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Geometrical Methods for Mismatched Formation Control

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***Abstract*— Formation shape control for a collection of point agents is concerned with devising decentralized control laws which will move the formation so that certain inter-agent distances reach prescribed desired values. Standard algorithms such as that proposed by [1] perform steepest descent of a smooth error function, ensuring that the formations will always converge to equilibrium points for the gradient flow. The convergence to equilibrium points of these algorithms depends critically on the fact that there is no mismatch in two neighboring agents' understandings of what the desired distance between them is supposed to be. If mismatches occur then the limiting dynamics will typically become periodic, as has been explored in several recent papers such as, e.g., [2]–[5]. The goal then becomes to develop methods to count such relative equilibria and characterize their local stability properties. In this paper we apply basic Lie group methods to analyze the relative equilibria in the presence of mismatches, thus simplifying earlier proofs in the literature.**

I. Introduction

Formations of specified shape may be useful for sensing and localizing objects, and formations of fixed shape can be contemplated for moving massive objects placed upon them. One problem of interest therefore is *formation shape control*. This is the task of specifying *distributed* control laws, normally using notions from graph theory, for the individual agents so that the formation takes up a prescribed shape, with the center of gravity and rotational orientation considered irrelevant. Thus assume a finite, undirected graph $\Gamma = (\mathscr{V},\mathscr{E})$ is given. Then each agent defines a vector $p_i \in \mathbb{R}^d$ that is associated with the i-th vertex element $i \in \mathscr{V}$. Similarly, edges in the graph correspond to those pairs of agents between which there is a distance specification. Thus $\mathscr{E}$ denotes the set of edges (i,j) in the underlying graph between which a desired distance d_{ij} is specified. Distributed laws are those which use, at any one agent, measurements just from neighboring agents. Literature on formation shape control via distances and graph concepts includes important early work by Olfati-Saber and Murray [6], work of Francis and coworkers, e.g. [1], [7] as well as [8]–[11].

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Almost all the presently known algorithms for formation control use steepest descent methods for optimizing a suitable cost function. Typically, the cost function on the space of all formations of N points $p_1,\dots,p_N$ in $\mathbb{R}^d$ is used, given as

$$V(p) = \frac{1}{4}\sum_{(i,j)\in\mathscr{E}} (\|p_i - p_j\|^2 - d_{ij}^2)^2. \tag{1}$$

An important feature of the cost function (1), as well as of most potential functions used for formation control, is the invariance with respect to arbitrary translations and rotations of the formation. On the basis of moving V along a steepest descent trajectory, one has the gradient descent flow

$$\dot{p} = -\nabla V(p)$$

on $\mathbb{R}^{dN}$ where

$$p = \mathrm{col}(p_1,\dots,p_N) := \begin{bmatrix} p_1 \\ \vdots \\ p_N \end{bmatrix} \tag{2}$$

It follows that

$$\dot{p}_i = -\sum_{j\in\mathscr{N}_i} (\|p_i - p_j\|^2 - d_{ij}^2)(p_i - p_j), \quad i = 1,\dots,N. \tag{3}$$

where $\mathscr{N}_i$ denotes the set of agent i's neighbors. Note that (3) is a decentralized control law since each $\dot{p}_i$ involves the relative positions of its neighbors and desired distances to its neighbors, but not more. The dynamics of such gradient descent algorithms is well studied and due to the real analytic nature of the gradient flows it can be shown that all solutions converge to single equilibrium points; see e.g. [12]. Comparatively little information is available on the number of such critical formations, although a first analysis of this has been done in [13].

In this paper we explore what happens with the dynamics of the system if there is a bias in the measurements of mutual interagent distances. This is a standard problem if, e.g., all agents are equipped with an individual sensor that measures the relative distances. It is then quite realistic to assume that a measurement bias occurs in each of the sensors of the mutual agents, thus leads to an effective mismatch in the interagent distances. We refer to [2] for first investigations and early results on this problem which is sometimes also referred to as the robustness problem for formation control gradient laws. Introducing mismatches due to measurement errors leads to the perturbed dynamical system in the form of

$$\dot{p} = -\nabla V(p) - S^\top \mu \tag{4}$$

where S and μ will be defined later. The nice feature of this flow is that it is invariant under rotations and translations, although the inclusion of the second term $S^\top \mu$ drastically changes the dynamics. In fact, as has been shown in [3]–[5], both theoretically and supported by simulations, the trajectories of (4) generically approach asymptotically periodic behavior. The main goal of this paper is to apply basic Lie group methods to analyze the relative equilibria in the presence of mismatches, thereby simplifying earlier proofs in [2]–[5]. Our results are almost all independent of the dimension of the underlying ambient space.

