if $p(t_0)$ is not in $\mathcal{C}$. Since $\mathcal{P}_i \subset \mathcal{C}$, it must be true that $\lim_{t\to\infty} \Delta(p(t)) = 0$. Let us define a subset $\mathcal{T}$ such that

$$\mathcal{T} = \left\{p \in \mathbb{R}^{3|\mathcal{V}|} :\ \operatorname{trace} E(p) < 0\right\},$$
$$\mathcal{T} \supsetneq \mathcal{P}_i \cap (\mathcal{C}_0 \cup \mathcal{C}_2), \quad \mathcal{T} \cap \mathcal{P}_d = \emptyset.$$

Then, there exists a finite time t_f such that $p(t_f) \notin \mathcal{P}_i \cap (\mathcal{C}_0 \cup \mathcal{C}_2)$, and $p(t) \in \mathcal{T}$ for each $t \in [t_f,\infty)$. From (6), we have

$$\begin{aligned}\Delta(p(t)) &= \exp\left[-\int_{t_f}^{t} \operatorname{trace}[E(p(s))]\mathrm{d}s\right] \\ &\quad \times \exp\left[-\int_{t_0}^{t_f} \operatorname{trace}[E(p(s))]\mathrm{d}s\right]\Delta(p(t_0)), \quad t \geq t_f \geq t_0. \end{aligned}$$

However, $\exp\left[-\int_{t_0}^{t_f} \operatorname{trace}[E(p(s))]\mathrm{d}s\right]\Delta(p(t_0))$ is a nonzero constant, and $\exp\left[-\int_{t_f}^{t} \operatorname{trace}[E(p(s))]\mathrm{d}s\right]$ is greater than or equal to 1 because $\operatorname{trace}[E(p(t))] < 0$ for all $t \in [t_f,\infty)$. Thus we have a contradiction, and $p(t)$ would not approach $\mathcal{P}_i \cap (\mathcal{C}_0 \cup \mathcal{C}_2)$ with $p(t_0) \notin \mathcal{C}$. ■

Lemma 3 means that for all initial conditions forming non-coplanar formations, the corresponding solution trajectories of (1) will not approach $\mathcal{P}_i \cap (\mathcal{C}_0 \cup \mathcal{C}_2)$. Now we focus on a possibility that there exists a solution trajectory approaching $\mathcal{P}_i \cap \mathcal{C}_1$ with a particular initial condition that forms a non-coplanar formation.

Without loss of generality, assume that an incorrect equilibrium formation of $p^* \in \mathcal{P}_i \cap \mathcal{C}_1$ is in the x–axis. Hence, we have $p_i^* = [x_i^* \quad 0 \quad 0]^T$ except for the cases such that $x_1^* = x_2^* = x_3^* = x_4^*$. In this case, M in (7) has rank 2 so $E(p^*)$ has rank at most 2. Although $E(p^*)$ has at least one negative eigenvalue from Lemma 1, we have yet to guarantee that the other nonzero eigenvalue (provided that $\operatorname{rank} E(p^*) = 2$) is non-positive. If $E(p^*)$ has such a positive eigenvalue, then we cannot guarantee that $\operatorname{trace} E(p^*) < 0$ so the analysis on $\Delta(p(t))$ used in the proof of Lemma 3 cannot be applied to the case of $\mathcal{P}_i \cap \mathcal{C}_1$. To show that any non-coplanar initial condition does not result in a solution approaching $\mathcal{P}_i \cap \mathcal{C}_1$, we examine the behavior of $p(t)$ near p^*. We have the linearized system at p^* by

$$\begin{aligned}\dot{p} &= -H_V(p^*)p \\ &= -T\bar{H}_V(p^*)T^T p,\end{aligned}$$

where the transformed Hessian $\bar{H}_V(p^*)$ is given by

$$\bar{H}_V(p^*) = \begin{bmatrix} 2R_x^T R_x + E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & E \end{bmatrix}.$$

Lemma 4: Let $p^* \in \mathcal{P}_i \cap \mathcal{C}_1$. Near p^*, any p which is not in $\mathcal{C}$ cannot lie on a trajectory for which p^* is attractive.

