53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Formation movements in minimally rigid formation control with mismatched mutual distances

Z. Sun, S. Mou, B. D. O. Anderson, A. S. Morse

***Abstract*— When a gradient descent control law is employed for stabilizing undirected minimally rigid formations, mismatched desired distances between neighboring agent pairs will cause additional motions of the whole formation. By reviewing and extending the results in [1], [2], we show that in general rotational motions and helical motions will occur for 2-D formations and 3-D formations, respectively. We then consider the problem of how to compute the formulas for the motions caused by constant mismatches. A novel idea based on the angular-momentum concept in rigid body dynamics is proposed for deriving the formation formulas, e.g., angular velocity, rotational radius, etc. in terms of the distance mismatch terms. This has implications on steering and controlling rigid formation motions.**

I. INTRODUCTION

Formation control of networked multi-agent systems has received considerable attention in recent years due to its extensive applications. One of the key problems in this area is how to design a controller to maintain a geometrical shape of the formation. By using rigid graph theory, the formation shape can be achieved by controlling a certain set of inter-agent distances [3], [4]. A commonly-used control method is the gradient descent control law, which is derived from some specified potential function which involves the squared errors between actual distances and desired distances. The stability analysis of this gradient control has been studied extensively in the literature; see e.g. [5], [6], [7], [8].

The stabilization of formation shape for multi-agent systems is a typical distributed and cooperative task, in which each agent pair associated with one prescribed inter-agent distance needs to work cooperatively to achieve that desired distance. This cooperative task requires that agent pairs should have the same view of the desired distance, and need to measure correctly the actual distance between them. One may ask what will happen if the mutual distances between two joint agents are not equal, i.e. if there exist unequal views for the distance that two associated agents are required to maintain, and whether it is possible or effective to achieve the distance control by using gradient control in such scenarios. It has been briefly mentioned in [9] that such distance mismatches may lead to formation control failure. The authors also introduced the concept *information-based instability* to illustrate this control problem arising in distributed control; see also the review in [10]. Recently, the papers [1], [2] have presented more elaborate discussions on these robustness issues arising in undirected formation shape control. In fact there are several dynamical systems involved, and the stability concept may have different interpretations. It has been shown that the formation shape will converge but additional motions will occur due to mismatched distances. Some interesting formation movements for 2-D triangular formations and 3-D tetrahedral formations have been discussed in [1] and [2], respectively.

In this paper the results in [1] and [2] are reviewed and extended and some more general results are presented by addressing general formation shapes. Then the main focus of this paper is to derive formulas for the motion parameters in terms of mismatched distance values, which have been regarded as open problems in [1], [2]. As a by-product, these motion formulas also suggest a possible formation motion control method, which deliberately introduces the mismatches as control inputs so that the formation orientations, angular velocity, etc., can be controlled.

The paper is organized as follows. Section II introduces the background and problem description, and sets up several main equations. Then some general motions for minimally and infinitesimally rigid formations caused by mismatched distances are reviewed and discussed. In Section III, we focus on the motion formula calculations in terms of distance mismatches. Finally, Section IV concludes this paper.

II. FORMATION MOTIONS INDUCED BY MISMATCHED MUTUAL DISTANCES

A. Problem setup and motion equations

We omit here some background introduction of graph theory and rigidity theory, but more details can be found in e.g. [4], [11]. The formation is often modeled by a graph, with vertices corresponding to agents and edges corresponding to specified inter-agent distances which should be maintained. Consider an undirected graph with m edges and n vertices, denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \cdots, n\}$ and edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. The neighbor set $\mathcal{N}_i$ of agent i is defined as $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Let $p_i \in \mathbb{R}^d$ denote a point in d-dimensional space that is assigned to $i \in \mathcal{V}$. The stacked vector $p = [p_1^T, p_2^T, \cdots, p_n^T]^T \in \mathbb{R}^{dn}$ represents the realization of $\mathcal{G}$ in $\mathbb{R}^d$. Let $z_{k_{ij}} \in \mathbb{R}^d$ denote the relative position vector for the vertex pair i and j defined by the k-th edge. The direction of the vector $z_{k_{ij}}$ is defined as $z_{k_{ij}} = p_i - p_j$ when $i > j$.