The rest of the paper is organized as follows. Section II casts the translation and rotation invariance property of a formation in a geometric framework, and recalls the definition of rigidity in such a framework. In Section III, we explain the mismatched formation control flow problem. The new material appears in Sections IV and V, where the notion of relative equilibria is explained (which relates to an equilibrium property of an orbit) and the associated error equations are studied using this concept. This leads to the main convergence theorem of Section VI. Brief concluding remarks appear in Section VII.

II. Geometry of formation spaces

In order to analyze the critical points of costs functions of interest in formation control, it becomes important to describe the underlying geometry of the problem. The manifolds that appear here are best described as quotient spaces of Euclidean space with respect to the Euclidean group. We begin with some basic terminology. Throughout this paper, formations are thought of evolving in ambient d-dimensional standard Euclidean space $\mathbb{R}^d$, endowed with the usual Euclidean inner product. Let $O(d)$ denote the compact matrix Lie group of all real orthogonal $d \times d$-matrices. The Euclidean group $E(d)$ then parameterizes all Euclidean group transformations of the form $p \mapsto gp + v$, where $g \in O(d)$ and $v \in \mathbb{R}^d$ denotes an arbitrary translation vector. Thus $E(d)$ is a Lie group of dimension $\frac{d(d+1)}{2}$, which is in fact a semidirect product of $O(d)$ and $\mathbb{R}^d$.

Since formations of N mobile agents are best described in terms of graph theory, we give a brief descriptions of some of the basic facts and definitions needed. Let $\Gamma = (\mathscr{V}, \mathscr{E})$ denote any finite, connected and undirected graph with vertex set $\mathscr{V} = \{1, \ldots, N\}$ and set of edges $\mathscr{E} = \{1, \ldots, M\}$ edges. Fix any orientation on the graph so that any edge $k \in \mathscr{E}$ has a well-defined head j and tail i, respectively. We then have the *oriented incidence matrix* $B = (b_{ik}) \in \mathbb{R}^{N \times M}$ which is defined as

$$b_{ik} = \begin{cases} -1 & \text{if} \quad i \text{ is the tail of the } k\text{th edge} \\ 1 & \text{if} \quad i \text{ is the head of the } k\text{th edge} \\ 0 & \quad \text{otherwise} \end{cases}$$

Its columns are labelled by the M undirected edges $(i, j) \in \mathscr{E}$ of Γ. Since the graph Γ is assumed to be connected, the incidence matrix B is known to have rank $N-1$ and the kernel of B has dimension $M - N + 1$. Likewise, we define a *head incidence matrix* $H = (h_{ik}) \in \mathbb{R}^{N \times M}$ as

$$h_{ik} = \begin{cases} 1 & \text{if} \quad i \text{ is the head of the } k\text{th edge} \\ 0 & \quad \text{otherwise} \end{cases}$$

Thus $\mathbf{1}^\top H = \mathbf{1}^\top$, where $\mathbf{1}$ denotes the vector with all entries equal to 1.

Given a vertex element $i \in \mathscr{V}$ we associate to it a point p_i of Euclidean space $\mathbb{R}^d$. The column vector $p = \mathrm{col}(p_1, \ldots, p_N)$ thus describes a *formation* (p, Γ) of N points, labelled by the set of vertices of Γ. For any edge $k \in \mathscr{E}$ with head j and tail i consider the associated edge vector defined as $z_k = p_j - p_i$. Let

$$\begin{aligned} z &= \mathrm{col}(z_1, \cdots, z_M) \in \mathbb{R}^{dM} \\ D(z) &= \mathrm{diag}\,(z_1, \cdots, z_M) \in \mathbb{R}^{dM \times M} \end{aligned}$$

denote the associated column vector and block diagonal matrix, respectively. With this notation at hand, we consider the smooth distance map

$$\mathscr{D} : \mathbb{R}^{dN} \longrightarrow \mathbb{R}^M, \mathscr{D}(p) = (\|p_i - p_j\|^2)_{(i,j) \in \mathscr{E}} = D(z)^\top z. \quad (5)$$