Proof: Suppose that $E(p^*)$ has a positive eigenvalue λ_+ with relevant eigenvector $s = [s_1 \quad s_2 \quad s_3 \quad s_4]^T$ of unit length. Then λ_+ is also an eigenvalue of $H_V(p^*)$ so there are two relevant eigenvectors of $H_V(p^*)$, which are given by

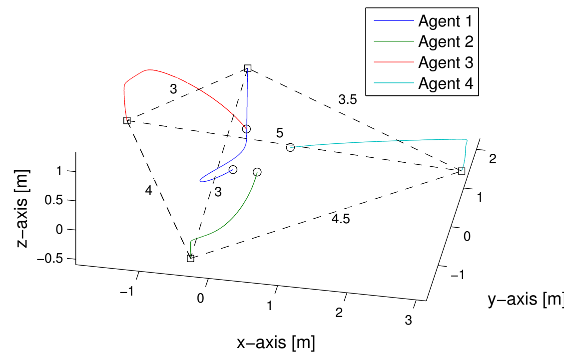
$$\begin{aligned} h_y &= [0 \quad s_1 \quad 0 \quad 0 \quad s_2 \quad 0 \quad 0 \quad s_3 \quad 0 \quad 0 \quad s_4 \quad 0]^T, \\ h_z &= [0 \quad 0 \quad s_1 \quad 0 \quad 0 \quad s_2 \quad 0 \quad 0 \quad s_3 \quad 0 \quad 0 \quad s_4]^T. \end{aligned}$$

Thus, a point p in a subset in which the solution trajectory converges to p^* can be expressed by $p = p^* + \varepsilon_y h_y + \varepsilon_z h_z$ with sufficiently small nonzero coefficients ε_y and ε_z.[2] Then we have $p_1 = [x_1^* \quad \varepsilon_y s_1 \quad \varepsilon_z s_1]^T$, $p_2 = [x_2^* \quad \varepsilon_y s_2 \quad \varepsilon_z s_2]^T$, $p_3 = [x_3^* \quad \varepsilon_y s_3 \quad \varepsilon_z s_3]^T$, and $p_4 = [x_4^* \quad \varepsilon_y s_4 \quad \varepsilon_z s_4]^T$. Obviously, all of them are in a plane which is orthogonal to the space spanned by $[0 \quad -\varepsilon_z \quad \varepsilon_y]^T$. Therefore, p must be in $\mathcal{C}$. ■

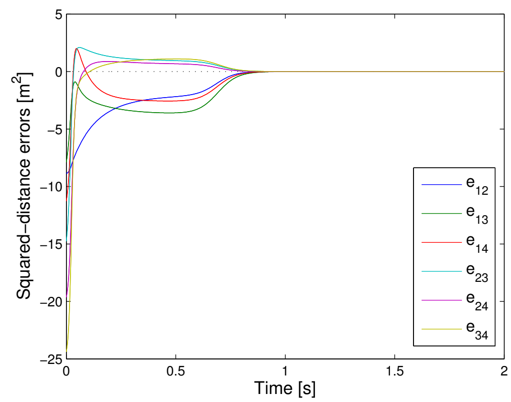
Corollary 1: The region of attraction for the desired equilibrium set $\mathcal{P}_d$ is $\mathbb{R}^{3|\mathcal{V}|} \setminus \mathcal{C}$.

Proof: We already know that any solution trajectory of (1) finally approaches either $\mathcal{P}_d$ or $\mathcal{P}_i$. From Lemma 3 and Lemma 4, any initial condition not in $\mathcal{C}$ does not produce the trajectory approaching the incorrect equilibrium set, and $\mathcal{C}$ is positively invariant. Therefore, the region of attraction for $\mathcal{P}_d$ is $\mathbb{R}^{3|\mathcal{V}|} \setminus \mathcal{C}$. ■

[2]One may think that we should consider also the eigenvectors relevant to the positive eigenvalues from 1–1 block of $\bar{H}_V(p^*)$, but those eigenvectors would not affect the conclusion because they contribute only to the x–components of p_i.