Z. Sun is with Shandong Computer Science Center, SCSC (National Supercomputer Center in Jinan), Jinan, China, and B. D. O. Anderson is a visiting expert with SCSC. Z. Sun and B. D. O. Anderson are with National ICT Australia and Research School of Engineering, The Australian National University, Canberra ACT 0200, Australia. ({zhiyong.sun, brian.anderson}@anu.edu.au)

S. Mou and A. S. Morse are with the Department of Electrical Engineering, Yale University, New Haven, CT, USA ({shaoshuai.mou, as.morse}@yale.edu)

978-1-4673-6088-3/14/$31.00 ©2014 IEEE

We assume in this paper that each agent's dynamics are described by a simple kinematic model in the form $\dot{p}_i = u_i$, where $u_i \in \mathbb{R}^d$ is the control input for agent i. The control goal in formation shape stabilization is to drive all the agents to reach a configuration, such that the desired distances between the specified agent pairs can be satisfied. We denote the desired distance for the k-th edge as $d_{k_{ij}}$, which agent pairs i and j should work cooperatively to achieve. However, due to measurement bias or differing views of the same distance, agent pairs may unknowingly use mismatched distances in their control law. Without loss of generality, we suppose that when $i < j$, $d_{k_{ij}}$ is the desired distance perceived by agent i, while the same distance perceived by agent j may be different. Thus, for $i < j$, the actual desired distances used by agents i and j in formulating their controls are

$$d_{ij}^2 = d_{k_{ij}}^2, d_{ji}^2 = d_{k_{ij}}^2 + \mu_{k_{ij}} \tag{1}$$

where μ_k denotes the distance mismatch corresponding to edge k between its two associated agents. We further define the squared distance error as

$$e_{k_{ij}} = z_{k_{ij}}^2 - d_{k_{ij}}^2 = \|p_i - p_j\|^2 - d_{k_{ij}}^2 \tag{2}$$

For ease of notation we will use $e_{k_{ij}}$ and e_k interchangeably; this also applies to $d_{k_{ij}}$, $\mu_{k_{ij}}$ and $z_{k_{ij}}$ in the following context. The error vector, distance vector and mismatched value vector are constructed as $e = [e_1, e_2, \cdots, e_m]^T$, $d = [d_1, d_2, \cdots, d_m]^T$ and $\mu = [\mu_1, \mu_2, \cdots, \mu_m]^T$, respectively.

The control law is the commonly-used gradient descent control [6], derived from the potential function

$$V = \frac{1}{4} \sum_{(i,j)\in\mathcal{E}} \left(\|p_i - p_j\|^2 - d_{k_{ij}}^2 \right)^2 \tag{3}$$

In the absence of any mismatch, the control input for agent i is obtained as

$$\begin{aligned} \dot{p}_i &= -\sum_{j\in\mathcal{N}_i} (p_i - p_j) e_{k_{ij}}(z) \\ &= -\sum_{j\in\mathcal{N}_i, j<i} z_{k_{ij}} e_{k_{ij}}(z) + \sum_{j\in\mathcal{N}_i, j>i} z_{k_{ij}} e_{k_{ij}}(z) \end{aligned} \tag{4}$$

Note that for the case $j > i$ in the right hand side of the above equation, the term has a plus sign $+$, which is in accordance with the definition of the vector $z_{k_{ij}}$.

Similar to [1] and [2], in the presence of mismatched distances, the equations of agent motions should be modified as

$$\dot{p}_i = -\sum_{j\in\mathcal{N}_i, j<i} z_{k_{ij}} (e_{k_{ij}}(z) - \mu_{k_{ij}}) + \sum_{j\in\mathcal{N}_i, j>i} z_{k_{ij}} e_{k_{ij}}(z) \tag{5}$$

In the following, we will use similar techniques as in [2] to obtain some compact forms of the equations. According to [2], [11], we denote the incidence matrix corresponding to the formation graph as $H \in \mathbb{R}^{m\times n}$, with the orientation in accordance with each vector $z_{k_{ij}}$. By defining $\bar{H} = H \otimes I_d$, the staked relative position vector is denoted as $z = [z_1^T, z_2^T, \cdots, z_m^T]^T \in \mathbb{R}^{dm}$. Then the vector z and rigidity matrix R can be expressed as $z = \bar{H}p$ and $R = Z^T \bar{H}$, where $Z = \text{diag}\{z_1, z_2, \cdots, z_m\}^T$; see [11]. Define J and $\bar{J}$ to be the matrices obtained from H and $\bar{H}$ by replacing all -1 entries by zeros, which means that $\bar{J} = J \otimes I_d$. With the definition of $\bar{J}$, we can define a $m \times dn$ matrix $S(z)$ by $S(z) = Z^T \bar{J}$. By doing this, we can obtain a compact form of the motion equation:

$$\dot{p} = -R^T(z)e + S^T(z)\mu \tag{6}$$

According to the construction of relative position vectors $z = \bar{H}p$, multiplying both sides of (6) by $\bar{H}$ yields the following equation

$$\dot{z} = -\bar{H}R^T(z)e(z) + \bar{H}S^T(z)\mu \tag{7}$$

Furthermore, with the definition of $e_{k_{ij}}$ in (2) and the equation (7) for the relative positions z , it is straightforward to obtain the differential equation for the vector e

$$\dot{e} = -2R(z)R^T(z)e + 2R(z)S^T(z)\mu \tag{8}$$

***Remark** 1:* It can be proven that the non-zero entries of the coefficient matrix $R(z)R^T(z)$ and $R(z)S^T(z)$ in (8) can be expressed as some smooth functions of e, when the formation shape is close to the target formation shape. Details can be found in [12]. Thus, the distance error system is a self-contained system, which can be rewritten as $\dot{e} = g(e, \mu) = A(e)e + B(e)\mu$, where $A(e) = -2R(z)R^T(z)$ and $B(e) = 2R(z)S^T(z)$, and $g(.)$ is a smooth function.

B. Exponential stability and its consequences

Our first result shows an exponential convergence of the distance error system (8) without and with distance mismatch terms μ. Note that the result holds locally for a minimally and infinitesimally rigid formation.

***Theorem** 1:* The nominal distance error system (8), i.e. $\dot{e} = A(e)e$, converges locally to zero *exponentially fast*. Furthermore, for small and constant $\mu \neq 0$, the perturbed error system (8) (i.e. $\dot{e} = g(e, \mu) = A(e)e + B(e)\mu$) will approach an exponentially stable equilibrium which is close to $e = 0$.

The above Theorem can be seen as an extended result of the 2-D triangular formation case treated in [1] and 3-D tetrahedron case reported in [2]. The proof of Theorem 1 is omitted here and will be reported in [12]. We denote the equilibrium as $\bar{e}(\mu)$, or shortly as $\bar{e}$. Thus, with small μ, the agents will form a formation which is close to the desired one. However, in general the formation will not actually come to rest when the error system converges to $\bar{e}$. We call the formation motion at the equilibrium state $e(t) = \bar{e}$ an *equilibrium motion*. Different equilibrium motions for 2-D and 3-D formations are summarized as follows.

***Lemma** 1:* Additional motions will occur due to constant and small mismatched interagent distances. Specifically,

- for 2-D formations, generically all the agents and the final formation will undergo a rotational motion with a specific radius and constant angular velocity; in special cases translation motion can occur which is not generic;

- for 3-D formations, the motion involves a rotational motion with constant angular velocity and a fixed axis, and a simultaneous translation motion with constant translational velocity whose direction is parallel to the rotational vector, which, in fact, indicates a helical movement. In special cases translation-only movement and rotation-only movement can occur which are not generic.

The proof for the above results requires quite lengthy analysis, which is omitted here and is reported in [12]. A geometrical method for the stability and motion analysis in this mismatched formation control problem can be found in [13].

III. Motion formula calculations

This section aims to provide another perspective of the mismatch problem. By assuming that the distance mismatch terms for each edge are known and constant, we would like to derive some formulas for the angular velocity, rotational radius, etc., in terms of the distance mismatch values. The basic idea of the formula derivation method in this section is inspired by the rigid body angular momentum concept in classical mechanics [14]. This may have implications to actively steer the formation (change of orientation, control of rotation motion, etc) by using small number of inputs. A first attempt of this formation manipulation method is reported in [15].