The **rigidity matrix** then is defined as the Jacobi matrix $R(z) = Jac_{\mathscr{D}}(p)$. By inspection, $R(z)$ is the $M \times dN$ matrix given as

$$R(z) = D(z)^\top (B^\top \otimes I_d). \quad (6)$$

There is some well known a priori information on the kernel of the rigidity matrix. In fact, since $\mathscr{D}$ is invariant under Euclidean group transformations

$$\mathscr{D}(gp_1 + v, \cdots, gp_N + v) = \mathscr{D}(p_1, \cdots, p_N),$$

the tangent space to such a group orbit is always contained in the kernel of the rigidity matrix $R(z)$. Since the Euclidean group $E(d)$ acts freely on all formations of points $p_1, \cdots, p_N$ whose affine span equals $\mathbb{R}^d$, we conclude

Lemma 1: The kernel of the rigidity map $R(z)$ always contains the tangent space $T_p(E(d) \cdot p)$. Assume that the affine span of $p_1, \ldots, p_N$ has dimension $r \leq d$. Then the kernel of $R(z)$ has at least dimension $\frac{(r+1)(2d-r)}{2}$.

Proof. The first statement is an immediate consequence of the invariance of $\mathscr{D}$ under the group of Euclidean transformations $(p_1, \ldots, p_N) \mapsto (gp_1 + v, \ldots, gp_N + v)$. Note that the dimension of the stabilizer group $Stab_{E(d)}(p)$ coincides with the subgroup of $E(d)$ that leave the elements of the affine span $< p_1, \ldots, p_N >$ pointwise invariant. Thus a straightforward computation reveals that the dimension of $Stab_{E(d)}(p)$ is equal to $\frac{(d-r)(d-r-1)}{2}$. Therefore the dimension of the group orbit $E(d) \cdot p$ is equal to $\frac{d(d+1)}{2} - \frac{(d-r)(d-r-1)}{2} = \frac{(r+1)(2d-r)}{2}$. This completes the proof. ■

A formation is called **infinitesimally rigid** if the kernel of the rigidity matrix coincides with the tangent space $T_p(E(d) \cdot p)$. Equivalently, infinitesimal rigidity holds if and if the following rank condition is satisfied

$$\mathrm{rk}\, R(z) = d(N-1) - \frac{1}{2} r(2d - r - 1),$$

where r is the affine dimension spanned by $p_1,\dots,p_N$. Note that from the structure of $R(z)$, one can easily check that for $r=1$ and any d we have $\mathrm{rk}R(z)\le N-1$. Likewise one can check that for $r=2$ and any d with $M\ge 2N-3$ there holds $\mathrm{rk}R(z)\le 2N-3$. With the aid of these bounds, one can then verify using the rank condition that a formation of N points in the plane $\mathbb{R}^2$ is infinitesimally rigid if and only if $r=2$ and the rank of $R(z)$ is equal to $2N-3$. Similarly, a formation of $N\ge 4$ points in $\mathbb{R}^3$ is infinitesimally rigid if and only if $r=3$ and the rank of $R(z)$ is equal to $3N-6$. A formation p is called **rigid** (as opposed to infinitesimally rigid) whenever the orbit $E(d)\cdot p$ is isolated in the fibre $\mathscr{D}^{-1}(\mathscr{D}(p))$. Any infinitesimally rigid formation is rigid, but the converse does not hold. A **rigid graph** in $\mathbb{R}^d$ is one for which almost every $p\in\mathbb{R}^{dN}$ is infinitesimally rigid. Thus Γ is rigid in $\mathbb{R}^d$ if and only if the rigidity matrix $R(z)$ has generic rank equal to $dN-\frac{d(d+1)}{2}$. A rigid graph is called **minimally rigid** if has exactly $dN-\frac{d(d+1)}{2}$ edges. An example of a rigid graph is the complete graph K_N that has an edge between any pair of the N vertices. K_N is minimally rigid if and only if $N=2,3$, but not for $N\ge 4$. In contrast, the graph with 4 vertices and 5 edges realized in the 2-D ambient space is minimally rigid.