(a) The initial and the final locations of the agents are denoted by circles and squares, respectively.



(b) Corresponding squared-distance errors

Fig. 2. Simulation on four agent tetrahedral formation

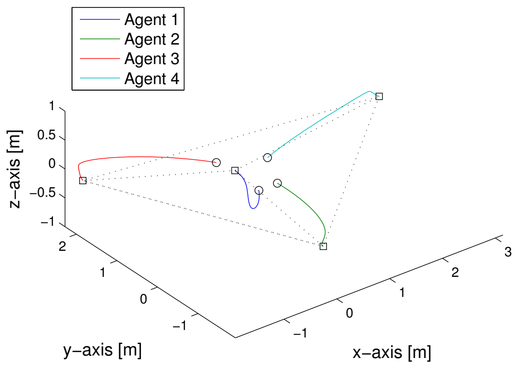
IV. Simulation

In this section, we provide a simulation result which shows the behavior of four mobile agents. The desired inter-agent distances are specified as $\bar{d}_{12} = 3$, $\bar{d}_{13} = 3$, $\bar{d}_{14} = 3.5$, $\bar{d}_{23} = 4$, $\bar{d}_{24} = 4.5$, and $\bar{d}_{34} = 5$. To observe the extreme behavior of the agents near $\mathcal{C}$, the initial conditions are set as $p_1(0) = [0 \quad 0 \quad 0]^T$, $p_2(0) = [0.5 \quad 0 \quad 0]^T$, $p_3(0) = [0 \quad 1.5 \quad 0]^T$, and $p_4(0) = [1 \quad 1 \quad 0.001]^T$. Thus, the trajectory starts out near x–y plane. In Fig. 2, we see that the squared-distance errors finally converge to 0 as time goes on, and the agents achieve the prescribed tetrahedral formation shape.

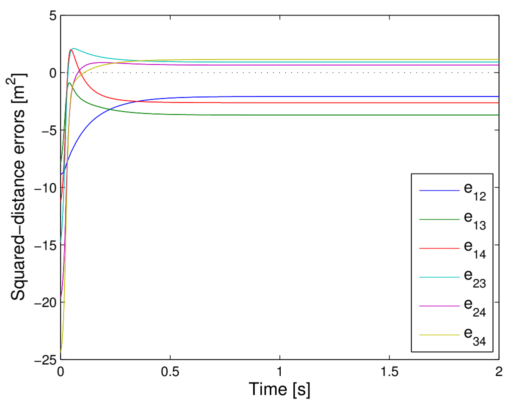
We carried out another simulation with the same desired distance set, while $p_4(0)$ is replaced by $[1 \quad 1 \quad 0]^T$. Thus, the initial formation exists on the x–y plane and has zero volume. Then the agents form a coplanar formation ever after, and the squared-distance errors never converge to 0 (see Fig. 3).

V. Conclusion

In this paper, we explored the four agent tetrahedral formation in 3-dimensional space. We used the gradient control law to maneuver the mobile agents so that they achieve the desired formation shape by controlling the inter-agent distances. Although, under the proposed control law, there



(a) The initial and the final locations of the agents are denoted by circles and squares, respectively.



(b) Corresponding squared-distance errors

Fig. 3. Simulation on four agent tetrahedral formation

exists an incorrect equilibrium set in which the agents do not achieve the desired inter-agent distances, it is proved that the incorrect equilibrium set is unstable. Further, by investigating the behavior of the agents near the incorrect equilibrium set, we characterized the exact region of attraction for the desired equilibrium set corresponding to the desired formation shape. As a result, we conclude that for any initial condition forming a tetrahedral formation with nonzero volume, we can achieve the desired formation shape under the proposed control law. Starting from this four agent problem, we hope to explore more complicated formations than the tetrahedral formation.

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