A. Method description in 2-D case

We denote the centroid of the formation as $p_c \in \mathbb{R}^2$, and the vector r_i denotes the relative position vector $r_i = p_i - p_c$. Let Ω denote a vector orthogonal to the plane containing the agents, with upward direction corresponding to counterclockwise angular velocity in the plane (see Fig. 1). With this definition and the derivation of particle velocity in rigid body dynamics (Chapter 16, [14]), agent i's motion can be described as:

$$\dot{p}_i = \dot{p}_c + \Omega \times r_i, \quad i = 1, 2, ..., n \tag{9}$$

where $\times$ denotes the vector cross product operation. Note that the cross product $\times$ is formally defined for operating vectors in $\mathbb{R}^3$, and here in 2-D case we can think of p_i, p_c and r_i as vectors in $\mathbb{R}^3$ by simply adding a zero third element into each vector. We also emphasize that equation (9) holds whether or not p_c is stationary, rotating on a circle around a fixed point, translating, etc. From (9) one has

$$\sum_{i=1}^{n} r_i \times \dot{p}_i = \sum_{i=1}^{n} r_i \times \dot{p}_c + \sum_{i=1}^{n} r_i \times (\Omega \times r_i) \tag{10}$$

Note that because $\sum_{i=1}^{n} r_i = 0$, one has $\sum_{i=1}^{n} r_i \times \dot{p}_c = 0$. Furthermore,

$$\sum_{i=1}^{n} r_i \times (\Omega \times r_i) = \sum_{i=1}^{m} (\Omega \|r_i\|^2 - r_i(r_i^T \Omega)) = \sum_{i=1}^{n} \Omega \|r_i\|^2$$

Therefore

$$\sum_{i=1}^{n} r_i \times \dot{p}_i = \Omega \sum_{i=1}^{n} \|r_i\|^2 \tag{11}$$

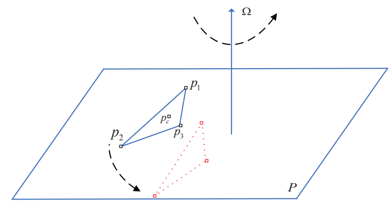

Fig. 1. Formation movements in the plane induced by constant mismatches (a triangular shape example)

The above equation resembles the angular-momentum formula in mechanics. Here each agent can be seen as a particle with unit mass. The left hand side sums the contribution from each of the point masses to the overall angular momentum, and the right side is the usual expression of the angular momentum involving the inertia. In the following, we aim to expand the left side and right side of (11) to obtain more explicit expressions in terms of μ and formation shape information. Some straightforward calculations will show that the left hand side of (11) does not involve the distance error vector e, which can be simplified as

$$\sum_{i=1}^{n} r_i \times \dot{p}_i = \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i, j<i} r_i \times z_{k_{ij}} \mu_{k_{ij}} \tag{12}$$

For the right hand side of (11), the following equality holds

$$\sum_{i=1}^{n} \|r_i\|^2 = \frac{1}{n} \sum_{1 \le i < j \le n} \|p_i - p_j\|^2 \tag{13}$$

The proof for the above equalities will be presented elsewhere due to space limit.

By combining the results of the above two lemmas, one can use the following equation to derive Ω in terms z_k and μ_k.

$$\Omega = \frac{\sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i, j<i} r_i \times z_{k_{ij}} \mu_{k_{ij}}}{\frac{1}{n} \sum_{1 \le i < j \le n} \|p_i - p_j\|^2} \tag{14}$$

Furthermore, the velocity for the centroid can be obtained as

$$\dot{p}_c = \frac{1}{n} \sum_{i=1}^{n} \dot{p}_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i, j<i} z_{k_{ij}} \mu_{k_{ij}} \tag{15}$$

Thus the radius r^c_{radius} of the rotation with respect to the formation centroid is derived as

$$r^c_{\text{radius}} = \|\dot{p}_c\| / \|\Omega\| \tag{16}$$

From the above equation, it is obvious that r^c_{radius} is also independent of e.

B. Method description in the 3-D case

As shown in Lemma 1, the motion in the 3-D case is generally a combination of rotation and translation, and the axis around which all the agents rotate is in the same direction of the translation (see Fig. 2). The following equation which describes the motion of each agent still holds in the 3-D case:

$$\dot{p}_i = \dot{p}_c + \Omega \times r_i, \tag{17}$$

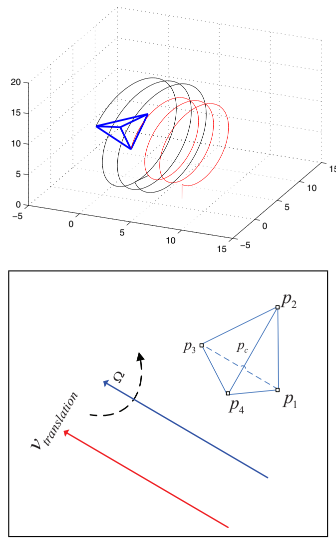

Fig. 2. Formation movements in 3-D space induced by constant mismatches (a tetrahedron shape example). Above: simulation result showing helical movements of the formation caused by distance mismatches; below: decomposition of the movements which involve a rotational motion and a simultaneous translation in a direction parallel to the rotation vector.