III. The Mismatched Formation Control Flow

We now consider the dynamical system on the space of agents that represents our formation control law in the presence of mismatched uncertainties in the desired distances. Thus let $d_1>0,\dots,d_M>0$ denote desired distances that we wish to achieve by following the dynamics of the flow. As mentioned in the introduction, the system

$$\dot p_i=-\sum_{j\in\mathcal{N}_i}(\|p_i-p_j\|^2-d_{ij}^2)(p_i-p_j),\quad i=1,\dots,N. \tag{7}$$

defines the steepest descent gradient flow of the potential function

$$V(p)=\frac{1}{4}\sum_{(i,j)\in\mathscr{E}}(\|p_i-p_j\|^2-d_{ij}^2)^2.$$

This cost function and associated gradient flow has been extensively studied in the literature; see e.g. [1], [7]–[9], [13], [14].

It seems natural to study the robustness properties of this gradient flow under perturbations since two neighboring agents may have different understandings of what the target distance between them is supposed to be. We refer to this as **distance mismatch**, which may arise for several reasons: First no two positioning controls can be expected to move agents precisely to specified positions because of inevitable imprecision in the physical comparators used to compute the positioning errors. Second, different agents may obtain different readings even when they are measuring the same distance due to a steady state bias of its onboard sensor.

For the kth edge in the oriented Γ with i as the tail and j as the head, we let

$$z_k=p_j-p_i,\qquad e_k=||z_k||^2-d_k^2$$

and suppose the tail agent i knows the desired distance d_k, that is,

$$d_{ij}=d_k \tag{8}$$

and the head agent j knows d_{ji} such that

$$d_{ji}^2=d_k^2-\mu_k \tag{9}$$

where $\mu_k\neq 0$ denotes the distance mismatch. Then (7) becomes

$$\dot p_i=-\sum_{k\in\mathscr{E}_i^+}z_k(e_k+\mu_k)+\sum_{k\in\mathscr{E}_i^-}z_ke_k \tag{10}$$

where $\mathscr{E}_i^+,\mathscr{E}_i^-$ denote the set of labels of all edges with i as the head vertex and the tail vertex, respectively. One might argue, that assuming a bias of measurements for both agents as $d_{ij}^2=d_k^2-\nu_k,\quad d_{ji}^2=d_k^2-\mu_k$ is more natural. However, using $\bar d_{ij}^2:=d_{ij}^2+\nu_k,\bar d_{ji}=d_{ji}$ we can transform the second case equivalently into (8, 9). Thus, without loss of generality we consider the scenario described by (8, 9). In order to rewrite this system in more compact form we use the following notation.

$$\begin{aligned}
R(z)&=D(z)^\top B^\top\otimes I_d\\
S(z)&=D(z)^\top H^\top\otimes I_d\\
e(z)&=\mathrm{col}(e_1(z),\dots,e_M(z))\\
\mu&=\mathrm{col}(\mu_1,\dots,\mu_M)\\
d^*&=\mathrm{col}(d_1,\dots,d_M).
\end{aligned}$$

We then obtain the equivalent form to (10) as

$$\dot p=-R^\top(z)e(z)-S^\top(z)\mu. \tag{11}$$

We refer to this system as the **mismatched gradient dynamics**, although it does not actually define a gradient flow. We note that this equation consists of two terms. The first term $R^\top(z)e(z)$ is the gradient of V and is therefore always orthogonal to the tangent space at the $E(d)$-orbit of p. In contrast, the second summand $S^\top(z)\mu$ is in general not orthogonal to $T_p(E(d)\cdot p)$ and will contain a tangent component to $E(d)\cdot p$. Let

$$\bar p=\frac{1}{N}(p_1+\cdots+p_N)=\frac{1}{N}\mathbf{1}^\top p$$

denote the center of mass. Using (9), the **center of mass dynamics** is described as

$$\dot{\bar p}=-\frac{1}{N}\sum_{k=1}^{M}\mu_kz_k. \tag{12}$$

and the entries of $e(z)$ do not explicitly appear.

In the sequel we will also consider the induced flows on the edge vectors z and error vectors e, respectively. Their dynamics are given as follows. Since $z=(B^\top\otimes I_d)p$ holds we obtain the dynamics for z expressed as

$$\dot z=-(B^\top\otimes I_d)(R^\top(z)e(p)+S^\top(z)\mu). \tag{13}$$

Similarly, using $e=D(z)^\top z$ we find the error dynamics given as

$$\dot e=-2(R(z)R^\top(z)e(p)+R(z)S^\top(z)\mu). \tag{14}$$

Note that in contrast with (11) these flows do not necessarily define autonomous differential equations in z, e respectively (see however below for an analysis of the e equation). Before passing on to a more detailed analysis we observe that the system (11) is invariant under the action of the Euclidean group on p. This simple observation will play a crucial role in our subsequent analysis. (11) has a further invariance property that concerns the collinearity properties of the agents. We next show that property.

***Proposition** 1:* Let $r(p) \leq d$ denote the dimension of the subspace spanned by $p_1, \ldots, p_N \in \mathbb{R}^d$. Then $r(p(t))$ and $r(z(t))$ are constant along any solution $p(t)$ of (11).

Proof. Let $P = (p_1, \ldots, p_N)$ denote the $d \times N$ matrix whose columns are defined by $p_1, \ldots, p_N$. Then $r(p) = \mathrm{rk} P$. By inspection, the mismatched flow is given as

$$\dot{p} = -\left((BD(e)B^\top + HD(\mu)B^\top) \otimes I_d\right) p$$

and therefore is equivalent to the flow on the matrix space $\mathbb{R}^{d\times N}$

$$\dot{P} = -P(BD(e)B^\top + BD(\mu)H^\top). \qquad (15)$$

Thus the matrix differential equation (15) is of the form $\dot{P} = PA(P)$, with a smooth matrix-valued function $A(P)$ of the variable P. It is shown in [15] that any matrix flow of the form $\dot{P} = PA(P)$ is rank preserving. This implies therefore that the matrix differential equation (15) is rank preserving. Similarly, we obtain for $Z(t) = (z_1(t), \ldots, z_M(t))$ that

$$\dot{Z} = -Z(D(e)B^\top B + D(\mu)H^\top B), \qquad (16)$$

which is again rank preserving. This proves the result. ■

IV. Relative Equilibria

Following a well established tradition in geometric mechanics, we next turn to the analysis of relative equilibrium points. These points play the same role for our system as the critical points of a potential function for a gradient flow.

***Definition** 1:* A formation $p_1, \ldots, p_N$ in $\mathbb{R}^d$ is called a **relative equilibrium point** for the mismatched system (11) whenever the associated relative equilibrium orbit

$$E(d) \cdot p \subset \mathbb{R}^{dN}$$

is invariant under the flow of (11). The restriction of the flow to the relative equilibrium orbit $E(d) \cdot p$ is called the **relative equilibrium dynamics**.

The notion of relative equilibria stems from celestial mechanics and has been introduced in a convenient geometric framework by S. Smale [16]. We refer also to related work by [17]–[19] although the results there do not cover exactly the situation we are studying here. Our main goal in this paper is to study the relative equilibria and determine the structure of the relative equilibrium dynamics of (11). Thus let $\mathscr{C} \subset \mathbb{R}^{dN}$ denote the union of all relative equilibria of (11). It is straightforward to see that $\mathscr{C}$ defines a closed semialgebraic subset of $\mathbb{R}^{dN}$. An interesting open question though is whether –at least for generic choices of d^*, μ-the set $\mathscr{C}$ consists only of a finite number of $E(d)$-orbits. In [13] it has been shown for $\mu = 0$ that the set of all distance vectors d^*, for which (11) has only finitely many nondegenerate equilibrium points, is open and dense. Moreover, the number of such equilibrium points were shown to be upper bounded by 3^{N-1}. It is desirable to extend such results for $\mu \neq 0$, but this will not be done here for reasons of space. We just state the following general result.

***Lemma** 2:* A formation $p \in \mathbb{R}^{dN}$ is a relative equilibrium point if and only if there exists a skew-symmetric $d \times d$ matrix Ω such that

$$(B^\top \otimes I_d)(R^\top(z)e(p) + S^\top(z)\mu) = (I_M \otimes \Omega)z$$

holds. In particular, the set of relative equilibria forms a semialgebraic set. For $d = 1$ the relative equilibria define a real algebraic set given as

$$\left(BD(e)B^\top + HD(\mu)B^\top\right) p - v\mathbf{1}_N = 0$$

where $v = \frac{1}{N}(\mathbf{1}_N^\top BD(e)^2 B^\top p + \mu^\top B^\top p)$.

Proof. Suppose $d \geq 2$. p is a relative equilibrium if and only if $\dot{z}$ is in the tangent space $T_z(O(d) \cdot z)$. But this is equivalent to the condition that $(B^\top \otimes I_d)(R^\top(z)e(p) + S^\top(z)\mu) = (I_M \otimes \Omega)z$ for some skew symmetric Ω. If $d = 1$, p is a relative equilibrium if and only if all entries of $\dot{p}$ are equal, which corresponds to the stated condition. This completes the proof. ■

We now explore the dynamics of the mismatched flow when restricted to a relative equilibrium orbit.