where the definitions of p_i, p_c and r_i are the same as those in previous sections, except that they are all 3-vectors. The vector Ω is the angular velocity vector indicating the rotational motion about a fixed axis. By doing some similar calculations as in the 2-D calculation, one can obtain

$$\sum_{i=1}^{n} r_i \times \dot{p}_i = \sum_{i=1}^{n} r_i \times \dot{p}_c + \sum_{i=1}^{n} r_i \times (\Omega \times r_i) = \sum_{i=1}^{n} r_i \times (\Omega \times r_i) \tag{18}$$

where $\sum_{i=1}^{n} r_i = 0$ by the definition of the centroid. Similar to the 2-D case, the left hand summation term $\sum_{i=1}^{n} r_i \times \dot{p}_i$ in (18) does not involve the error vector e, which can be shown as

$$\sum_{i=1}^{n} r_i \times \dot{p}_i = \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i, j<i} r_i \times z_{k_{ij}} \mu_{k_{ij}} \tag{19}$$

We would like to examine the last term in the right hand side of (18). To this end we follow a similar way of defining the inertia matrix [Chapter 18, [14]] by introducing the skew-symmetric matrix R_i, which is constructed from the vector $r_i = [r_{ix}, r_{iy}, r_{iz}]^T$:

$$R_i = \begin{bmatrix} 0 & -r_{iz} & r_{iy} \\ r_{iz} & 0 & -r_{ix} \\ -r_{iy} & r_{ix} & 0 \end{bmatrix}$$

The above skew-symmetric matrix R_i is used to perform the cross product operation: $r_i \times \Omega = R_i \Omega$. Thus one has

$$\sum_{i=1}^{n} r_i \times (\Omega \times r_i) = -\sum_{i=1}^{n} R_i (r_i \times \Omega) = -\sum_{i=1}^{n} R_i^2 \Omega \tag{20}$$

where

$$R_i^2 = -R_i^T R_i = -\begin{bmatrix} r_{iy}^2 + r_{iz}^2 & -r_{ix} r_{iy} & -r_{ix} r_{iz} \\ -r_{iy} r_{ix} & r_{ix}^2 + r_{iz}^2 & -r_{iy} r_{iz} \\ -r_{iz} r_{ix} & -r_{iz} r_{iy} & r_{ix}^2 + r_{iy}^2 \end{bmatrix} \tag{21}$$

The following Lemma shows the non-singularity of the matrix $R_{\text{sum}} := \sum_{i=1}^{n} R_i^2$.

***Lemma** 2:* The matrix R_{sum} is non-singular and its inverse exists for 3-D formation objects with positive volume.

Proof: The proof is based on some facts of the kernel of the matrix R_i^2. Note that the matrix R_i^2 is negative semidefinite and $\text{kernel}(R_i^2) = \text{kernel}(R_i^T R_i) = \text{kernel}(R_i)$, where the kernel of the skew symmetric matrix R_i is spanned by the vector r_i. Further note that the matrices R_i^2 and R_j^2 do not share the same kernel if the vectors r_i and r_j are linearly independent. In any 3-D formation objects with positive volume, there always exist three linearly independent vectors r_i, r_j, r_k which span the 3-D space. Hence the kernel of the matrix R_{sum} is trivial, i.e. R_{sum} is negative definite and its inverse exists. ■

By using the above results, the angular velocity can be calculated as

$$\Omega = -R_{\text{sum}}^{-1} \left(\sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i, j<i} r_i \times z_{k_{ij}} \mu_{k_{ij}} \right) \tag{22}$$

which involves the shape geometrical information and mismatch terms μ, but not e.

Since the axis along which the agents rotate is also the same direction in which all the agents translate, we can find the translational part of the velocity by projecting the velocity onto the rotational axis. To this end, we first obtain the velocity of the centroid described as

$$\dot{p}_c = \frac{1}{n} \sum_{i=1}^{n} \dot{p}_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_i, j<i} z_{k_{ij}} \mu_{k_{ij}} \tag{23}$$

Then the translational velocity can be obtained as

$$v_{\text{translation}} = \frac{\Omega \Omega^T}{\Omega^T \Omega} \dot{p}_c \tag{24}$$

The rotational velocity v_{rotation} and the radius r_{radius}^c of the rotation with respect to the centroid are

$$v_{\text{rotation}} = \dot{p}_c - v_{\text{translation}}, \quad r_{\text{radius}}^c = \|v_{\text{rotation}}\| / \|\Omega\| \tag{25}$$

Note that the above motion formulas (23), (24) and (25) are determined by the formation geometry and mismatch terms μ, while the error vector e is not involved. In the following we will show how to use the methods just described to study the motions in 2-D and 3-D cases by two examples.

C. Example: 2-D triangle formation

According to Eq. (5) of [1], the motion equation for each agent is

$$\begin{aligned} \dot{p}_1 &= -z_1 e_1 + z_3 (e_3 + \mu_3) \\ \dot{p}_2 &= -z_2 e_2 + z_1 (e_1 + \mu_1) \\ \dot{p}_3 &= -z_3 e_3 + z_2 (e_2 + \mu_2) \end{aligned}$$

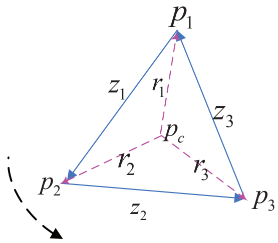

Fig. 3. A triangular formation example: rotational movement caused by distance mismatches. Note that the orientation in each edge indicates the direction of the relative position.

The notations and directions of each z_i are shown in Fig. 3. Note that in [1] the definition of the incidence matrix is slightly different to the definition in this paper, but this does not effect the main result shown below.

Inserting the above equations for $\dot{p}_i$ in (11), one can rewrite the left hand side of (11) as $\sum_{i=1}^{3} r_i \times \dot{p}_i = \mu_1 r_2 \times z_1 + \mu_2 r_3 \times z_2 + \mu_3 r_1 \times z_3$. In a triangle, by using the fact that the centroid divides the triangle area into three equal areas (the magnitude of $r_2 \times z_1$, $r_3 \times z_2$ or $r_1 \times z_3$ is two thirds of the area of the triangle), one can show that the magnitude of the vector on the right side of the above equation is in fact equal to $\frac{2}{3}(\mu_1 + \mu_2 + \mu_3)S$, where S is the area of the triangle. Furthermore, according to Lemma 5, there holds $\|r_1\|^2 + \|r_2\|^2 + \|r_3\|^2 = 1/3(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2)$. Thus one has

$$\omega = \frac{2S(\mu_1 + \mu_2 + \mu_3)}{\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2} \tag{26}$$

The rotational radius can be obtained as

$$r^c_{\text{radius}} = \frac{\|(\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2)(z_1\mu_1 + z_2\mu_2 + z_3\mu_3)\|}{|6S(\mu_1 + \mu_2 + \mu_3)|} \tag{27}$$

***Remark** 2:* We make some remarks about the motions in the triangle case:

- From (27) it can be seen that for an arbitrary triangle the radius can be arbitrarily large even with very small μ.
- Note that $\sum_{i=1}^{3} \mu_i = 0$ corresponds to $\omega = 0$, which indicates an infinite radius and also corresponds to a translation-only movement. In this case the translation velocity is equal to $\dot{p}_c = \frac{z_3\mu_3 + z_1\mu_1 + z_2\mu_2}{3}$ which should be constant.
- For fixed μ, it can be shown that for an equilateral triangle, the formation can achieve fastest rotation among all triangle shapes. In this case the frequency is $\omega^{max} = (1/\sqrt{3})\frac{\mu_1 + \mu_2 + \mu_3}{2}$.

D. Example: 3-D tetrahedron formation

We record here the motion equations for the 3-D tetrahedron formation from [2]:

$$\dot{p}_i = -\sum_{j<i} z_{k_{ij}}(e_{k_{ij}}(z) - \mu_{k_{ij}}) + \sum_{j>i} z_{k_{ij}} e_{k_{ij}}(z) \tag{28}$$

where $i, j \in \{1, 2, 3, 4\}$. Some notations are shown in Fig. 4. The left hand side of (18) can be calculated as

$$\begin{aligned} \sum_{i=1}^{4} r_i \times \dot{p}_i &= \mu_1 r_1 \times z_1 + \mu_4 r_1 \times z_4 + \mu_6 r_1 \times z_6 \\ &\quad + \mu_2 r_2 \times z_2 + \mu_5 r_2 \times z_5 + \mu_3 r_3 \times z_3 \end{aligned} \tag{29}$$

The rotational velocity, translational velocity, etc. can be calculated by using (22)-(25).