***Proposition** 2:* Let p denote any relative equilibrium point of (11). Then there exists a skew-symmetric matrix $\Omega \in \mathbb{R}^{d\times d}$ and $v \in \mathbb{R}^d$ such that the relative equilibrium dynamics of p are given as

$$p(t) = (I_M \otimes e^{t\Omega})p + \mathbf{1} \otimes \frac{e^{t\Omega} - I}{\Omega} v$$

Proof. Since the Lie group orbit $E(d) \cdot p$ is invariant under the flow of (11), and since the flow of (11) is itself invariant under the $E(d)$-action, it follows that the trajectories of (11) are orbits $e^{t\xi} \cdot p$ of one-parameter subgroups $t \mapsto e^{t\xi}$ of the Euclidean group $E(d)$. Here ξ denotes a suitable element of the Lie algebra of $E(d)$. But the Lie algebra elements of $E(d)$ are of the form

$$\xi = \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix}$$

and thus

$$e^{t\xi} = \begin{pmatrix} e^{t\Omega} & \frac{e^{t\Omega} - I}{\Omega} v \\ 0 & 1 \end{pmatrix}.$$

Note that the expansion

$$\frac{e^{t\Omega} - I}{\Omega} = tI + \frac{t^2\Omega}{2} + \frac{t^3\Omega^2}{3!} + \cdots$$

is well defined and real analytic in t for any matrix Ω, even for the zero matrix. This completes the proof. ■

Of course, the skew symmetric matrix Ω and translation vector v in Proposition 2 still need to be determined and they depend on the parameters in the mismatched flow (11). In particular it is of interest to see when $\Omega = 0$ in which case the equilibrium dynamics describe a pure translation. We will consider this question for $d = 2$ in a subsequent section.

V. ERROR DYNAMICS

We now analyze the behavior of the mismatched flow near a realization of d^*. Thus assume that $p^* \in \mathbb{R}^{dN}$ is a formation with $e(p^*) = 0$. We next refine a result from [14]. The second claim implies that around any infinitesimally rigid formation p of a rigid graph one can express any distance function $\|p_i - p_j\|^2$ as a real analytic function of the M distances defined by the graph. This has been stated implicitly in [20]; see also the forthcoming paper [3] for a different proof.

***Theorem** 1:* (i) Let Γ be a rigid graph in $\mathbb{R}^d$ and let $\mathbb{R}^{dN}_{\text{reg}}$ denote the dense open subset of infinitesimally rigid formations p. Then $\mathscr{D}(\mathbb{R}^{dN}_{\text{reg}})$ is a smooth manifold of dimension $d(N-1) - \frac{1}{2}d(d-1)$ and

$$\mathscr{D} : \mathbb{R}^{dN}_{\text{reg}} \longrightarrow \mathscr{D}(\mathbb{R}^{dN}_{\text{reg}})$$

defines a smooth $E(d)$- fibre bundle.

(ii) In particular, given any infinitesimally rigid formation $p^* \in \mathbb{R}^{dN}$ with distance vector $d^* := \mathscr{D}(p^*)$ there exists an open neighborhood $\mathscr{U} \subset \mathbb{R}^{dN}$ and an open subset $V \subset \mathbb{R}^M$ such that for any $i \neq j$

$$\|p_i - p_j\|^2 = f_{ij}(\mathscr{D}(p)) \quad \forall p \in \mathscr{U}$$

holds for a suitable real analytic function $f_{ij} : V \longrightarrow \mathbb{R}$.

Proof. It has been observed in [14] that the map $\mathscr{D} : \mathbb{R}^{dN}_{\text{reg}} \longrightarrow \mathscr{D}(\mathbb{R}^{dN}_{\text{reg}})$ defines a smooth locally trivial fibre bundle. In fact, this is easily shown first for a minimally rigid graph, in which case the map $\mathscr{D}$ defines a submersion on $\mathbb{R}^{dN}_{\text{reg}}$ and we can thus argue via the local normal form of any submersion. Since any rigid graph contains a minimally rigid subgraph the result follows. The second claim can be deduced from the first one. Explicitly, we can assume without loss of generality that Γ is minimally rigid, i.e. we have $M = dN - d(d+1)/2$. Moreover, let p^* denote any infinitesimally rigid formation such that the first d agents $p_1^*, \ldots, p_d^*$ are linearly independent. Thus there exists a neighborhood U of p^* such that $p_1, \ldots, p_d$ are linearly independent for all $p \in U$. By the well-known smoothness of the QR-factorization in $GL_d(\mathbb{R})$ we conclude that there exists a smooth map $g : U \longrightarrow E(d), p \mapsto g_p$, such that for any $p \in U$ we have

$$g_p \cdot p = (0, p_2, \ldots, p_{d+1}, \ldots, p_N)$$

where $(p_2, \ldots, p_{d+1})$ is upper triangular with positive diagonal entries. The zero entry in place of p_1 comes from the translation aspect of $E(d)$, and the upper triangular property from the QR factorization. Thus we have shown that the quotient space $U/E(d)$ is a smooth manifold of dimension $M = d(N-1) - d(d-1)/2$. The map $\mathscr{D} : U \longrightarrow \mathbb{R}^M$ is a submersion that is invariant under the Euclidean group and therefore defines a submersion $\overline{\mathscr{D}} : U/E(d) \longrightarrow \mathbb{R}^M$. Since $\dim U/E(d) = M$ it follows that $\overline{\mathscr{D}}$ is a local diffeomorphism. Let Φ denote the (locally defined) inverse of $\overline{\mathscr{D}}$. Thus Φ is real analytic and satisfies $\Phi \circ \mathscr{D} = Id$, locally. Therefore $f := \mathscr{D}_N \circ \Phi$ fulfills locally $f \circ \mathscr{D} = \mathscr{D}_N$. This proves the result.

■

***Proposition** 3:* Assume that Γ is minimally rigid and that $p^* \in \mathbb{R}^{dN}$ is an infinitesimally formation $r(p^*) = d$ with $e(p^*) = 0$. For μ sufficiently small there exists an $E(d)$-invariant neighborhood $U \subset \mathbb{R}^{dN}$ of p^* such that the following assertions hold:

(i) U is invariant under the flow of (11).

(ii) There exists a relative equilibrium formation $\overline{p}_\mu \in U$ such that every solution $p(t)$ of (11) in U satisfies

$$\lim_{t \to \infty} e(p(t)) = \overline{e}(\mu) := e(\overline{p}_\mu).$$

$\overline{p}_\mu$ is unique up to Euclidean group transformations and can be chosen to depend real analytic on μ.

(iii) The map $\mu \mapsto \overline{e}(\mu)$ is real analytic with derivative

$$D\overline{e}(0) = -\left(R(z^*)R(z^*)^\top\right)^{-1} R(z^*)S(z^*)^\top.$$

Proof. The error dynamics of (11) are

$$\dot{e} = -2(R(z)R^\top(z)e(z) + R(z)S^\top(z)\mu).$$

We first prove that the right hand side is locally around p^* a real analytic function of $e(z)$. In fact, the first summand is equal to $D(z)^\top(B^\top B \otimes I_d)D(z)e(z)$. Each entry of $D(z)^\top(B^\top B \otimes I_d)D(z)$ is a sum of inner products $z_k^\top z_\ell$. But $z_k^\top z_\ell = (p_i - p_j)^\top(p_r - p_s)$ is a sum of inner products of the form $p_u^\top p_v = \frac{1}{2}(\|p_u - p_v\|^2 - \|p_u\|^2 - \|p_v\|^2)$. After a suitable translation of one agent to the origin we can assume without loss of generality that the last agent $p_N = 0$. By Theorem 1 we can express any term $\|p_u - p_v\|^2, \|p_u\|^2, \|p_v\|^2$ as a real analytic function of $e(z)$. This shows that each entry of the first term $D(z)^\top(B^\top B \otimes I_d)D(z)e(z)$ can be expressed as a real analytic function of $e(z)$. Similarly one argues for the second summand $R(z)S^\top(z)\mu = D(z)^\top(B^\top H \otimes I_d)D(z)\mu$. This shows that there is a matrix $A(e, \mu)$ of real analytic functions in $e(z)$ such that

$$\dot{e} = A(e, \mu) \tag{17}$$

holds. To inspect the equilibrium of (17), we first study the unperturbed error system $\dot{e} = A(e, 0)$, which is