In the following, we will explore some conditions which ensure translation-only movement and rotation-only movement. For translation-only movement which corresponds to $\Omega = 0$, one needs to ensure that $\sum_{i=1}^{4} r_i \times \dot{p}_i = 0$. By observing the expression in the right hand side of (29) and using the fact that the vectors z_4, z_5, z_6 can be described by three linearly independent vectors z_1, z_2, z_3, all the six cross product terms in (29) can be reduced to three linearly independent ones. Then the condition for ensuring a translation-only movement can be described in the following equations (any three implying the fourth):

$$\begin{aligned} \mu_1 + \mu_2 &= \mu_4 \\ \mu_3 + \mu_4 &= \mu_6 \\ \mu_1 + \mu_5 &= \mu_6 \\ \mu_2 + \mu_3 &= \mu_5 \end{aligned} \tag{30}$$

which is independent of the formation shape. An intuitive interpretation of the above condition is that the mismatch values in each triangular face should sum to zero in order to have a translation-only movement. In this case, the translational velocity is described by the velocity of the centroid $\dot{p}_c = \frac{1}{4}\sum_{k=1}^{6} \mu_k z_k$. We mention that the above condition has also been recorded in [2] but without proof.

For the rotation-only movement, one needs to explore the condition $v_{\text{translation}} = 0$ in (24). By considering the expression of the $\dot{p}_c$ in (23), the necessary and sufficient condition can be expressed as $\sum_{k=1}^{6} \mu_k(\Omega^T z_k) = 0$, which involves both the selection of μ and formation shape terms. However, from the expression of $v_{\text{translation}} = 0$ in (24), a simple condition ensuring no translation movement is that $\dot{p}_c = 0$, which can be described in terms of the following equations involving only μ (any three implying the fourth):

$$\begin{aligned} \mu_1 + \mu_4 + \mu_6 &= 0 \\ \mu_2 + \mu_5 - \mu_1 &= 0 \\ \mu_2 + \mu_4 - \mu_3 &= 0 \\ \mu_3 + \mu_5 + \mu_6 &= 0 \end{aligned} \tag{31}$$

This set of conditions ensures that for any tetrahedron formation at all, there will be no translation movement. Hence these conditions are sufficient in relation to a particular tetrahedron. To ensure no tetrahedron of any shape will translate, these conditions are both necessary and sufficient. A proof will be reported in an extended version of this paper. An intuitive interpretation of the above rotation-only condition (31) is that any three μ_k corresponding to three adjacent edges should algebraically sum to zero for each vertex (the sign being obtained from the direction of the edges).

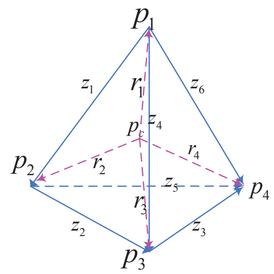

Fig. 4. A tetrahedron formation example. Note that the orientation in each edge indicates the direction of the relative position.

E. Steering formations by manipulating mismatches

From previous subsections, we have obtained formulas on rotational speed, rotation radius, translational velocity, etc, in terms of μ and shape geometry. Furthermore, we also show that in the 2-D case, a special choice of μ can guarantee a translation only movement. In the 3-D case, by using a tetrahedron shape as an example, we show some conditions for ensuring rotation-only movements and translation-only movements. All these discussions suggest that one can assign different values of μ to obtain the control objective in relation to formation orientations, angular velocity, translation direction, etc.

Let us consider the triangle formation as an example. In the special case when $\mu_3 = 0$ and $\mu_1 = -\mu_2$, i.e. the two edges incident on vertex 2 have associated mismatch values, it is easy to check that $\dot{p}_c = \frac{1}{3}(z_1 - z_2)\mu_1 = (p_2 - p_c)\mu_1$, i.e. the motion is in a direction defined by the vector $p_2 - p_c$, which is a fixed direction in body coordinates.

This analysis can be of course extended to more general formations. For example in the 2-D case, with one μ_k nonzero and the others zero, one can use the resulting expression of Ω to change the formation orientation. With two μ_k corresponding to two adjacent edges chosen so that the associated Ω is zero, the motion will be along a direction defined with respect to body centred coordinates. In the 3-D case, by using the expression of Ω derived in (22), one can manipulate two μ_k corresponding to two adjacent edges successively and set the others to zero to change the rotational axis and formation orientation. Furthermore, with three μ_k from three adjacent edges chosen so that the associated Ω is zero, the formation will translate along a fixed direction defined in body coordinates.