$$\dot{e} = -2Qe \tag{18}$$

where $Q = R(z)R(z)^\top$ is a real analytic function of e. Note that at $e = 0$, the formation is infinitesimally and minimally rigid, which implies that $R(z)$ has full row rank. Thus $Q(0)$ is positive definite. Note that the linearization of (18) is

$$\dot{e} = -2Q(0)e$$

Then $e = 0$ is an exponentially stable equilibrium of the unperturbed error system (18). It follows that for a sufficiently small μ, there exists a constant $\bar{e}_\mu$ such that $e = \bar{e}_\mu$ is an exponentially stable equilibrium of the error system (18). By the implicit function theorem, $\bar{e}_\mu$ is a real analytic function of μ, which is close to 0 when μ is sufficiently small. This in turn implies in the dynamics of p that there exists an $E(d)$-invariant neighborhood $U \subset \mathbb{R}^{dN}$ of p^* such that every solution $p(t)$ of (11) in U converges to $\overline{p}_\mu \in U$ such that

$$\lim_{t\to\infty} e(p(t)) = \overline{e}_\mu := e(\overline{p}_\mu). \tag{19}$$

The last claim in (ii) follows immediately from the real analyticity of $\mu \mapsto \overline{e}(\mu)$ together with part (ii) in Theorem 1. To prove (iii) we consider the real analytic map in (e,μ) defined as

$$A(e,\mu) = -2(R(z)R(z)^\top e(z) + R(z)S(z)^\top \mu).$$

The partial derivative of A with respect to e at $(e,\mu) = (0,0)$ is the linear map with matrix $-2R(z^*)R(z^*)^\top$, which is invertible. Thus by the implicit function theorem there exists locally a unique real analytic function $\mu \mapsto \overline{e}(\mu)$ with $\overline{e}(0) = 0$ that satisfies $A(\overline{e}(\mu),\mu) = 0$. Therefore

$$\begin{aligned} 0 &= D_1A(0,0)D\overline{e}(0) + D_2A(0,0) \\ &= -2(R(z^*)R(z^*)^\top D\overline{e}(0) + R(z^*)S(z^*)^\top) \end{aligned}$$

holds, which completes the proof. ■

VI. Main Convergence Theorem and Discussion

Now we state our main convergence theorem as follows:

***Theorem** 2:* Assume that Γ is minimally rigid and that $p^* \in \mathbb{R}^{dN}$ is an infinitesimally formation $r(p^*) = d^*$ with $e(p^*) = 0$. For μ sufficiently small there exists an $E(d)$-invariant neighborhood $U \subset \mathbb{R}^{dN}$ of p^* such that for any $p(0) \in U$, the solution of the mismatched gradient dynamics (11) converges to the relative equilibrium $\bar{p}(t)$,

$$\bar{p}(t) = (I_M \otimes e^{t\Omega})\bar{p}(0) + \mathbf{1} \otimes \frac{e^{t\Omega} - I}{\Omega} v. \tag{20}$$

for some $v \in \mathbb{R}^d$.

Proof. For reasons of space we can only sketch the main arguments. Following the notation in the proof of Proposition 3 we see that the error dynamics at $\mu = 0$ is $\dot{e} = -2Q(0)e$, with $Q(0)$ positive definite. This shows that the critical orbit of p^* is normally hyperbolic. Thus the center manifold theorem [21] applies, implying the existence of an open invariant neighborhood $U \subset \mathbb{R}^{dN}$ of p^* such that for any $p(0) \in U$, the distance from the solution trajectory of the mismatched gradient dynamics (11) to the critical formation orbit goes exponentially fast to zero. Moreover, the center manifold theorem also implies the convergence to a single trajectory $\bar{p}(t)$ on the critical formation orbit. By Proposition 2, the trajectories on the critical formation orbit are of the form (20). This completes the proof. ■

Theorem 2 implies that the limiting trajectory $\bar{p}(t)$ is a combination of rotation and translation. When $d = 1$, the skew symmetric matrix Ω must be 0. Then $\bar{p}(t) = \bar{p}(0) + tv$, and hence only translation occurs. When $d = 2$, only rotation is observed for generic μ and only translation is observed for some special μ as shown in [2], [3]. For $d = 3$, the combination of rotation and translation is observed for generic μ, for which $\bar{p}(t)$ is a helical motion [4].

VII. Conclusion

In this paper, we have analyzed a mismatched gradient dynamics proposed in [2]–[4]. Application of basic Lie group methods not only leads to simplification of the analysis in earlier works but also to more general results in high dimensional systems.

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