IV. CONCLUDING REMARKS

In this paper we have considered the shape control problem for minimally rigid formations in the presence of mismatched distances. The results in the previous work [1] and [2] have been generalized and we show some general formation motions induced by distance mismatches. We have then discussed in detail how to derive the movement formulas to describe the motions in terms of mismatches for 2-D formations and 3-D formations, respectively. The method resembles the concept of angular-momentum in rigid body dynamics. This has potential applications on controlling formation motions by using small number of control inputs.

ACKNOWLEDGMENT

This work was supported by NICTA, which is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council (ARC) through the ICT Centre of Excellence program. B. D. O. Anderson was also supported by the ARC under grant DP110100538. The research of S. Mou and A. S. Morse is supported by the US Air Force Office of Scientific Research and the National Science Foundation. Z. Sun is also supported by the Prime Minister's Australia Asia Incoming Endeavour Postgraduate Award.

REFERENCES

[1] M. A. Belabbas, S. Mou, A. S. Morse, and B. D. O. Anderson, "Robustness issues with undirected formations", in *Proc. of the 51st Conference on Decision and Control*. IEEE, 2012, pp. 1445–1450.

[2] Z. Sun, S. Mou, B. D. O. Anderson, and A. S. Morse, "Non-robustness of gradient control for 3-d undirected formations with distance mismatch", in *Proc. of the 3rd Australian Control Conference*. IEEE, 2013, pp. 369–374.

[3] R. Olfati-Saber and R. M. Murray, "Graph rigidity and distributed formation stabilization of multi-vehicle systems", in *Proc. of the 41st Conference on Decision and Control*. IEEE, 2002, vol. 3, pp. 2965–2971.

[4] B. D. O. Anderson, C. Yu, B. Fidan, and J. M. Hendrickx, "Rigid graph control architectures for autonomous formations", *Control Systems Magazine, IEEE*, vol. 28, no. 6, pp. 48–63, 2008.

[5] D. V. Dimarogonas and K. H. Johansson, "Stability analysis for multi-agent systems using the incidence matrix: quantized communication and formation control", *Automatica*, vol. 46, no. 4, pp. 695–700, 2010.

[6] L. Krick, M. E. Broucke, and B. A. Francis, "Stabilisation of infinitesimally rigid formations of multi-robot networks", *International Journal of Control*, vol. 82, no. 3, pp. 423–439, 2009.

[7] F. Dorfler and B. A. Francis, "Geometric analysis of the formation problem for autonomous robots", *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2379–2384, 2010.

[8] K.-K. Oh and H.-S. Ahn, "Distance-based undirected formations of single-integrator and double-integrator modeled agents in n-dimensional space", *International Journal of Robust and Nonlinear Control*, vol. 24, no. 12, pp. 1809–1820, 2014.

[9] J. Baillieul and A. Suri, "Information patterns and hedging brockett's theorem in controlling vehicle formations", in *Proc. of the 42nd Conference on Decision and Control*. IEEE, 2003, vol. 1, pp. 556–563.

[10] J. Baillieul and P. J. Antsaklis, "Control and communication challenges in networked real-time systems", *Proceedings of the IEEE*, vol. 95, no. 1, pp. 9–28, 2007.

[11] Z. Sun, S. Mou, M. Deghat, B. D. O. Anderson, and A. S. Morse, "Finite time distance-based rigid formation stabilization and flocking", in *Proc. of the 19th IFAC World Congress*, 2014, pp. 9183—9189.

[12] S. Mou, A. S. Morse, M. A. Belabbas, Z. Sun, and B. D. O. Anderson, "Undirected rigid formations are problematic", *IEEE Transactions on Automatic Control, conditionally accepted*, 2014.

[13] U. Helmke, S. Mou, Z. Sun, and B. D. O. Anderson, "Geometrical methods for mismatched formation control", in *Proc. of the 53rd Conference on Decision and Control*. IEEE, 2014, p. accepted.

[14] R. D. Gregory, *Classical Mechanics*, Cambridge University Press, 2006.

[15] H. G. de Marina, M. Cao, and B. Jayawardhana, "Rigid motion of undirected infitesimally rigid formations by manipulating disagreements", in *Proc. of 32nd Benelux Meeting on Systems and Control*, 2013, p. 190.